New Trends in Equilibrium Systems, , 1–15 (2000)

Equal Circles Packing in a Square I — Problem Setting and Bounds for Optimal Solutions *

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Received 31.7.99

Abstract. In this paper, a short review of the problem of finding the densest packing of n equal circles in a square is made. On the basis of regular arrangements new lower bounds for this problem will be defined. Based on the computation of the areas of circles and minimum gap between circles and between circles and sides of the square, new upper bounds will be established. The paper also contains all the known exact values of optimal packings and the corresponding minimal polynomials.

Keywords: Optimal Packing, Minimal Polynomials, Regular Patterns.

1. Introduction

The general problem of finding the densest packing of n equal objects in a bounded space is a classical one which arises in many scientific and engineering fields. For the two-dimensional case, this problem has been studied for several different shapes of the bounded space; e.g. packing n equal circles in a circle [6, 7, 8, 11, 14, 19, 27, 41, 42], or circles in an equilateral triangle [12, 25, 26, 28, 37] or circles in a square, where circles are non-overlapping and the radius of equal circles should be maximized. In this paper all the discussions will only concern the packing circles in a square problem.

The packing problem has a long history in the mathematical literature [3, 4, 5], but the packing circles in a square problem is only 39 years old. In 1960, Leo Moser was the first who studied it, when he wrote the next conjecture [31]: eight points in or on a unit square determine a distance $\leq \frac{1}{2} \sec 15^o$. This problem for up to nine circles ($n = 2, \ldots, 9$) was solved in 1965 [43, 45], although the first proof for $n = 6$ was reported in 1970 [47] and for $n = 7$ in 1996 [33]. Between 1970 and 1990, at least ten papers have reported solutions for $n = 10$ [10, 15, 16, 17, 29, 30, 40, 44, 46, 48], but the optimal solution was given in 1990 [16]. R. Peikert *et al.* [39] proposed an elimination procedure and found optimal results for $n = 10, \ldots, 20$. Until recently the problem has been solved for $n \leq 27$ [34, 39] using computer-aided methods for

* This work was supported by the SOCRATES-ERASMUS program (25/ERMOB/1998-99), by the Ministry of Education of Spain (CICYT TIC96-1125-C03-03) and by the Consejería de Educación de la Junta de Andalucía (07/FSC/MDM), and by the Grants FKFP 0739/97, OTKA T 016413 and T 017241.

optimality proofs [33, 39] and the $n = 36$ [18] case was proven with mathematical tools. Optimality proofs of packings obtained by hand (without using a computer) exist up to 9 [24, 43, 45, 47] and for the $n = 14$ [50], 16 [49], 25 [51], and 36 [18] cases, albeit Schaer and Meir in [43] claimed that for $n = 7$ they had found the optimality proof, but they did not provide it and until now it has not been published. A more detailed history of the packing equal circles in a square problem can be found in [52].

In addition to these optimal solutions, good approximations to the problem solution, for $n \leq 52$ and n =54, 55, 56, 60, 61, 62, 72, and 78, have been published in several papers. These numerical results were obtained using several different strategies; for instance, using billiard simulation [13, 20, 21], minimization of the energy function [32], standard BFGS quasi-Newton algorithm [39], nonlinear programming solver (MINOS 5.3) [23] or the Cabri-Géomètre software in [30]. Although it is not known whether these solutions are optimal, in some cases these numerical results can help to find better solutions (if any) when they are used as lower bounds of the optimal (maximal) solutions. For instance, several strategies described in the packing literature, as that proposed by R. Peikert *et al.* in [39], are based on the knowledge of a good lower bound of the solution. In this sense, lower bounds are useful for the problem solution.

The present paper provides new results, lower and upper bounds on the radius of circles for up to one hundred circles. The paper is organized as follows: in Section 2 four equivalent definitions of the packing n circles in a square problem as well as notations and parameters used are given. Section 3 is devoted to set up several proposals for defining upper and lower bounds on the radius for the packing n circles in a unit square problem. Upper bounds are based on the computation of the circle areas as well as on the minimum gap areas between circles and between circles and the sides of the square. Our proposal for lower bounds are based on specific regular patterns to arrange n equal circles in a unit square. In Section 4, for several specific values of n, theoretical solutions and exact results of the optimal values of the problem at hand are given.

2. Definitions and notation

The packing circles in a square problem can be described by the following equivalent problem settings:

1. Locate n points in a unit square, such that the minimum distance m_n between any two points is maximal.
2. Find the value of the maximum circle radius, r_n, such that n equal non-overlapping circles can be placed in a unit square.
3. Give the smallest square of side s_n, which contains n equal and non-overlapping circles, where the radius of circles is 1.
4. Determine the smallest square of side σ_n, that contains n points with mutual distance of at least 1.

From these statements it is easy to prove that for the defined parameters m_n, r_n, s_n and σ_n the following relations hold:

$$r_n = \frac{m_n}{2(m_n+1)}, \quad m_n = \frac{2r_n}{1-2r_n}, \quad s_n = \frac{1}{r_n}, \quad \sigma_n = \frac{1}{m_n}. \tag{1}$$

The problem of maximizing the minimal pairwise distance of n points which are contained in a unit square (Problem 1) can be formulated as the following continuous global optimization problem:

$$\begin{gathered} \mu(x,y) = \min_{1\le i<j\le n} \sqrt{(x_i-x_j)^2+(y_i-y_j)^2} \\ \frac{\max \mu(x,y)}{x,y \in [0,1]^n,\ n>1 \text{ integer}} \end{gathered} \tag{2}$$

where x_i, y_i are the coordinates of the *i-th* point. The goal is not only to obtain the maximum of the minimum distance between any two points ($\max \mu(x,y)$), but also to find the respective locations of the n points in the unit square (the coordinates x_i, y_i; $1 \le i \le n$).

Definition 1. Given an $\epsilon \ge 0$ (tolerance error), it is said that *two circles* i, j (Problem 1) located at (x_i, y_i) and (x_j, y_j) *are in contact* if:

$$\sqrt{(x_i-x_j)^2+(y_i-y_j)^2} \le m_n + \epsilon$$

Similarly, a circle i is in contact to a side of the square if $x_i \le \epsilon$, $y_i \le \epsilon$, $x_i \ge 1-\epsilon$, or $y_i \ge 1-\epsilon$.

The exact mathematical definition of the contact for two circles is that m_n is equal to $\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}$, and also when a point is on the side of the square. We used this exact form of definition in the theoretical results and Definition 1 in numerical results [1], since usually a finite arithmetic is used on computers. For the present study we set $\epsilon = 10^{-10}$.

Definition 2. Given a solution of the problem (2) we call *a circle to be free*, if the center of this circle can be moved in a positive distance point without causing overlapping with other circles.

If a packing contains one or more free circles then the solution is obviously not unique and the locations of the center of any free circles form a non-empty interior and connected set. In this paper, the number of contacts will be denoted by c_n, the number of free circles by f_n. In all the figures, a contact will be represented by a short line section and free circles will be denoted by dark shading.

Definition 3. The density d_n of a packing can be given by:

$$d_n = n\pi r_n^2 = \frac{n\pi m_n^2}{4(m_n+1)^2}.$$

3. Lower and upper bounds of the maximized minimum distance between n points in a unit square

An approximate solution of (2) can be used as a lower bound, m_n^l, of the optimal solution m_n. Let suppose that it is possible to determine a good value for the upper bound, m_n^u, of the packing problem, so an approximate solution m_n of (2) can be considered a good solution of the packing problem if it is close enough to m_n^u.

On the other hand, it has been proved in [5] that the densest packing of circles in the plane is a regular (hexagonal) arrangement (see Figure 1) and the density of this packing is $\sqrt{3}\pi/6 \approx 0.9068996821$. In addition, it can be proved that this value can provide an upper bound for the solution of (2), although better upper bound values can also be proposed. Using this hexagonal pattern an asymptotic formula for m_n, $m_n \approx \sqrt{\frac{2}{\sqrt{3}n}}$, can be obtained [3].

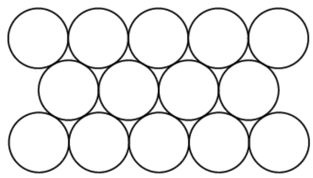

Figure 1. The arrangement of the densest packing in the plane.

It is also known that, for several values of n, the optimal solutions of the packing circles in a unit square problem are regular arrangements. With these idea in mind, it easy to guess that for the problem (2), other regular arrangements may also provide at least a good lower bound of the optimal solution. In this section improved lower and upper bounds are discussed.

3.1. Lower bounds of m_n using patterns

THEOREM 1 *Consider the packing circles in a unit square problem according to Problem 1. The maximum of the minimal distance between n points in the unit square m_n is not smaller than*

$$m_n^l = \max(L_1(n), L_2(n), L_3(n), L_4(n), L_5(n))$$

with

$$
\begin{aligned}
L_1(n) &= \frac{1}{\lceil\sqrt{n}\rceil - 1},\\
L_2(n) &= \frac{1}{\lceil\sqrt{n+1}\rceil - 3 + \sqrt{2+\sqrt{3}}},\\
L_3(n) &= \frac{1}{\lceil\sqrt{n+2}\rceil - 5 + 2\sqrt{2+\sqrt{3}}},\\
L_4(n) &= \frac{k^2 - k - \sqrt{2k}}{k^3 - 2k^2}, \text{ if } n = k(k+1), \text{ and 0 otherwise,}\\
L_5(n) &= \sqrt{\frac{1}{p^2} + \frac{1}{q^2}}, \text{ if } n = \left\lceil \frac{(p+1)(q+1)}{2} \right\rceil, \ p^2 \le 3q^2, \text{ and } q^2 \le 3p^2;\\
&\quad \text{and 0 otherwise.}
\end{aligned}
$$

(Here $\lceil . \rceil$ denotes the smallest integer not smaller than the argument)

Proof: The new lower bounds are based on the use of some patterns proposed in [13, 22, 32], on our own new patterns and on the assumption that $m_{n+1} \le m_n$, thus m^l_{n+1}, or any value of the m_{n+1} of (2), is also a lower bound of m_n. In this way lower bounds for all the values of n $(2 \le n \le 100)$ can be obtained. The set of patterns we will take into account, for defining lower bounds, are (see them illustrated on Figure 3):

PAT1 For $n = k^2$ $(k \ge 2,\ k \in \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers) the packings have an obvious square grid pattern of $k \times k$ points [13], for which the minimum distance between the points is $L_1(n) = \frac{1}{k-1}$. This lower bound can be applied also for n values not conforming the equation $n = k^2$ with the $k = \lceil\sqrt{n}\rceil$ substitution.

PAT2 For $n = k^2 - 1 = (k-1)^2 + 2(k-1)$ $(k \ge 2,\ k \in \mathbb{N})$ the pattern can be considered as a square pattern like PAT1 of $(k-1)^2$ circles to which one row and one column of shifted circles are inserted [13]. In this case $L_2(n) = \frac{1}{k-3+\sqrt{2+\sqrt{3}}}$, and $k = \lceil\sqrt{n+1}\rceil$, because $m_n(k - 3 + 2\cos(15^o)) = 1$.

PAT3 For $n = k^2 - 2 = (k-1)^2 + 2(k-2) + 1$ $(k \ge 3,\ k \in \mathbb{N})$ the pattern is similar to PAT2, but here there are two shifted columns and two shifted rows of fitting length [13]. In this case $L_3(n) = \frac{1}{k-5+2\sqrt{2+\sqrt{3}}}$, and $k = \lceil\sqrt{n+2}\rceil$, because $m_n(k - 5 + 2\cos(15^o) + 2\cos(15^o)) = 1$.

PAT4 For $n = k(k+1)$ $(k \ge 2,\ k \in \mathbb{N})$ the pattern consists of $(k-1)$ alternating columns with k circles each [13]. In Figure 3 this pattern is demonstrated for $n = 20$. In this case

$$L_4(n) = \frac{k^2 - k - \sqrt{2k}}{k^3 - 2k^2},$$

because $m_n(k - 1 + \sin(\alpha)) = 1$, where $\cos(\alpha) = \frac{1}{m_n k}$.

PAT5 In this new pattern class the centers of the circles are located in a grid which is built by composing two shifted rectangular grids. An example of this pattern is shown in Figure 2 and in Figure 3. The number of non-overlapping circles (n) which can be arranged in a square using this pattern is a function of two integers $p, q \in \mathbb{N}$, $p \neq q$, subject to the following constraints:

$$p^2 \leq 3q^2,$$
$$q^2 \leq 3p^2.$$

For the sake of clarity, three cases will be distinguished:

- p and q are both even; then $p = 2p^*$; $q = 2q^*$, and $n = 2p^*q^* + p^* + q^* + 1$
- p is even and q is odd; then $p = 2p^*$; $q = 2q^* + 1$, and $n = (2p^* + 1)(q^* + 1)$
- both p and q are odd, then $p = 2p^* + 1$; $q = 2q^* + 1$, and $n = 2(p^* + 1)(q^* + 1)$

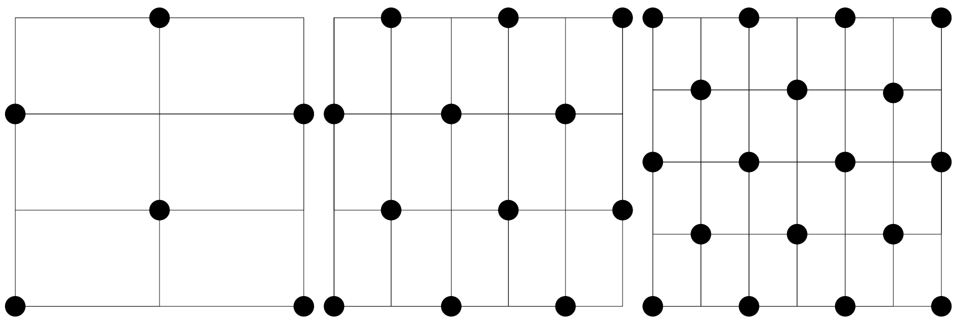

Figure 2. Examples (p,q,n) using PAT5, (2,3,6), (5,3,12) and (6,4,18).

and for all the cases $L_5(n) = \sqrt{\frac{1}{p^2} + \frac{1}{q^2}}$ holds for this lower bound.

The maximum of the derived lower bounds provide the statement of the theorem with the suggested substitutions wherever necessary. ■

Table 1 summarizes the results obtained by applying the discussed patterns. With patterns PAT1 to PAT5 only 48 out of 99 values of n can directly be modelled. In columns with headers: n, pattern and m_n^l, the number of circles, the pattern used and the value of the lower bound of m_n, respectively, are contained. If two different patterns exist for the same value of n, then the better one has been chosen. In column n, boldface means that it is known that this packing is the optimal solution of the given packing circles in a square problem. All the values of m_n^l in Table 1 are those calculated by the respective formula of the specific pattern.

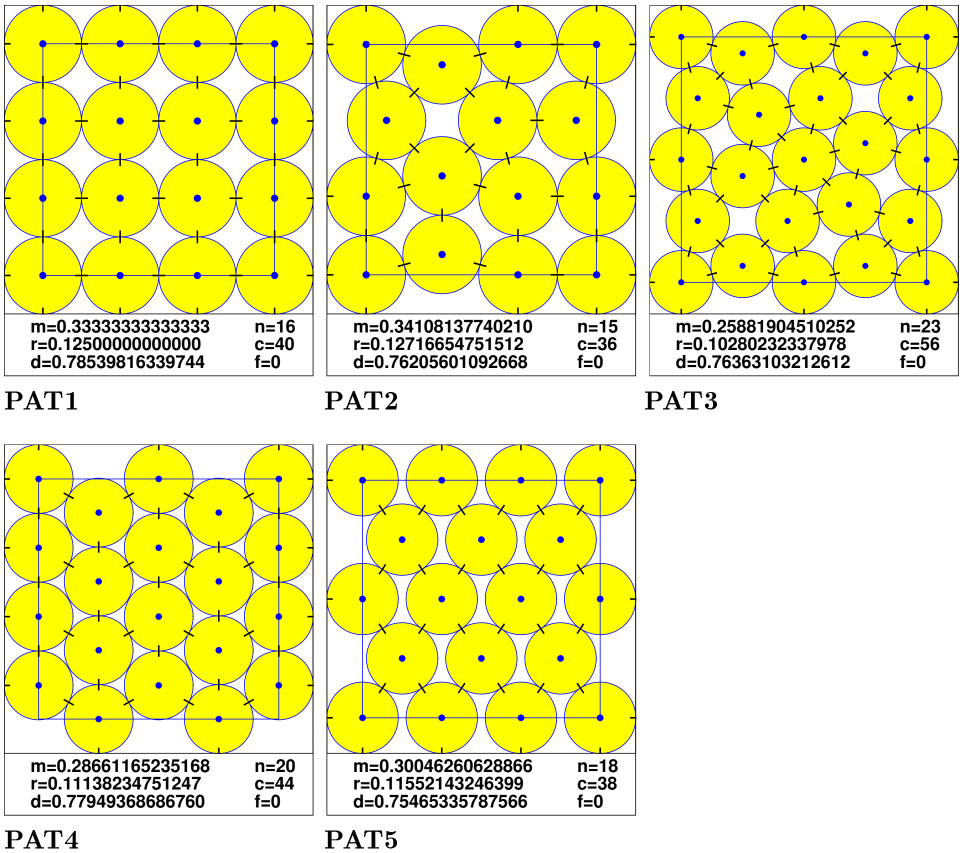

Figure 3. Examples of patterns for $n = 16$ (PAT1), $n = 15$ (PAT2), $n = 23$ (PAT3), $n = 20$ (PAT4) and $n = 18$ (PAT5). These examples are optimal packings.

In [35], using a similar pattern as PAT5, it was proved, that if $N_p(\sigma)$ is the maximum number of points with mutual distance of at least 1 that can be placed into a square of side σ, then

$$\sigma^2 + \frac{1-\sqrt{3}}{2}\sigma \leq \frac{\sqrt{3}}{2}N_p(\sigma).$$

It is easy to see that from this inequality, the following lower bound of m_n can be given:

$$\frac{1-\sqrt{3}+\sqrt{4-2\sqrt{3}+8\sqrt{3}n}}{2\sqrt{3}n} \leq m_n \tag{3}$$

After an analysis of (3) and m_n^l, it can be seen, that for $2 \leq n \leq 100$ m_n^l is always greater than the lower bound given by (3).

Table 1. Lower bounds of m_n obtained using repeated patterns. The proven optimal solutions are denoted by boldface in columns n.

n	pattern	m_n^l	n	pattern	m_n^l	n	pattern	m_n^l
2	PAT5	1.414213	21	PAT5	0.260341	52	PAT5	0.165386
3	PAT2	1.035276	**23**	PAT3	0.258819	56	PAT4	0.156156
4	PAT1	1.000000	**24**	PAT2	0.254333	59	PAT5	0.150231
5	PAT5	0.707106	**25**	PAT1	0.250000	63	PAT5	0.146772
6	PAT5	0.600925	**27**	PAT5	0.235849	64	PAT1	0.142857
7	PAT3	0.535898	30	PAT4	0.224502	65	PAT5	0.138888
8	PAT2	0.517638	32	PAT5	0.208333	72	PAT4	0.135416
9	PAT1	0.500000	34	PAT3	0.205604	75	PAT5	0.132089
10	PAT5	0.416666	35	PAT2	0.202763	80	PAT5	0.129576
12	PAT5	0.388730	**36**	PAT1	0.200000	81	PAT1	0.125000
13	PAT5	0.353553	39	PAT5	0.194365	83	PAT5	0.122890
14	PAT3	0.348915	42	PAT4	0.184277	88	PAT5	0.120185
15	PAT2	0.341081	44	PAT5	0.174379	90	PAT4	0.119501
16	PAT1	0.333333	47	PAT3	0.170540	94	PAT4	0.117924
18	PAT5	0.300462	48	PAT5	0.169329	99	PAT4	0.116018
20	PAT4	0.286611	49	PAT1	0.166666	100	PAT1	0.111111

3.2. An upper bound of $\boldsymbol{m_n}$

THEOREM 2 *Consider the packing circles in a unit square problem according to Problem 2. The maximum of the minimal distance between n points in the unit square m_n is not greater then*

$$m_n^u = \min(U_1(n), U_2(n))$$

with

$$\begin{aligned}
U_1(n) &= \frac{2}{\sqrt{n\pi + C_n(\sqrt{3}-\frac{\pi}{2}) + (4\lfloor\sqrt{n}\rfloor - 2)(2-\frac{\pi}{2}) - 2}}, \text{ where}\\
&\quad C_n = n-2 \qquad\qquad \text{if } 3 \le n \le 6,\\
&\quad C_n = n-1 \qquad\qquad \text{if } 7 \le n \le 9, \text{ and}\\
&\quad C_n = 3\left\lfloor\frac{n}{2}\right\rfloor - 5 + n \bmod 2 \text{ otherwise.}\\
U_2(n) &= \frac{1+\sqrt{1+(n-1)\frac{2}{\sqrt{3}}}}{n-1}.
\end{aligned}$$

(Here $\lfloor.\rfloor$ denotes the greatest integer not greater than the argument.)

Proof: Let us describe an analytical expressions for the upper bounds of the packing problem. It is based on the area of n circles and on that of their neighbourhood within the unit square:

- U_1: The area of n circles ($nr^2\pi$) plus the area of the gaps between circles and between circles and the sides of the square must be less than 1. The convex hull of the centers of the circles divide the square to two regions. Let us compute now the area of the minimal gaps in these regions.

  Circles-to-side case. When packing n circles in a square, it is evident that the minimum gap between circles and the sides of the square is obtained when $4\lfloor\sqrt{n}\rfloor - 2$ gaps (with the respective corrections at the corners) are in contact to the sides of the square with $r_n^2(2-\pi/2)$ areas per gaps (see Figure 4).

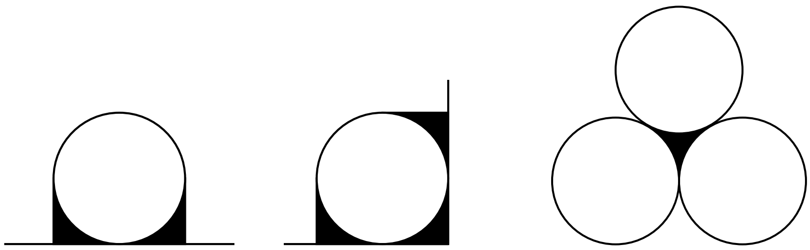

Figure 4. The minimal gaps between a circle and the sides of a square and between three circles.

  Circles-to-circles case In a similar way, the minimal gap between circles occurs when three circles in the packing have all the possible three contact, as it is shown in Figure 4. For a packing of n circles the minimal number of circle-to-circle gaps is given by:

$$C_n = \begin{cases} n-2 & 3 \le n \le 6 \\ n-1 & 7 \le n \le 9 \\ 3\left\lfloor \frac{n}{2} \right\rfloor - 5 + n \bmod 2 & \text{otherwise} \end{cases}$$

  when the circles are in the densest (hexagonal) arrangement. The area of this minimum gap between three circles is $r_n^2(\sqrt{3}-\frac{\pi}{2})$, thus the sum of the area of the circles and gaps is:

$$S = r_n^2\left(n\pi + C_n(\sqrt{3}-\frac{\pi}{2}) + (4\lfloor\sqrt{n}\rfloor - 2)(2-\frac{\pi}{2})\right).$$

  Due to $S \le 1$ then

$$r_n^u \le \left(n\pi + C_n(\sqrt{3}-\frac{\pi}{2}) + (4\lfloor\sqrt{n}\rfloor - 2)(2-\frac{\pi}{2})\right)^{-\frac{1}{2}},$$

$$m_n^u \le U_1(n) = \frac{2}{\sqrt{n\pi + C_n(\sqrt{3}-\frac{\pi}{2}) + (4\lfloor\sqrt{n}\rfloor - 2)(2-\frac{\pi}{2})} - 2}.$$

- U_2: From Oler theorem [36] the following statement implies [9]: if X is a compact convex subset of the plane, then the number of points with mutual distance of at least 1 is at most

$$\frac{2}{\sqrt{3}}A(X) + \frac{1}{2}P(X) + 1 \tag{4}$$

where $A(X)$ is the area and $P(X)$ is the perimeter of X. If X is a σ side of square, then $A(X) = \sigma^2$ and $P(X) = 4\sigma$.
In this case from (4) and (1) implies

$$m_n \le U_2(n) = \frac{1 + \sqrt{1 + (n-1)\frac{2}{\sqrt{3}}}}{n-1}.$$

■

Using the values of the lower and upper bounds of m_n, described in this section, and our own numerical results, obtained by the algorithm described in [1], Figure 5 has been built.

4. Minimal polynomials of optimal packings and exact results

For several values of n, the exact values of the solution of the packing circles in a unit square problem are known. The set of known exact values is given in Table 2.

Table 2. Exact values of m_n for some n.

n	m_n	n	m_n
2	$\sqrt{2}$	15	$(1+\sqrt{2}-\sqrt{3})/2$
3	$\sqrt{6}-\sqrt{2}$	16	$1/3$
4	1	18	$\sqrt{13}/12$
5	$\sqrt{2}/2$	20	$(6-\sqrt{2})/16$
6	$\sqrt{13}/6$	23	$(\sqrt{6}-\sqrt{2})/4$
7	$2(2-\sqrt{3})$	24	$4+2\sqrt{3}-\sqrt{26+15\sqrt{3}}$
8	$(\sqrt{6}-\sqrt{2})/2$	25	$1/4$
9	$1/2$	27	$\sqrt{89}/40$
12	$\sqrt{34}/15$	36	$1/5$
14	$2(4-\sqrt{3})/13$		

When the optimal solution for the problem of packing n circles in a square is known, it is possible to define the corresponding minimal polynomial. The procedure to obtain this minimal polynomial consists of setting up a system of equations representing the corresponding optimal packing. Variables of the system of equations are the coordinates of the centers of the circles and the maximum minimal distance, m_n. So the number of variables for the system is $2n+1$. For every two points (centers of the circles) whose distance is equal to m_n, an equation can be written. In addition, for every point located at the border of the square, it is known

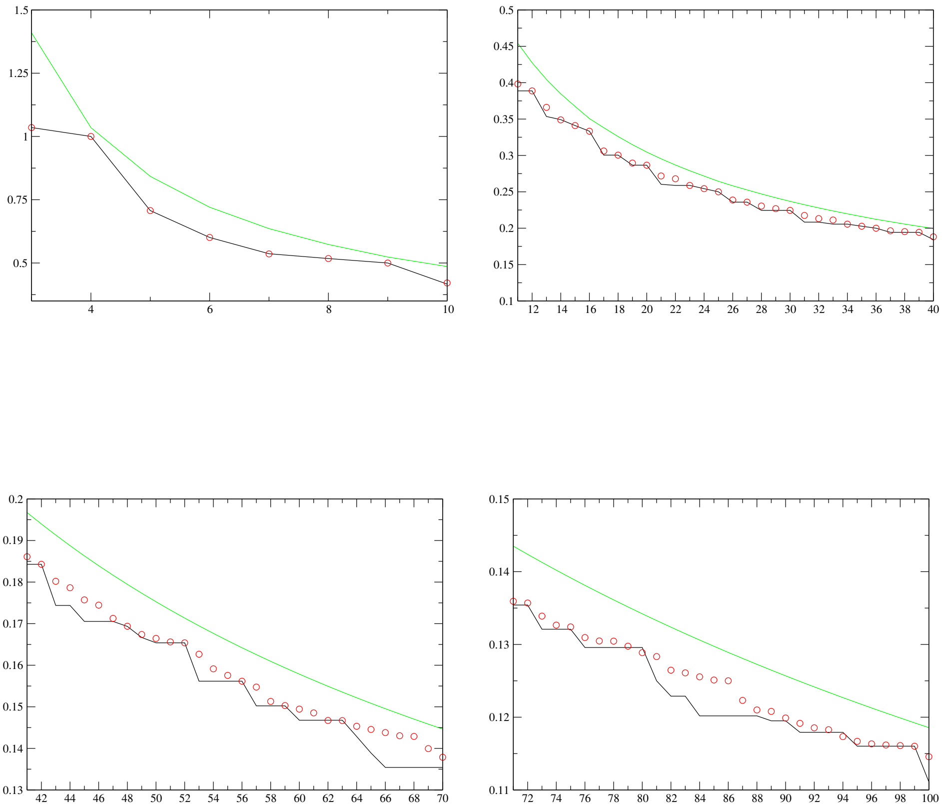

Figure 5. Lower (m_n^l) and upper bound (m_n^u) on the m_n values and the best m_n values found by our program [1] (denoted by circles) are displayed.

that at least one of their coordinates are equal 0 or 1, so the corresponding equation(s) can be written. After reducing the system of equations, a symbolic algebra package like MAPLE, MATHEMATICA, etc., can be used to calculate the Gröbner basis of the system [39] and the minimal polynomial for which the smallest positive root is m_n.

The minimal polynomial can be obtained for most of the cases, but e.g. for $n = 13$ this method was unable [39] to provide the minimal polynomial. In Table 3, the minimal polynomials for several values of n are shown [16, 38]. The degrees of these polynomials indicate how difficult it can be to find the corresponding solution of the packing problem. When the degree of the minimal polynomial was less than

Table 3. The minimal polynomials of certain optimal solutions. Here * denotes cases with unknown minimal polynomials.

n	Minimal polynomials
2	$m^2 - 2$
3	$m^4 - 16m^2 + 16$
4	$m - 1$
5	$2m^2 - 1$
6	$36m^2 - 13$
7	$m^2 - 8m + 4$
8	$m^4 - 4m^2 + 1$
9	$2m - 1$
10	$1180129m^{18} - 11436428m^{17} + 98015844m^{16} - 462103584m^{15} + 1145811528m^{14} +$ $-1398966480m^{13} + 227573920m^{12} + 1526909568m^{11} - 1038261808m^{10} - 2960321792m^9 +$ $+7803109440m^8 - 9722063488m^7 + 7918461504m^6 - 4564076288m^5 + 1899131648m^4 -$ $-563649536m^3 + 114038784m^2 - 14172160m + 819200$
11	$m^8 + 8m^7 - 22m^6 + 20m^5 + 18m^4 - 24m^3 - 24m^2 + 32m - 8$
12	$225m^2 - 34$
13	$5322808420171924937409m^{40} + 586773959338049886173232m^{39} +$ $+13024448845332271203266928m^{38} - 12988409567056909990170432m^{37} + \ldots +$ $+2960075719794736758784m^2 - 174103532094609162240m + 4756927106410086400$
14	$13m^2 - 16m + 4$
15	$2m^4 - 4m^3 - 2m^2 + 4m - 1$
16	$3m - 1$
17	$m^8 - 4m^7 + 6m^6 - 14m^5 + 22m^4 - 20m^3 + 36m^2 - 26m + 5$
18	$-144m^2 + 13$
19	$242m^{10} - 1430m^9 - 8109m^8 + 58704m^7 - 78452m^6 - 2918m^5 + 43315m^4 + 39812m^3 -$ $-53516m^2 + 20592m - 2704$
20	$128m^2 - 96m + 17$
21	*
22	*
23	$16m^4 - 16m^2 + 1$
24	$m^4 - 16m^3 + 20m^2 - 8m + 1$
25	$4m - 1$
26	*
27	$1600m^2 - 89$
36	$5m - 1$

five, we were able to calculate the exact value of m_n. These values can be found in Table 2. For $n = 10$, 11, 17 and 19, we were able to provide close bounds of m_n using a reliable algorithm for finding the first root of a function [2]. This algorithm is based on a branch and bounds technique using interval arithmetic. Numerical results for these four values of n are given in interval form in Table 4.

5. Summary

In the first of this series of papers a short introduction is given to the problem of equal circles packing in a unit square. The most important earlier results are summarised, and new and improved lower and upper bounds on the maximal distance of the centers of the circles are given. These bounds are based on geometrical

Table 4. Reliable numerical results for $n = 10, 11, 17, 19$.

n	Lower and upper bounds for m_n
10	[0.421279541378085, 0.421279544064904]
11	[0.398207310236837, 0.398207310236850]
17	[0.306153985300327, 0.306153985300338]
19	[0.289541991994965, 0.289541991994996]

considerations and on an extended set of patterns of the related circles. These bounds can be applied directly in some computational procedures for finding the best packings, but they can also be well utilised in the evaluation of the approximate solutions provided by our numerical algorithm discussed in the second paper of this series. The known exact optimal values are comprised in a table together with the related minimal polynomials for some cases.

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