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New Approaches to Circle Packing in a Square

With Program Codes

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Preface

The problem of finding the densest packing of congruent circles in a square is obviously an interesting challenge of discrete and computational geometry with all its surprising structural forms and regularities. It is easy to understand what the problem is: to position a given number of equal circles in such a way that the circles fit fully in a square without overlapping.

With a large number of circles to be packed, the solution is very difficult to find. This difficulty is highlighted by many features. On one hand it is clear that an optimal solution can be rotated, reflected, or the circles can be reordered, and hence the number of equivalent optimal solutions blows up as the number of circles increases. On the other hand, in several cases there even exists a circle in an optimal packing that can be moved slightly while retaining the optimality. Such free circles (or rattles) mean that there exist not only a continuum of optimal solutions, but the measure of the set of optimal solutions is positive!

One nice aspect of circle packing is that the problem is clear without any further explanation, and a solution can be provided by a figure. In a certain sense, the number of circles for which we know the optimal packing with certainty indicates how sophisticated the available theoretical and computational tools are. One can think that it is a kind of measure of our technological and scientific capabilities.

Beyond the theoretically challenging character of the problem, there are several ways in which the solution methods can be applied to practical situations. Direct applications are related to cutting out congruent two-dimensional objects from an expensive material, or locating points within a square in such a way that the shortest distance between them is maximal. Circle packing problems are closely related to the 'obnoxious facility location' problems, to the Tammes problem (locate a given number of points on a sphere in such a way that the minimal distance among them is maximal), and less closely related to the Kissing Number Problem (determine how many unit spheres can touch a unit sphere in an n-dimensional space). The emerging computational algorithms can also be well utilized in other hard-to-solve optimization problems like molecule conformation.

The involvement of the authors with circle packing began in 1993 when one of the authors, Tibor Csendes, visited the University of Karlsruhe in Germany. His colleague, Dietmar Ratz, showed him a paper written in the IBM Nachrichten (42(1992) 76–77) about the difficulties involved in packing ten circles in the unit square. The problem seemed to fit the interval arithmetic-based reliable optimization algorithms they were investigating.

However, the first tries were disappointing: not even the trivial three-circle problem instance could be solved with the hardware and software environments available at that time. It became clear that symmetric equivalent solutions made the problem hard to solve.

A couple of years later a PhD student called Péter Gábor Szabó asked Tibor Csendes for an operations research subject. Among those offered was the circle packing problem, since Péter graduated as a mathematician (and not as a computer science expert, as do the majority of the PhD students at the Institute of Informatics of the University of Szeged). He began with the investigation of the structural questions of circle packing.

The Szeged team collaborated with the Department of Computer Architecture and Electronics at the University of Almería in Spain in optimization first in the framework of a Tempus project, and then for some years within a European Erasmus / Socrates programme. The head of the Spanish department, Inmaculada García, sent two PhD students to Szeged for a few months. One of the PhD students called Leocadio González Casado was interested in the circle packing problem and invested some time in a new approximating solution algorithm. Péter Gábor Szabó visited the Almería team, and the joint work resulted in the first double article reporting six world best packings at that time. These candidate optimal packings were all beaten later by the results of Eckard Specht at the University of Magdeburg. His collaboration with the Hungarian team brought other publications on the subject.

The last member joining the present team was Mihály Csaba Markót. He was a PhD student supervised by Tibor Csendes, but was working on another topic of improving the efficiency of interval arithmetic-based global optimization algorithms. His interest turned relatively late to circle packing, when the interval optimization methods seemed to be effective enough to tackle such tough problems. It was very interesting to see what kind of utilization of the problem specialities was necessary to build a numerically reliable computer procedure that cracked the next unsolved problem instances of 28, 29, and 30 circles. That produced a kind of happy end for this nearly 10-year story – and a return to the first hopeless-looking technique.

The authors had intended this volume to be a summary of the results achieved in the past few years, providing the reader with a comprehensive view of the theoretical and computational achievements. One of the major aims was to publish all the programming codes used. The checking performed by the wider scientific community helps in having the computational proofs accepted. The open source codes will enable the interested reader to improve on them and solve problem instances that still remain challenging, or to use them as a starting point for solving related application problems.

The present book can be recommended for those who are interested in discrete geometrical problems and their efficient solution techniques. The volume is also worth reading by operations research and optimization experts as a report or as a case study of how the utilization of the problem structure and specialities enabled the verified solution of the previously hopeless high-dimensional nonlinear optimization problems with nonlinear constraints. The outlined history of the whole solution procedure provides a balanced picture of how theoretical results, like repeated patterns, lower and upper bounds on possible optimum values, and approximate stochastic optimization techniques, supported the final resolution of the original problem.

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Glossary of Symbols

n	the number of circles/points to be packed/arranged, page 1
$P(r_n, S)$	a circle packing, page 13
r_n	the common radius of a circle packing, page 13
S	the size of the enclosing square of a circle packing, page 13
$\overline{r}_n$	the optimum value of for the circle packing problem, page 13
$A(m_n, \varSigma)$	a point arrangement, page 14
m_n	the minimal pairwise distance in a point arrangement, page 14
$\varSigma$	the size of the enclosing square for a point arrangement, page 14
$\overline{m}_n$	the optimum value of the point arrangement problem, page 14
$P'(R, s_n)$	an associated circle packing, page 14
s_n	the size of the enclosing square for an associated circle packing, page 14
R	the common radius for an associated circle packing, page 14
$\overline{s}_n$	the optimum value for the associated circle packing, page 14
$A'(M, \sigma_n)$	an associated point arrangement, page 14
σ_n	the size of the enclosing square for an associated point arrangement, page 14
M	the minimal pairwise distance for an associated point arrangement, page 14
$\overline{\sigma}_n$	the optimum value for the associated point arrangement problem, page 14
$\mathcal{P}_i^n$	Problem 2.i ($1 \leq i \leq 5$), page 14
$d_n(r_n, S)$	the density of a circle packing $P(r_n, S)$, page 17
T_h	threshold level in Threshold Accepting Algorithm, page 33
ξ	the variance of the perturbation size, page 35
d_{ij}	the squared distance between two points for a point arrangement, page 52
$d(x, y)$	the squared distance between $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, page 52
$f_n(x, y)$	the objective function for Problem 6.2, page 52
$\mathbb{I}$	the set of real, compact intervals, page 53
$\mathrm{lb}(A)$	the lower bound of the interval A, page 53
$\mathrm{ub}(A)$	the upper bound of the interval A, page 53
$\mathbb{I}(S)$	the set of intervals in $S \subseteq \mathbb{R}$, and the set of boxes in $S \subseteq \mathbb{R}^n$, page 53
$w(A)$	the width of the interval A, page 53
$F(X)$	an interval inclusion function of a real function f over the box X, page 53

$f(X)$	the range of a real function f over the box X, page 53
D_{ij}	the natural interval extension of d_{ij}, page 56
$F_n(X, Y)$	an interval inclusion function of $f_n(x, y)$, page 56
$\tilde{f}$	the best known lower bound for the maximum of Problem 6.2, page 57
$\tilde{f}_0$	the best known lower bound for the maximum of Problem 6.1, page 57
$D(x, y)$	the natural interval extension of $d(x, y)$, page 79
$S^m_{s..f}$	input sets of tile combinations for the B&B method, page 89
$\bar{S}^m_{s..f}$	output sets of tile combinations for the B&B method, page 89
${}^{m_1}_{s_1..f_1}S^m_{s..f}$	input sets of tile combinations for the B&B method, page 89
${}^{m_1}_{s_1..f_1}\bar{S}^m_{s..f}$	output sets of tile combinations for the B&B method, page 89
$\mathbf{PAT}(f(k))$	a pattern class, where $n = f(k)$, page 115
$\mathbf{STR}(f(k))$	a structure class, where $n = f(k)$, page 123
$[[p, q]]$	a grid packing, page 132
GP	the set of grid packings, page 132
$p_n^I(x)$	a generalized minimal polynomial, page 145
$P_n(x)$	$p_n^1(x)$, page 145

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Related websites

Program codes

Demonstration program for circle packing in Java:
`http://www.inf.u-szeged.hu/~csendes/pack.html`

Source code and the control scripts of the sophisticated discussed interval algorithm:
`http://www.sztaki.hu/~markot/packcirc.htm`

Source code, Makefile and the 'nice' library of the PDS algorithm:
`ftp://hydra.nat.uni-magdeburg.de/pub/packing/csq/csq.c`
`ftp://hydra.nat.uni-magdeburg.de/pub/packing/csq/Makefile`
`ftp://hydra.nat.uni-magdeburg.de/pub/nice/nice-2.0.tgz`

Packing websites

Packing website of E. Specht:
`http://www.packomania.com`

Packing website of P. G. Szabó:
`http://www.inf.u-szeged.hu/~pszabo/Packing_circles.html`

Papers

Papers by L. G. Casado:
`http://www.ace.ual.es/~leo/curr.html`

Papers by T. Csendes:
`http://www.inf.u-szeged.hu/~csendes/publ.html`

Papers by I. García:
http://www.ace.ual.es/~inma/

Papers by M. Cs. Markót:
http://www.sztaki.hu/~markot/

Papers by P. G. Szabó:
http://www.inf.u-szeged.hu/~pszabo/publikacios_jegyzek.html

Some more web pages

Erich's Packing Center:
http://www.stetson.edu/~efriedma/packing.html

M. Locatelli's home page:
http://www.di.unito.it/~locatell/curr.html

Circle packings by R. Peikert:
http://www.cg.inf.ethz.ch/~peikert/personal/CirclePackings/

H. Pfoertner, The Densest Packing of Equal Spheres in a Cube:
http://www.randomwalk.de/sphere/incube/

Zs. Ruttkay, Growing water lilies:
http://wwwhome.cs.utwente.nl/~zsofi/sangaku/SanSim.html

T. Tarnai's home page:
http://www.epito.bme.hu/me/htdocs/dolgozok/
/dolgozo_reszlet.php?felhasznalonev=ttarnai

T. C. Woo's works:
http://depts.washington.edu/ie/people/woo.htm
http://depts.washington.edu/ie/people/faculty/
twoo/NickelToolbox.exe

Al Zimmermann's Programming Contests:
http://www.recmath.org/contest/

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