# GLOBAL ATTRACTIVITY OF THE ZERO SOLUTION FOR WRIGHT'S EQUATION*

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**Abstract.** In a paper published in 1955, E.M. Wright proved that all solutions of the delay differential equation $u'(t) = -\alpha u(t-1)(1 + u(t))$ converge to zero for $\alpha \in (0, 1.5]$, and conjectured that this is even true for $\alpha \in (0, \frac{\pi}{2})$. The present paper provides a computer-assisted proof of the conjecture for $\alpha \in [1.5, 1.5706]$ (compare with $\frac{\pi}{2} = 1.570796...$).

**Key words.** Delayed logistic equation, slowly oscillating periodic solution, Wright's conjecture, verified computational techniques, computer-assisted proof, interval arithmetic

**AMS subject classifications.** 34K13, 34K20, 37L15, 65G20, 65L03

**1. Introduction.** In 1955, Edward M. Wright [35], motivated by an unpublished note of Lord Cherwell about a heuristic approach to the density of prime numbers (see also [10], [36]), studied the delay differential equation

$$\dot{u}(t) = -\alpha u(t-1)[1 + u(t)], \quad \alpha > 0. \tag{1.1}$$

An equivalent form, the so-called delayed logistic equation or Hutchinson's equation

$$\dot{v}(t) = \alpha v(t)[1 - v(t-1)],$$

was introduced by Hutchinson [11] in 1948 for ecological models.

Considering only those solutions of equation (1.1) which have values in $(-1, \infty)$, the transformation $x = \log(1 + u)$ leads to the equation

$$\dot{x}(t) = f_\alpha(x(t-1)) \tag{1.2}$$

with $f_\alpha(\xi) = -\alpha(e^\xi - 1)$, $\xi \in \mathbb{R}$. Throughout this paper (1.2) is also called Wright's equation. It is one of the simplest nonlinear delay differential equations. Wright [35] was the first who obtained deep results for equation (1.2). He proved among others that all solutions of (1.2) approach zero as $t \to \infty$ provided $\alpha \le \frac{3}{2}$, and he made the following remark: *My methods, at the cost of considerable elaboration, can be used to extend this result to* $\alpha \le \frac{37}{24}$ *and, probably to* $\alpha < 1.567...$ *(compare with* $\frac{\pi}{2} = 1.570796...$*). But the work becomes so heavy for the last step that I have not completed it.*

For every $\alpha > \frac{\pi}{2}$, Wright [35] proved the existence of bounded solutions of equation (1.2) which do not tend to zero. If $\alpha < \frac{\pi}{2}$ then the roots of the characteristic equation $z + \alpha e^{-z} = 0$ of the linear variational equation $\dot{y}(t) = -\alpha y(t-1)$ of (1.2) have negative real parts. Thus the zero solution of (1.2) is locally attractive.

Based on the above facts the question of the global attractivity of the zero solution of (1.2) for parameter values $\alpha < \frac{\pi}{2}$ arises naturally, and it is known as Wright's conjecture.

*This work was partially supported by the Hungarian National Science Foundation Grants OTKA No. T 048377, T 046822, and K 75517, and also by the Grants TÁMOP-4.2.2/08/1/2008-0008, TÁMOP-4.2.1/B-09/1/KONV-2010-0005 and TÁMOP-4.2.2.A-11/1/KONV-2012-0060.

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CONJECTURE 1. *For every $\alpha < \frac{\pi}{2}$, the zero solution of equation (1.2) is globally attractive, i.e., all solutions approach zero as $t \to \infty$.*

The problem is still open, and, as far as we know, Wright's result, i.e., $\alpha \leq \frac{3}{2}$, is still the best one for global attractivity of the zero solution. Walther [32] proved that the set of parameter values $\alpha$, for which 0 is globally attracting, is an open subset of $(0, \frac{\pi}{2})$.

We mention that Wright's equation motivated the development of a wide variety of deep analytical and topological tools (see e.g. the monographs [7], [9]) to get more information about the dynamics of (1.2). For example, Jones [12] proved the existence of slowly oscillating periodic solutions of (1.2) for $\alpha > \frac{\pi}{2}$, where slow oscillation means that $|z_1 - z_2| > 1$ for each pair of zeros $z_1, z_2$ of the periodic solution. Chow and Mallet-Paret [4] showed that there is a supercritical Hopf bifurcation of slowly oscillating periodic solutions form the zero solution at $\alpha = \frac{\pi}{2}$.

Applying the Poincaré–Bendixson type result of Mallet-Paret and Sell [23] it can be shown that any solution of equation (1.2) approaches either a nontrivial periodic solution or zero as $t \to \infty$. Mallet-Paret and Walther [24] verified that slow oscillation is generic for equation (1.2) for all $\alpha > 0$, that is, for an open dense set of initial data from the phase space the solutions are eventually slowly oscillating. Mallet-Paret [21] obtained a Morse decomposition of the global attractor of (1.2); see also McCord and Mischaikow [26]. For several other results we refer to the monographs [7], [9]. Despite of the simplicity of equation (1.2) and the very intensive investigation since 1955, it seems that we are still far from the complete understanding of the dynamics of (1.2).

Conjecture 1 is not the only open question for equation (1.2). Recently, Lessard [20] made some progress toward the proof of Jones' conjecture [12]:

CONJECTURE 2. *For every $\alpha > \frac{\pi}{2}$, equation (1.2) has a unique slowly oscillating periodic orbit.*

In the work [19] we defined the set $U(\alpha)$, in the space of continuous functions from $[-1, 0]$ into $\mathbb{R}$, as the forward extension (by the semiflow) of a local unstable manifold at zero. Then we described the dynamic and geometric structure of its closure $\overline{U(\alpha)}$. The results of [19] are valid for equations including (1.2). In [17] for equation (1.2) we formulated the so called generalized Wright's conjecture:

CONJECTURE 3. *For every $\alpha > 0$, the set $\overline{U(\alpha)}$ is the global attractor for equation (1.2).*

An affirmative answer for Conjecture 3 would mean a more or less complete understanding of the dynamics of equation (1.2). For example, for the equation $\dot{x}(t) = -ax(t) - b\tanh(cx(t-1))$ with $a \geq 0$, $b > 0$, $c > 0$, the analogue of Conjecture 3 is known to be valid [16].

In this paper we prove that Wright's conjecture is equivalent to the nonexistence of slowly oscillating periodic solutions, and we develop a reliable computational tool to exclude the existence of slowly oscillating periodic solutions with amplitude greater than a certain constant $\epsilon_0 > 0$. For $\frac{3}{2} \leq \alpha < \frac{\pi}{2}$, following a geometric idea of Walther [33], we project slowly oscillating periodic solutions of (1.2) and periodic solutions of $\dot{y}(t) = -\frac{\pi}{2} y(t-1)$ into the plane $\mathbb{R}^2$, and explicitly construct $\epsilon(\alpha) > 0$ so that for every slowly oscillatory periodic solution $p$ of (1.2), $\max_{t\in\mathbb{R}} p(t) > \log \frac{\pi}{2\alpha}$ holds. Since $\log \frac{\pi}{2\alpha} \to 0$ as $\alpha \to \frac{\pi}{2}$, we are able to prove Wright's conjecture only for those values of $\alpha$ for which $\log \frac{\pi}{2\alpha} > \epsilon_0$. These results combined verify Wright's conjecture for

$$\alpha \leq 1.5706.$$

Important parts of the proof are based on verified numerical calculations applying interval arithmetic, the respective inclusion functions and guaranteed reliability

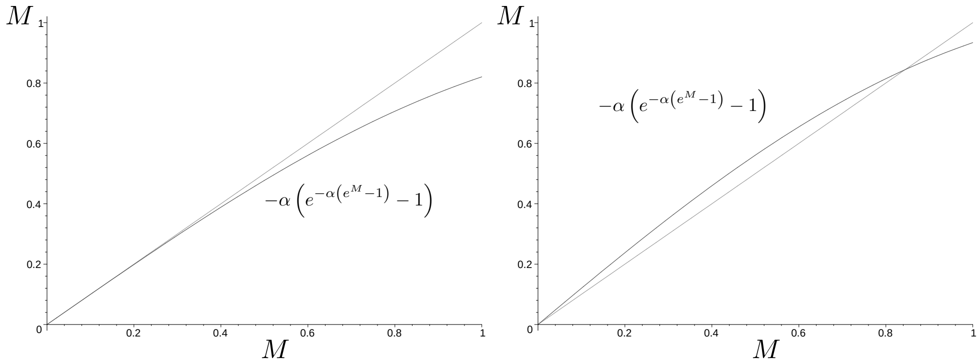


FIG. 1.1. *The simplest bounding function values compared to $M$ for $\alpha = 1.0$ (left) and $\alpha = 1.1$ (right).*

bounds for the solutions of the involved delay differential equation [1, 30].

The idea of Wright's original proof is the following. For the sake of simplicity, we formulate it to exclude the existence of periodic solutions. We show in Section 3 that the nonexistence of slowly oscillatory periodic solutions and the global attractivity of 0 are equivalent statements. Assume that $y$ is a slowly oscillatory periodic solution of equation (1.1) with maximal value $M$ and with minimal value $-m$, where $m > 0$, $M > 0$. Let $z$ be a time with $y(z) = 0$. Then, by the fact [23] that for periodic solutions of (1.2) there is a unique zero of $y'$ between two consecutive zeros of $y$, and three consecutive zeros of $y$ determine the minimal period, $z+1$ must be a minimum or a maximum point of $y(t)$. Consider first the case $y(z+1) = M$. Then

$$M = \int_z^{z+1} y'(t)\,dt = -\alpha \int_z^{z+1} \left(e^{y(t-1)} - 1\right)\,dt = -\alpha \int_{z-1}^{z} \left(e^{y(t)} - 1\right)\,dt$$

$$\leq -\alpha \int_{z-1}^{z} \left(e^{-m} - 1\right)\,dt = -\alpha\left(e^{-m} - 1\right).$$

In other words, $M$ can be bounded as $M \leq -\alpha\left(e^{-m}-1\right)$, and in a similar way $m \leq \alpha\left(e^{M}-1\right)$. Now one can derive the inequality $M \leq \overline{M} := -\alpha\left(e^{-\alpha\left(e^{M}-1\right)}-1\right)$ which is illustrated on Figure 1.1. For $\alpha = 1$, the respective bounding function is below $M$ for positive values of $M$, i.e., $\overline{M} < M$ holds, which is a contradiction. Therefore, no slowly oscillating periodic solutions can exist. But for $\alpha > 1$ (illustrated in the figure for $\alpha = 1.1$) the inequality does not imply this statement, since not all positive values of $M$ can be discarded.

The above reasoning was strengthened by Wright utilizing a better estimation of the possible solution up to $z$. Applying the bounds $-m \leq y(t) \leq M$, we obtain

$$-\alpha\left(e^{M}-1\right) \leq y'(t) \leq -\alpha\left(e^{-m}-1\right) \tag{1.3}$$

from (1.1) for all $t$. The bound (1.3) for the derivative of $y$ allows a wider set of values $\alpha$ for which slowly oscillating periodic solutions cannot exist. The first and the improved allowed regions for the slowly oscillating periodic solutions are given in Figure 1.2.

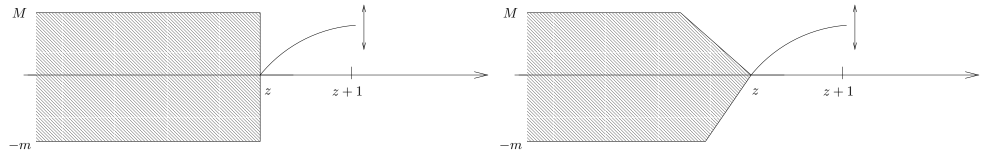


FIG. 1.2. *Regions where the solutions can be before z, according to the first and second bounding scheme.*

We skip here the technical details from Wright's paper, and just give the conditions obtained by him:

$$M \le -\alpha\left(e^{-m}-1\right)+(-m)\frac{e^{-m}}{e^{-m}-1}-1 \quad \text{if } \alpha\left(e^{-m}-1\right)\le -m, \tag{1.4}$$

$$M \le \alpha - \frac{1-e^{\alpha\left(e^{-m}-1\right)}}{\left(1-e^{-m}\right)}, \tag{1.5}$$

$$m \le \alpha\left(e^{M}-1\right) - M\frac{e^{M}}{e^{M}-1}+1. \tag{1.6}$$

These inequalities imply a new upper bound $\overline{M} = M$, i.e., $M \le \overline{M}$ holds for all $M > 0$.

Figure 1.3 shows the effects of the improved bounding inequalities. Compare the upper left bounding function with that on the Figure 1.1: it is obvious that the new, stronger bounding function excludes the possibility of a periodic solution with a positive amplitude. For $\alpha = 1.5$ one can see that the bounding function implies the nonexistence of slowly oscillating periodic solutions, while for $\alpha > 1.5$ (illustrated in the figure for $\alpha = 1.55$ and $\alpha = \pi/2$), no conclusion can be drawn on the basis of these bounding functions.

The present paper provides a new iterative bounding scheme that is based on the ideas of Wright. First we present the scheme based on theoretical notion (Section 5). Then a purely technical presentation is given in Sections 6 to 7 that starts directly from the anticipated ideas of Wright, and need only elementary considerations. The actual computer implementation also applies guaranteed reliability bounds on the trajectories, and safe bounds on the zeros. The main result of the paper is

THEOREM 1.1. *If $\alpha \in [1.5, 1.5706]$, then the zero solution of equation (1.2) is globally attractive.* We have also completed a computational proof of the following assertion.

THEOREM 1.2. *If $\alpha \in [1.5, \pi/2]$ and $p^\alpha : \mathbb{R} \to \mathbb{R}$ is a slowly oscillating periodic solution of (1.2), then* $\max_{t\in\mathbb{R}} |p^\alpha(t)| < 0.04$ *holds.*

Applying center manifold theory and local Hopf bifurcation techniques it is possible to find an $\epsilon_* > 0$ independently of $\alpha$ such that, for any $\alpha \in [1.5, \pi/2]$ and any slowly oscillating periodic solution $p^\alpha : \mathbb{R} \to \mathbb{R}$ of (1.2), the inequality $\max_{t\in\mathbb{R}} |p^\alpha(t)| > \epsilon_*$ holds [18]. However, $\epsilon_*$ is much smaller than the constant 0.04 obtained in Assertion 1, and thus the proof is still not complete for Conjecture 1.

**2. Preliminary results, notation.** The results which we mention below without references are all well known, and can be found e.g. in [7] or [9].

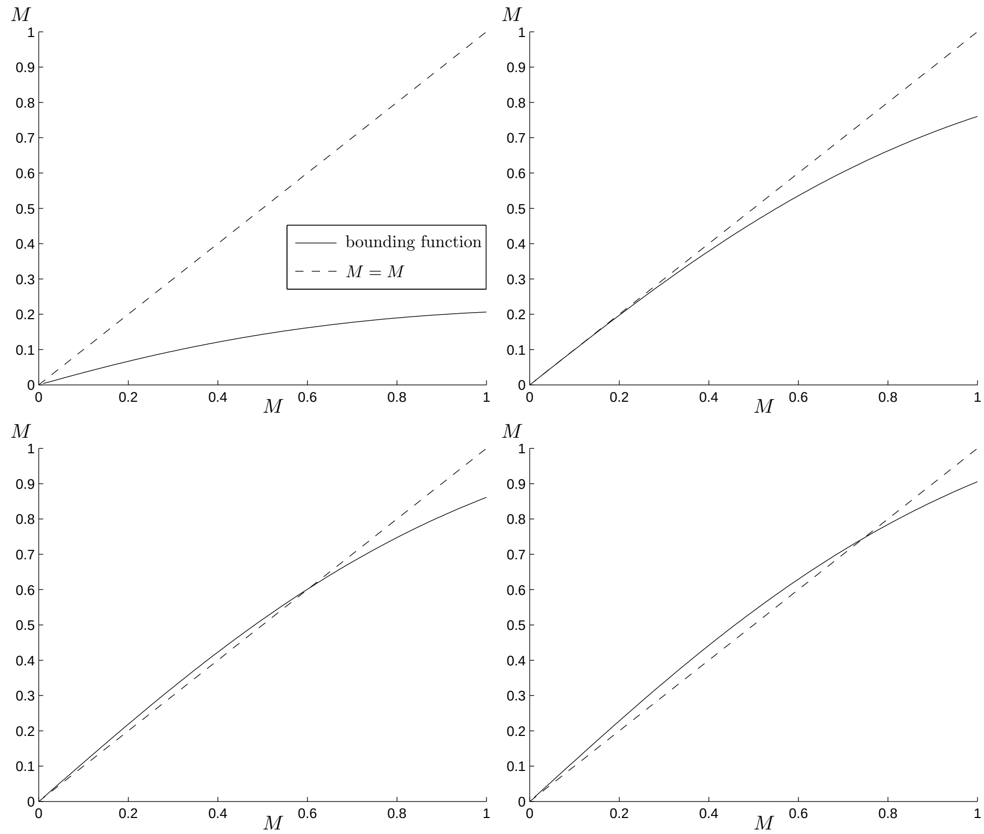


Fig. 1.3. *The second bounding function values compared to* $M$ *for* $\alpha = 1.1$ *(top left),* $\alpha = 1.5$ *(top right),* $\alpha = 1.55$ *(bottom left), and* $\alpha = \pi/2$ *(bottom right).*

The natural phase space for equation (1.2) is $C = C([-1,0],\mathbb{R})$ equipped with the supremum norm $||\cdot||$. By the method of steps, every $\phi \in C$ uniquely determines a solution $x = x^{\phi} : [-1,\infty) \to \mathbb{R}$ of (1.2), i.e., a continuous function $x$ so that $x|_{(0,\infty)}$ is differentiable, $x|_{[-1,0]} = \phi$, and $x$ satisfies (1.2) for all $t > 0$. $C^1$ is the Banach space of all $C^1$-maps $\phi : [-1,0] \to \mathbb{R}$, with norm $||\phi||_1 = ||\phi|| + ||\dot{\phi}||$. If $I \subset \mathbb{R}$ is an interval, $x : I \to \mathbb{R}$ is a continuous function, $t \in \mathbb{R}$ so that $[t-1,t] \subset I$, then the segment $x_t \in C$ is defined by $x_t(s) = x(t+s)$, $-1 \le s \le 0$.

For every $\phi \in C$ the unique solution $x^{\phi} : [-1,\infty) \to \mathbb{R}$ is bounded. The map

$$F : \mathbb{R}^{+} \times C \ni (t,\phi) \mapsto x_t^{\phi} \in C$$

defines a continuous semiflow. 0 is the only stationary point of $F$. All maps $F(t,\cdot) : C \to C$, $t \ge 0$ are injective. It follows that for every $\phi \in C$ there is at most one solution $x : \mathbb{R} \to \mathbb{R}$ of (1.2) with $x_0 = \phi$. We denote also by $x^{\phi}$ such a solution on $\mathbb{R}$ whenever it exists. For a given $\phi \in C$, the $\omega$-limit set of $\phi$ is defined as

$$\omega(\phi) = \{\psi \in C : \text{there is a sequence } (t_n)_0^{\infty} \subset [0,\infty) \text{ so that } t_n \to \infty \text{ and } F(t_n,\phi) \to \psi \text{ as } n \to \infty\}.$$

Each map $F(t,\cdot)$, $t \geq 0$, is continuously differentiable. The operators $D_2F(t,0)$, $t \geq 0$, form a strongly continuous semigroup. The spectrum of the generator of the semigroup $(D_2F(t,0))_{t\geq 0}$ consists of the solutions $\lambda \in \mathbb{C}$ of the characteristic equation

$$\lambda + \alpha e^{-\lambda} = 0.$$

In case $\frac{1}{e} < \alpha \leq \frac{\pi}{2}$, all points in the spectrum form a sequence of complex conjugate pairs $(\lambda_j, \overline{\lambda_j})_0^\infty$ with

$$0 \geq \text{Re } \lambda_0 > \text{Re } \lambda_1 > \text{Re } \lambda_2 > \dots,$$

$$0 < \text{Im } \lambda_0 \leq \frac{\pi}{2}, \quad 2j\pi < \text{Im } \lambda_j < (2j+1)\pi \ \text{ for all } j \in \mathbb{N} \setminus \{0\}.$$

Let $P$ denote the realified generalized eigenspace of the generator associated with the spectral set $\{\lambda_0, \overline{\lambda_0}\}$. Let $Q$ denote the realified generalized eigenspace given by the spectral set of all $\lambda_k, \overline{\lambda_k}$ with $k \geq 1$. Then $P$ and $Q$ are positively invariant under $D_2F(t,0)$ for all $t \geq 0$, and $C = P \oplus Q$.

We recall the definition and some properties of a discrete Lyapunov functional

$$V : C \setminus \{0\} \to \mathbb{N} \cup \{\infty\}.$$

The version which we use was introduced in Mallet-Paret and Sell [22]. The definition is as follows. First, set $\text{sc}(\phi) = 0$ whenever $\phi \in C \setminus \{0\}$ is nonnegative or nonpositive, otherwise, for nonzero elements of $C$, let

$$\begin{aligned}\text{sc}(\phi) = \sup\big\{ k \in \mathbb{N} \setminus \{0\} :\ & \text{there is a strictly increasing finite sequence} \\ & (s^i)_0^k \ \text{in } [-1,0] \ \text{with} \ \phi(s^{i-1})\phi(s^i) < 0 \ \text{ for all } i \in \{1,2,\dots,k\}\big\} \leq \infty.\end{aligned}$$

Then define

$$V(\phi) = \begin{cases} \text{sc}(\phi) & \text{if } \text{sc}(\phi) \text{ is odd or } \infty, \\ \text{sc}(\phi) + 1 & \text{if } \text{sc}(\phi) \text{ is even.} \end{cases}$$

Set

$$\begin{aligned} R = \{\phi \in C^1 : \quad & \phi(0) \neq 0 \text{ or } \dot{\phi}(0)\phi(-1) < 0, \\ & \phi(-1) \neq 0 \ \text{ or } \dot{\phi}(-1)\phi(0) > 0, \\ & \text{all zeros of } \phi \text{ in } (-1,0) \text{ are simple}\}. \end{aligned}$$

We list some basic properties of $V$ [22], [23].

PROPOSITION 2.1.

*(i) For every $\phi \in C \setminus \{0\}$ and for every sequence $(\phi_n)_0^\infty$ in $C \setminus \{0\}$ with $\phi_n \to \phi$ as $n \to \infty$,*

$$V(\phi) \leq \liminf_{n\to\infty} V(\phi_n).$$

*(ii) For every $\phi \in R$ and for every sequence $(\phi_n)_0^\infty$ in $C^1 \setminus \{0\}$ with $||\phi_n - \phi||_1 \to 0$ as $n \to \infty$,*

$$V(\phi) = \lim_{n\to\infty} V(\phi_n) < \infty.$$

*(iii) Let an interval* $I \subset \mathbb{R}$, *and continuous functions* $b : I \to (-\infty, 0)$ *and* $z : I + [-1, 0] \to \mathbb{R}$ *be given so that* $z|_I$ *is differentiable with*

$$\dot{z}(t) = b(t)z(t-1) \tag{2.1}$$

*for* $\inf I < t \in I$, *and* $z(t) \neq 0$ *for some* $t \in I + [-1, 0]$. *Then the map* $I \ni t \mapsto V(z_t) \in \mathbb{N} \cup \{\infty\}$ *is monotone nonincreasing. If* $t \in I$, $t - 3 \in I$ *and* $z(t) = 0 = z(t-1)$, *then* $V(z_t) = \infty$ *or* $V(z_{t-3}) > V(z_t)$. *If* $t \in I$ *with* $t - 4 \in I$ *and* $V(z_{t-4}) = V(z_t) < \infty$, *then* $z_t \in R$.

*(iv) If* $b : \mathbb{R} \to (-\infty, 0)$ *is continuous and bounded,* $z : \mathbb{R} \to \mathbb{R}$ *is differentiable and bounded,* $z$ *satisfies (2.1) for all* $t \in \mathbb{R}$, *and* $z(t) \neq 0$ *for some* $t \in \mathbb{R}$, *then* $V(z_t) < \infty$ *for all* $t \in \mathbb{R}$.

We remark that solutions of (1.2), differences of solutions of (1.2), and solutions of the linear variational equation satisfy an equation of the form (2.1) with a suitable coefficient $b(t)$. For example, for solutions $x$, $\hat{x}$ of (1.2), the difference $y = x - \hat{x}$ satisfies (2.1) with

$$b(t) = \int_0^1 f'_\alpha(sx(t-1) + (1-s)\hat{x}(t-1))\, ds.$$

The following result is a consequence of a more general Poincaré–Bendixson type theorem of Mallet-Paret and Sell [23] applied for equation (1.2). A nontrivial solution $x : \mathbb{R} \to \mathbb{R}$ and the corresponding orbit $\{x_t : t \in \mathbb{R}\}$ are called homoclinic to zero if $\lim_{|t|\to\infty} x(t) = 0$ and $\lim_{|t|\to\infty} x_t = 0$, respectively.

Proposition 2.2. *For every* $\phi \in C$, *the* $\omega$*-limit set* $\omega(\phi)$ *is either* $0 \in C$ *or a periodic orbit, or a set in* $C$ *containing 0 and orbits homoclinic to 0.*

**3. Attractivity and periodic solutions.** Recall that a solution $x$ of (1.2) is called slowly oscillatory if $|z_1 - z_2| > 1$ for each pair of zeros of $x$.

The aim of this section is to reduce Conjecture 1 to the nonexistence of slowly oscillating periodic solutions.

Theorem 3.1. *The zero solution of (1.2) is globally attracting if and only if (1.2) has no slowly oscillating periodic solution.*

For the proof of Theorem 3.1 we need the following result.

Proposition 3.2. *Suppose* $0 < \alpha \leq \pi/2$. *Then (1.2) has no homoclinic orbit to zero.*

*Proof.* If $\alpha < \pi/2$ then 0 is locally asymptotically stable, and there is no homoclinic orbit to zero.

Suppose $\alpha = \pi/2$, and assume that $x : \mathbb{R} \to \mathbb{R}$ is a nontrivial solution of (1.2) with $x(t) \to 0$ as $|t| \to \infty$.

Let $\hat{x} : \mathbb{R} \to \mathbb{R}$ be another solution of (1.2) with $\hat{x}(t) \to 0$ as $|t| \to \infty$, and $\hat{x} \not\equiv x$. For example, $\hat{x} \equiv 0$, or $\hat{x}(\cdot) = x(\cdot + \tau)$ for some $\tau \neq 0$.

The function $y = x - \hat{x}$ satisfies

$$\dot{y}(t) = b(t)y(t-1) \quad (t \in \mathbb{R})$$

with

$$b(t) = \int_0^1 f'_{\pi/2}(sx(t-1) + (1-s)\hat{x}(t-1))\, ds.$$

From $y(t) \to 0$ as $t \to -\infty$ and $y \not\equiv 0$ it follows that there is a sequence $(t_n)_0^\infty$ such that $t_n \to -\infty$ as $n \to \infty$, and

$$|y(t_n)| = \sup_{t \le 0} |y(t_n + t)| \quad (n \in \mathbb{N}).$$

For the functions

$$y^n(t) = \frac{y(t_n + t)}{|y(t_n)|} \quad (t \in \mathbb{R}),$$

we have

$$\dot{y}^n(t) = b(t_n + t) y^n(t-1) \quad (t \in \mathbb{R}) \tag{3.1}$$

and

$$1 = |y^n(0)| \ge |y^n(t)| \quad (t \le 0). \tag{3.2}$$

Observe that

$$b(t_n + t) \to -\frac{\pi}{2} \quad \text{as } n \to \infty \text{ uniformly in } t \in (-\infty, 0]. \tag{3.3}$$

By (3.1), (3.2), and (3.3) there is a uniform bound for $|\dot{y}^n(t)|$, $n \in \mathbb{N}$, $t \le 0$. Then the Arzela–Ascoli theorem and the diagonalization process gives a subsequence $(n_k)$ and a continuous function $z : (-\infty, 0] \to \mathbb{R}$ such that

$$y^{n_k}(t) \to z(t) \text{ as } k \to \infty$$

uniformly on compact subsets of $(-\infty, 0]$, and

$$|z(t)| \le |z(0)| = 1 \quad (t \le 0).$$

Considering (3.1) and (3.3), we find that

$$\dot{y}^{n_k}(t) \to -\frac{\pi}{2} z(t-1) \text{ as } k \to \infty$$

uniformly on compact subsets of $(-\infty, 0]$. It follows that $z$ is differentiable,

$$\dot{y}^{n_k}(t) \to \dot{z}(t) \text{ as } k \to \infty$$

uniformly on compact subsets of $(-\infty, 0]$, and

$$\dot{z}(t) = -\frac{\pi}{2} z(t-1) \quad (t \le 0).$$

Recall the decomposition $C = Q \oplus P$. Let $\mathrm{Pr}_Q$ denote the projection of $C$ onto $Q$ along $P$. It is well known [9] that there are $K \ge 1$ and $\kappa > 0$ so that

$$||D_2 F(t, 0) \mathrm{Pr}_Q \phi|| \le K e^{-\kappa t} ||\mathrm{Pr}_Q \phi||$$

for all $t \ge 0$ and $\phi \in C$. Then, for $-\infty < s < t \le 0$, we have

$$\mathrm{Pr}_Q z_t = \mathrm{Pr}_Q D_2 F(t-s, 0) z_s = D_2 F(t-s, 0) \mathrm{Pr}_Q z_s$$

and

$$||\mathrm{Pr}_Q z_t|| \leq K e^{-\kappa(t-s)} ||\mathrm{Pr}_Q z_s|| \leq K e^{-\kappa(t-s)} ||\mathrm{Pr}_Q||.$$

Letting $s \to -\infty$, $\mathrm{Pr}_Q z_t = 0$ follows. Therefore, $z_t \in P \setminus \{0\}$ for all $t \leq 0$. Subspace $P$ contains segments of $a \cos \frac{\pi}{2} t + b \sin \frac{\pi}{2} t$, $a, b \in \mathbb{R}$. Consequently, $V(\phi) = 1$ for all $\phi \in P \setminus \{0\}$, and $V(z_t) = 1$ for all $t \leq 0$. Proposition 2.1 (iii) gives $z_0 \in R$. As $||y_0^{n_k} - z_0||_1 \to 0$, Proposition 2.1 (ii) implies $V(y_0^{n_k}) = 1$ for all sufficiently large $k$. The definition of $y$ and the monotone property of $V$ in Proposition 2.1 (iii) combined yield

$$V(y_t) = 1 \quad \text{for all } t \in \mathbb{R}.$$

Hence, by Proposition 2.1 (iii) again,

$$(y(t), y(t-1)) \neq (0,0) \quad \text{for all } t \in \mathbb{R}. \tag{3.4}$$

Applying (3.4) with $\hat{x}(t) = x(t+\tau)$, $t \in \mathbb{R}$, for all $\tau \neq 0$, it follows that the curve

$$\gamma : \mathbb{R} \ni t \mapsto (x(t), x(t-1)) \in \mathbb{R}^2$$

is injective.

If (3.4) is used with $\hat{x} \equiv 0$ then $(x(t), x(t-1)) \neq (0,0)$, $t \in \mathbb{R}$, is obtained. Then by (1.2) all zeros of $x$ are simple. In addition, $\gamma$ transversally intersects the half line $v_+ = \{(0,v) : v > 0\}$. Indeed, if $\gamma(t) \in v_+$ for some $t \in \mathbb{R}$, then $x(t) = 0$, $x(t-1) > 0$, and $\dot{x}(t) < 0$.

The homoclinic solution $x$ has arbitrarily large negative zeros. Otherwise, there is $T \in \mathbb{R}$ so that either $x(t) > 0$ and $\dot{x}(t) < 0$ for all $t < T$, or $x(t) < 0$ and $\dot{x}(t) > 0$ for all $t < T$. Both cases contradict $\lim_{t \to -\infty} x(t) = 0$.

Let $t_1, t_2, t_3$ be consecutive zeros of $x$ with $\dot{x}(t_1) < 0$, $\dot{x}(t_2) > 0$, $\dot{x}(t_3) < 0$. Set

$$L = \{(1-s)\gamma(t_1) + s\gamma(t_3) : 0 < s < 1\}$$

and

$$\Gamma = \{\gamma(t) : t_1 \leq t \leq t_3\} \cup L.$$

Then $\Gamma$ is a simple closed curve. By the Jordan curve theorem, $\mathbb{R}^2 \setminus \Gamma$ has two disjoint, open and connected components. The bounded component is the interior $\mathrm{int}(\Gamma)$ of $\Gamma$, and the unbounded component is the exterior $\mathrm{ext}(\Gamma)$ of $\Gamma$. Clearly, $(0,0) \in \mathrm{int}(\Gamma)$. $\gamma(t_1) \neq \gamma(t_3)$ because of the injectivity of $\gamma$. Suppose $\gamma(t_3) < \gamma(t_1)$ in the natural ordering of $v_+$.

The transversal intersection of $\gamma$ and $v_+$ implies that $\gamma$ can cross $L$ only from outside of $\Gamma$ to inside of $\Gamma$, that is, if $\gamma(t) \in L$ for some $t \in \mathbb{R}$, then

$$\gamma(t-s) \in \mathrm{ext}(\Gamma), \quad \gamma(t+s) \in \mathrm{int}(\Gamma)$$

for all sufficiently small $s > 0$.

Observe $\gamma(t_1 - s) \in \mathrm{ext}(\Gamma)$ for all small $s > 0$. Combining this fact, the injectivity of $\gamma$, the Jordan curve theorem and the fact that through $L$ the curve $\gamma$ can only enter into $\mathrm{int}(\Gamma)$, we conclude

$$\gamma(t) \in \mathrm{ext}(\Gamma) \quad \text{for all } t < t_1.$$

This contradicts $\lim_{t\to-\infty}\gamma(t) = (0,0)$.

The case $\gamma(t_3) > \gamma(t_1)$ analogously leads to a contradiction. This completes the proof of Proposition 3.2. ∎

*Proof.* [of Theorem 3.1] 1. It is obvious that if (1.2) has a slowly oscillating periodic solution then not all solutions approach zero as $t \to \infty$.

2. Suppose that (1.2) has no slowly oscillating periodic solution. Our aim is to show $\lim_{t\to\infty} x(t) = 0$ for all solutions of (1.2).

It is known [12] that for $\alpha > \pi/2$ equation (1.2) has a slowly oscillating periodic solution. Therefore, in the remaining part of the proof we may assume $\alpha \le \pi/2$.

By Proposition 3.2, (1.2) cannot have an orbit which is homoclinic to zero. Using this fact, Proposition 2.2 implies that for any $\phi \in C$ the $\omega$-limit set $\omega(\phi)$ is either $0 \in C$ or a periodic orbit.

In order to complete the proof it suffices to show that (1.2) cannot have nontrivial periodic solutions.

2.1. Let $\psi \in C$ be given by $\psi(\theta) = \alpha$, $\theta \in [-1, 0]$. Consider the solution $x = x^\psi$ of (1.2). We claim that $x^\psi(t) \to 0$ as $t \to \infty$. Assume the contrary. Then $\omega(\psi) = \{q_t : t \in \mathbb{R}\}$ for some nontrivial periodic solution $q$ which cannot be slowly oscillating by our assumption. Clearly, $V(\psi) = 1$. The monotone property of $V$ implies $V(x_t^\psi) = 1$, $t \ge 0$. There are $\eta \in \omega(\psi)$ and a sequence $(t_n)$ with $t_n \to \infty$ such that $x_{t_n}^\psi \to \eta$ as $n \to \infty$. Proposition 2.1 (i) implies

$$\liminf_{n\to\infty} V(x_{t_n}^\psi) \ge V(\eta).$$

Therefore, $V(\eta) = 1$. The periodicity of $q$ and results of Proposition 2.1 combined yield $V(q_t) = 1$, $q_t \in R$ for all $t \in \mathbb{R}$. It follows that $q$ is a slowly oscillatory periodic solution, a contradiction. Consequently, $x^\psi(t) \to 0$ as $t \to \infty$.

2.2. Let $p : \mathbb{R} \to \mathbb{R}$ be a nontrivial periodic solution of (1.2). Set $M = \max_{t\in\mathbb{R}} p(t)$. Choose $t_0$ with $p(t_0) = M$. Then $\dot p(t_0) = 0$, and by (1.2), $p(t_0 - 1) = 0$. Therefore

$$M = p(t_0) - p(t_0 - 1) = \int_{t_0-1}^{t_0} \dot p(t)\,dt = \int_{t_0-2}^{t_0-1} f_\alpha(p(t))\,dt < \alpha.$$

By the definition of $\psi$ in 2.1 and $M < \alpha$, $V(x_0^\psi - p_0) = 1$ follows. The monotone property of $V$ gives $V(x_t^\psi - p_t) = 1$ for all $t \ge 0$. If $T > 0$ is the minimal period of $p$, then by part 2.1

$$x_{nT}^\psi - p_{nT} \to -p_0 \quad \text{as } n \to \infty,$$

and Proposition 2.1 (i) implies

$$1 = \liminf_{n\to\infty} V(x_{nT}^\psi - p_{nT}) \ge V(-p_0) = V(p_0).$$

From $V(p_0) = 1$, form the periodicity of $p$, and from the monotone property of $V$, $V(p_t) = 1$ follows for all $t \in \mathbb{R}$. Consequently, $p$ is a slowly oscillatory periodic solution, and it is a contradiction. This completes the proof. ∎

**4. Nonexistence of small slowly oscillating periodic solutions.** We prove a slightly more general result which was motivated by a paper of Walther [33].

THEOREM 4.1. *Suppose $a > 0$, $b > 0$, $g \in C^1((-a, b), \mathbb{R})$ with $g(0) = 0$ and*

$$0 < g'(\xi) < \frac{\pi}{2} \quad \text{for all } \xi \in (-a, b) \setminus \{0\}. \tag{4.1}$$

*Then equation*

$$\dot{x}(t) = -g(x(t-1)) \tag{4.2}$$

*has no slowly oscillating periodic solution $x$ with $x(\mathbb{R}) \subset (-a, b)$.*

*Proof.* Assume that $x$ is a slowly oscillatory periodic solution of (4.2) with $x(\mathbb{R}) \subset (-a, b)$. It is well known (see e.g. the paper of Mallet-Paret and Sell [23]) that the minimal period $T > 0$ of $x$ is given by 3 consecutive zeros of $x$, and thus $T > 2$. Moreover, if $t_0 \in \mathbb{R}$ with $x(t_0) = \min_{t\in\mathbb{R}} x(t)$, then there exists a unique $t_1 \in (t_0, t_0+T)$ with $x(t_1) = \max_{t\in\mathbb{R}} x(t)$, and $\dot{x}(t) > 0$ for all $t \in (t_0, t_1)$, $\dot{x}(t) < 0$ for all $t \in (t_1, t_0 + T)$. It follows that $(x(t), \dot{x}(t)) \neq (0, 0)$ for all $t \in \mathbb{R}$.

The function $\mathbb{R} \ni t \mapsto k \sin \frac{\pi}{2}(t + t_*) \in \mathbb{R}$ is a solution of

$$\dot{z}(t) = -\frac{\pi}{2} z(t-1) \tag{4.3}$$

for any $k \in \mathbb{R}$ and $t_* \in \mathbb{R}$.

Define the simple closed curves

$$X : [0, T] \ni t \mapsto (x(t), \dot{x}(t)) \in \mathbb{R}^2,$$

and

$$Y_l^\tau : [0, 4] \ni t \mapsto l \left( \sin \frac{\pi}{2}(t + \tau), \frac{\pi}{2} \cos \frac{\pi}{2}(t + \tau) \right) \in \mathbb{R}^2$$

for $l > 0$, $\tau \in \mathbb{R}$. Let $|X|$, $|Y_l^\tau|$ denote the traces of $X$, $Y_l^\tau$, respectively, and $\mathrm{int}(Y_l^\tau)$, $\mathrm{ext}(Y_l^\tau)$ the interior, exterior of $Y_l^\tau$, respectively. Clearly, $|Y_l^\tau| = \{(u, v) \in \mathbb{R}^2 : u^2 + \left(\frac{2}{\pi}\right)^2 v^2 = l^2\}$ is an ellipse, and it is independent of $\tau$.

Fix $\tau \in \mathbb{R}$. There exists a $k > 0$ so that

$$|X| \subset \mathrm{int}(Y_l^\tau) \quad \text{for all } l > k$$

and

$$|X| \cap |Y_k^\tau| \neq \emptyset.$$

Set $z(t) = k \sin \frac{\pi}{2}(t + \tau)$, $t \in \mathbb{R}$, and $Z = Y_k^\tau$. Clearly, $|X| \cap \mathrm{ext}(Z) = \emptyset$.

By the definition of $k$ there are $t_0, t_1$ in $\mathbb{R}$ with

$$X(t_0) = Z(t_1) \in |X| \cap |Z|. \tag{4.4}$$

Replacing $x(\cdot)$ and $z(\cdot)$ with $x(\cdot + t_0)$ and $z(\cdot + t_1)$, respectively, we may assume $t_0 = t_1 = 0$, that is

$$(x(0), \dot{x}(0)) = (z(0), \dot{z}(0)).$$

Obviously, $(z(0), \dot{z}(0)) \neq (0, 0)$.

Suppose $\dot{x}(0) = \dot{z}(0) = 0$. Then $x(0) = z(0) = c \neq 0$. We consider only the case $c > 0$ as the case $c < 0$ is analogous. Clearly, $c = k$. From equation (4.2), condition (4.1), and $\dot{x}(0) = 0$, one finds $x(-1) = 0$. The monotone property of $x$ and $x(0) = c > 0$, $\dot{x}(0) = 0$ imply

$$c = x(0) = \max_{t\in\mathbb{R}} x(t), \ \dot{x}(t) > 0 \quad \text{for all } t \in [-1, 0).$$

By $z(0) = c > 0$ and $\dot{z}(0) = 0$, we have $z(t) = c\cos\frac{\pi}{2}t$, $t \in \mathbb{R}$.

Let $\tau_x : [0, c] \to [-1, 0]$ and $\tau_z : [0, c] \to [-1, 0]$ denote the inverses of $x|_{[-1,0]}$ and $z|_{[-1,0]}$, respectively. The functions

$$\phi_x : [0, c] \ni u \mapsto \dot{x}(\tau_x(u)) \in \mathbb{R}, \ \ \phi_z : [0, c] \ni u \mapsto \dot{z}(\tau_z(u)) \in \mathbb{R}$$

satisfy $\phi_x(c) = \phi_z(c) = 0$, and $\phi_x(u) > 0$ for all $u \in [0, c)$, $\phi_z(u) > 0$ for all $u \in [0, c)$. The arcs

$$\Omega_x = \{X(t) : t \in [-1, 0]\} \ \text{ and } \Omega_z = \{Z(t) : t \in [-1, 0]\}$$

coincide with the graphs

$$\{(u, \phi_x(u)) : u \in [0, c]\} \text{ and } \{(u, \phi_z(u)) : u \in [0, c]\},$$

respectively. From the inclusions $|X| \subset \operatorname{int}(Z) \cup |Z|$ and $\Omega_x \subset |X| \cap \{(u, v) \in \mathbb{R}^2 : v \geq 0\}$, $\Omega_z \subset |Z| \cap \{(u, v) \in \mathbb{R}^2 : v \geq 0\}$ it follows that

$$0 \leq \phi_x(u) \leq \phi_z(u) \quad \text{for all } u \in [0, c].$$

From the definition of $\phi_x$ and $\phi_z$, we obtain

$$\dot{x}(s) = \phi_x(x(s)), \ \ \dot{z}(s) = \phi_z(z(s)) \quad s \in [-1, 0].$$

Hence

$$1 = \lim_{\epsilon \to 0+} (1 - \epsilon) = \lim_{\epsilon \to 0+} \int_{-1}^{-\epsilon} \frac{\dot{x}(s)}{\phi_x(x(s))}\, ds = \int_0^c \frac{du}{\phi_x(u)}$$

where the last integral is improper. Similarly,

$$1 = \int_0^c \frac{du}{\phi_z(u)}.$$

Then

$$\int_0^c \frac{du}{\phi_x(u)} = \int_0^c \frac{du}{\phi_z(u)}.$$

The last equality and $0 < \phi_x(u) \leq \phi_z(u)$, $0 \leq u < c$, combined imply

$$\phi_x(u) = \phi_z(u) \quad \text{for all } u \in [0, c].$$

As

$$\frac{d}{du}\tau_x(u) = \frac{1}{\dot{x}(\tau_x(u))} = \frac{1}{\phi_x(u)} = \frac{1}{\phi_z(u)} = \frac{d}{du}\tau_z(u)$$

for all $u \in [0, c)$, and $\tau_x(0) = -1 = \tau_z(0)$, we conclude $\tau_x(u) = \tau_z(u)$ for all $u \in [0, c]$, and $x(t) = z(t)$ for all $t \in [-1, 0]$.

As a consequence, $\dot{x}(-1) = \dot{z}(-1) = \frac{\pi}{2}c$. From equation (4.2) at $t = -1$, the equality $g(x(-2)) = -\frac{\pi}{2}c$ follows. Hence $x(-2) \in (-a, 0)$. By the mean value theorem there is $\xi \in (x(-2), 0) \subset (-a, 0)$ with $g'(\xi)x(-2) = g(x(-2)) = -\frac{\pi}{2}c$. By (4.1), $x(-2) < -c$ follows, that is, $X(-2) \in \operatorname{ext}(Z)$, a contradiction. Therefore $\dot{x}(0) = \dot{z}(0) \neq 0$.

Then, for sufficiently small $\delta > 0$, $x$ and $z$ have inverses $t_x$ and $t_z$ in $(-\delta, \delta)$, respectively. Define

$$\eta_x : x((-\delta,\delta)) \ni u \mapsto \dot{x}(t_x(u)) \in \mathbb{R}, \ \eta_z : z((-\delta,\delta)) \ni u \mapsto \dot{z}(t_z(u)) \in \mathbb{R}.$$

Then

$${\eta_x}'(u) = \frac{\ddot{x}(t_x(u))}{\dot{x}(t_x(u))}, \quad {\eta_z}'(u) = \frac{\ddot{z}(t_z(u))}{\dot{z}(t_z(u))}.$$

In particular at $u = d = x(0) = z(0)$,

$${\eta_x}'(d) = \frac{\ddot{x}(0)}{\dot{x}(0)}, \quad {\eta_z}'(d) = \frac{\ddot{z}(0)}{\dot{z}(0)}.$$

The smooth arcs

$$\{(u, \eta_x(u)) : u \in x((-\delta,\delta))\} \subset |X|, \quad \{(u, \eta_z(u)) : u \in z((-\delta,\delta))\} \subset |Z|$$

intersect at $u = d$. As $|X| \subset \mathrm{int}(Z) \cup Z$, it follows that ${\eta_x}'(d) = {\eta_z}'(d)$. So

$$\frac{\ddot{x}(0)}{\dot{x}(0)} = \frac{\ddot{z}(0)}{\dot{z}(0)}.$$

From $\dot{x}(0) = \dot{z}(0) \neq 0$, $\ddot{x}(0) = \ddot{z}(0)$ follows. Applying these equalities, from equations (4.2) and (4.3) one gets

$$g(x(-1)) = \frac{\pi}{2} z(-1) \tag{4.5}$$

and

$$g'(x(-1))\dot{x}(-1) = \frac{\pi}{2}\dot{z}(-1). \tag{4.6}$$

We have $x(-1) \neq 0$, $z(-1) \neq 0$, since otherwise $\dot{x}(0) = 0$, $\dot{z}(0) = 0$ from equations (4.2), (4.3), respectively.

Consequently, by (4.1) and (4.5),

$$0 < |z(-1)| < |x(-1)|, \quad x(-1)z(-1) > 0. \tag{4.7}$$

If $\dot{x}(-1) = 0$ or $\dot{z}(-1) = 0$, then by (4.1) and (4.6) we find $\dot{x}(-1) = \dot{z}(-1) = 0$, and (4.7) implies $X(-1) \in \mathrm{ext}(Z)$, a contradiction. Thus, $\dot{x}(-1) \neq 0$, $\dot{z}(-1) \neq 0$.

Then (4.1) and (4.6) combined yield

$$0 < |\dot{z}(-1)| < |\dot{x}(-1)|, \quad \dot{x}(-1)\dot{z}(-1) > 0. \tag{4.8}$$

It is easy to see that (4.7) and (4.8) lead to the contradiction

$$X(-1) \in \mathrm{ext}(Z).$$

This completes the proof. □

We can apply Theorem 4.1 in the case $g(\xi) = \alpha(e^{\xi} - 1)$. If $0 < \alpha < \frac{\pi}{2}$, then for $-\infty < \xi < \log \frac{\pi}{2\alpha}$ we have $g'(\xi) < \frac{\pi}{2}$. So, we obtained

Corollary 4.2. *If $0 < \alpha < \frac{\pi}{2}$ and $p^{\alpha} : \mathbb{R} \to \mathbb{R}$ is a slowly oscillating periodic solution of equation (1.2) then*

$$\max_{t \in \mathbb{R}} p^{\alpha}(t) \geq \log \frac{\pi}{2\alpha} > 1 - \frac{2\alpha}{\pi}.$$

The last inequality in the corollary is a consequence of the elementary inequality $\log \xi > 1 - \frac{1}{\xi}$, $\xi > 1$.

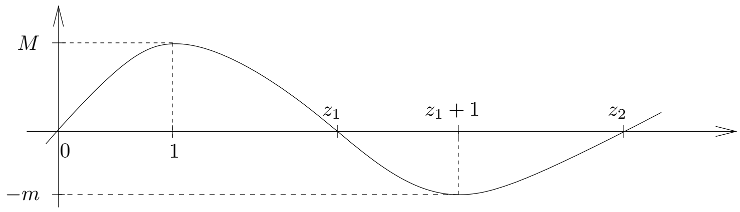


FIG. 5.1. *The considered typical shape for one period of the solution trajectory.*

**5. A rigorous numerical method.** For people familiar with the theoretical part of delay differential equations, we describe an algorithm which can be applied to prove that certain slowly oscillating periodic solutions of equation (1.2) cannot exist. In Section 6 technical details of the rigorous numerical part of the proof are given.

Define

$$B = \{\phi : [-1, 0] \to \mathbb{R} \mid \phi \text{ is bounded and integrable}\}.$$

If $I$ is an interval, $t \in I$, $t - 1 \in I$, and $u : I \to \mathbb{R}$ is a bounded and locally integrable function, then $u_t \in B$ is defined by $u_t(s) = u(t + s)$, $-1 \le s \le 0$. For elements $\phi, \psi$ in $B$ we write $\phi \le \psi$ provided $\phi(s) \le \psi(s)$ for all $s \in [-1, 0]$.

Suppose that $M, m$ are given positive numbers and $p : \mathbb{R} \to \mathbb{R}$ is a slowly oscillating periodic solution of (1.2) such that

$$\max_{t\in\mathbb{R}} p(t) = M, \quad \min_{t\in\mathbb{R}} p(t) = -m.$$

If we want to emphasize the dependence on $\alpha, M, m$ then we write $p(\alpha, M, m)$. By [23] without loss of generality we may assume that $p$ has the shape as shown on Figure 5.1. That is, there are reals $z_1 > 1$ and $z_2 > z_1 + 1$ such that $z_2$ is the minimal period of $p$, and

$$p(0) = 0,\ p(z_1) = 0,\ p(z_2) = 0,\ p(1) = M,\ p(z_1 + 1) = -m,$$

$$p'(t) > 0 \text{ for all } t \in (0, 1) \cup (z_1 + 1, z_2),$$

$$p'(t) < 0 \text{ for all } t \in (1, z_1 + 1).$$

The elements

$$y^0_+,\ y^0_-,\ y^1_+,\ y^2_+,\ y^2_-,\ y^3_-$$

of $B$ are called bounding functions of the periodic solution $p$ if

$$\begin{aligned} y^0_- \le\ & p_1 \le y^0_+ \\ & p_{z_1} \le y^1_+ \\ y^2_- \le\ & p_{z_1+1} \le y^2_+ \\ y^3_- \le\ & p_{z_2}. \end{aligned} \tag{5.1}$$

The idea is that we construct in an iterative way a finite sequence of bounding functions for $p$. In each step the bounding functions are improved, i.e., the inequalities (5.1) are sharpened. After each step we check whether

$$y^0_+(0) < M \quad \text{and} \quad y^2_-(0) > -m \tag{5.2}$$

hold or not. If at least one of inequalities (5.2) is satisfied, then we stop the iteration process. In this case the conclusion is that there is no slowly oscillating periodic solution $p(\alpha, M, m)$. If none of the inequalities in (5.2) holds, then we construct the next element of the sequence of bounding functions.

The iteration process goes as follows. Initially we set

$$\begin{aligned} y_-^0 &= y_+^2 = 0 \\ y_+^0 &= y_+^1 = M \\ y_-^2 &= y_-^3 = -m. \end{aligned} \tag{5.3}$$

Suppose that after $k$ steps we obtained the bounding functions $y_\pm^0,\ y_+^1,\ y_\pm^2,\ y_-^3$ in $B$ satisfying (5.1). We describe how to construct the new bounding functions

$$\hat{y}_\pm^0,\ \hat{y}_+^1,\ \hat{y}_\pm^2,\ \hat{y}_-^3.$$

For a $\phi \in C$ with $\phi(0) = 0$ the unique solution $x = x^\phi$ of equation (1.2) satisfies

$$x(t) = \int_0^t f_\alpha(x(u-1))\,du = \int_{-1}^{t-1} f_\alpha(\phi(u))\,du, \quad 0 \le t \le 1,$$

or equivalently

$$x_1(s) = \int_{-1}^{s} f_\alpha(\phi(u))\,du, \quad -1 \le s \le 0.$$

If $\psi \in B$ and $\eta \in B$ with $\psi \le \phi \le \eta$, then the monotone decreasing property of $f_\alpha$ can be used to obtain

$$\int_{-1}^{s} f_\alpha(\eta(u))\,du \le x_1(s) \le \int_{-1}^{s} f_\alpha(\psi(u))\,du, \quad -1 \le s \le 0. \tag{5.4}$$

Choosing $\phi = p_0 = p_{z_2}$ and $\psi = y_-^3$, we have $x(t) = p(t)$ and in this case (5.4) gives

$$p_1(s) \le \int_{-1}^{s} f_\alpha(y_-^3(u))\,du, \quad -1 \le s \le 0.$$

Similarly, if $\phi = p_{z_1}$ and $\eta = y_+^1$, then $x(t) = p(z_1 + t)$ and

$$\int_{-1}^{s} f_\alpha(y_+^1(u))\,du \le p_{z_1+1}(s), \quad -1 \le s \le 0.$$

Then the new bounds $\hat{y}_+^0$ and $\hat{y}_-^2$ are defined by

$$\hat{y}_+^0(s) = \min\left\{ y_+^0(s), \int_{-1}^{s} f_\alpha(y_-^3(u))\,du \right\}$$

and

$$\hat{y}_-^2(s) = \max\left\{ y_-^2(s), \int_{-1}^{s} f_\alpha(y_+^1(u))\,du \right\}$$

for each $s \in [-1, 0]$.

For $\phi \in C$ the unique solution $x = x^\phi$ of equation (1.2) satisfies

$$x(1) - x(t) = \int_t^1 \dot{x}(u)\, du = \int_{t-1}^0 f_\alpha(\phi(u))\, du, \quad 0 \le t \le 1,$$

or equivalently

$$x_1(s) = x(1) - \int_s^0 f_\alpha(\phi(u))\, du, \quad -1 \le s \le 0.$$

If $\psi$ and $\eta$ are in $B$ with $\psi \le \phi \le \eta$, then by using the monotone increasing property of $-f_\alpha$ we obtain

$$x(1) - \int_s^0 f_\alpha(\psi(u))\, du \le x_1(s) \le x(1) - \int_s^0 f_\alpha(\eta(u))\, du, \quad -1 \le s \le 0. \tag{5.5}$$

Applying inequality (5.5) in the cases $\phi = p_0$, $\psi = y_-^3$ and $\phi = p_{z_1}$, $\eta = y_+^1$, respectively, the new bounds $\hat{y}_-^0$ and $\hat{y}_+^2$ are defined by

$$\hat{y}_-^0(s) = \max\left\{ y_-^0(s), M - \int_s^0 f_\alpha(y_-^3(u))\, du \right\}$$

and

$$\hat{y}_+^2(s) = \min\left\{ y_+^2(s), -m - \int_s^0 f_\alpha(y_+^1(u))\, du \right\}.$$

The above definitions of $\hat{y}_\pm^0$ and $\hat{y}_\pm^2$ follow the original idea of Wright [35]. The construction of the bounds $\hat{y}_+^1$ and $\hat{y}_-^3$ is slightly more complicated. It seems to be new, it does not appear in Wright's paper [35]. The difficulty of the construction of $\hat{y}_+^1$ and $\hat{y}_-^3$ is that the zeros $z_1$ and $z_2$ of the periodic solution $p$ are not known. Below we describe the definition of $\hat{y}_+^1$. $\hat{y}_-^3$ can be obtained analogously.

Starting from the bounds $y_\pm^0$, and applying a reliable numerical integration method, we get lower and upper bounds $\underline{P}$ and $\overline{P}$ for $p$. The functions $\underline{P}$ and $\overline{P}$ are step functions as illustrated in Figure 6.2. Let $h > 0$ denote the step size of the numerical integration. Both $\overline{P}$ and $\underline{P}$ are nondecreasing functions on $[0, 1]$. $\overline{P}$ and $\underline{P}$ are nonincreasing at least on the intervals $[1, t_i]$ and $[1, t_j]$, respectively, where $i$ is the smallest positive integer so that $\underline{P}(t) < 0$ for all $t \in (t_i, t_i + h)$, and the positive integer $j$ is the smallest one such that $\overline{P}(t) < 0$ for all $t \in (t_j, t_j + h)$. Setting

$$\underline{z}_1 = t_i, \quad \overline{z}_1 = t_j,$$

obviously $[\underline{z}_1, \overline{z}_1]$ is a verified enclosing interval for the zero $z_1$ of $p$. Set $\Delta = \overline{z}_1 - \underline{z}_1$. For $\Delta \le 1$ define the function $q : [-1, \underline{z}_1] \to \mathbb{R}$ by

$$q(t) = \begin{cases} 0 & \text{if } -1 \le t < -\Delta \\ \overline{P}(t + \Delta) & \text{if } -\Delta \le t < 1 - \Delta \\ M & \text{if } 1 - \Delta \le t < 1 \\ \overline{P}(t) & \text{if } 1 \le t \le \underline{z}_1 \end{cases}$$

For the case $1 < \Delta \le 2$

$$q(t) = \begin{cases} \overline{P}(t + \Delta) & \text{if } -1 \le t < 1 - \Delta \\ M & \text{if } 1 - \Delta \le t < 1 \\ \overline{P}(t) & \text{if } 1 \le t \le \underline{z}_1 \end{cases}$$

For $2 < \Delta$

$$q(t) = \begin{cases} M & \text{if } -1 \le t < 1 \\ \overline{P}(t) & \text{if } 1 \le t \le \underline{z}_1 \end{cases}$$

Clearly,

$$q(t) \ge p(t) \quad \text{for all } t \in [-1, \underline{z}_1].$$

CLAIM 1.

$$q(\underline{z}_1 + s) \ge p(z_1 + s) \quad \text{for all } s \in [-2, 0].$$

*Proof.* We prove the statement only for the case $\Delta \le 1$. Setting $\delta = z_1 - \underline{z}_1$, the claim is equivalent to

$$q(t) \ge p(t + \delta) \quad \text{for all } t \in [\underline{z}_1 - 2, \underline{z}_1]. \tag{5.6}$$

From $z_1 \in [\underline{z}_1, \overline{z}_1]$, $\delta \in [0, \Delta]$ follows.

If $t \in [1, \underline{z}_1]$ then $q(t) = \overline{P}(t) \ge p(t)$. Function $p$ is decreasing on $[1, \underline{z}_1 + \delta] = [1, z_1]$. Therefore $p(t + \delta) \le p(t)$, $t \in [1, \underline{z}_1]$, and thus

$$q(t) \ge p(t + \delta) \quad \text{for all } t \in [1, \underline{z}_1]$$

follows. Inequality (5.6) clearly holds on $[1 - \Delta, 1]$ since $q(t) = M$ for $t \in [1 - \Delta, 1]$.

For $t \in [-\Delta, 1 - \Delta)$ inequality (5.6) is equivalent to

$$\overline{P}(t + \Delta) \ge p(t + \delta) \quad \text{for all } t \in [-\Delta, 1 - \Delta),$$

which is equivalent to

$$\overline{P}(t) \ge p(t - (\Delta - \delta)) \quad \text{for all } t \in [0, 1).$$

The last inequality is obvious since $p(s) < 0$ for $s \in (-\Delta, 0)$, $p$ is increasing on $[0, 1]$, $\Delta \ge \delta$, and thus

$$\overline{P}(t) \ge p(t) \ge p(t - (\Delta - \delta)) \quad \text{for all } t \in [0, 1].$$

If $\underline{z}_1 - 2 < -\Delta$ and $t \in [\underline{z}_1 - 2, -\Delta)$ then $t \in (-1, -\Delta)$, $t + \delta \in (-1, 0)$, and thus $q(t) = 0$ and $p(t + \delta) < 0$. This completes the proof of the Claim. $\square$

For any $s \in [-1, 0]$ one has

$$p(z_1 + s) = p(z_1 + s) - p(z_1) = -\int_{z_1 + s}^{z_1} \dot{p}(u)\, du = -\int_{s-1}^{-1} f_\alpha(p(z_1 + u))\, du.$$

Combining the above inequality, Claim 1 and the monotone increasing property of $-f_\alpha$, it follows that

$$p(z_1 + s) \le -\int_{s-1}^{-1} f_\alpha(q(\underline{z}_1 + u))\, du \quad \text{for all } s \in [-1, 0].$$

Now the new bounding function $\hat{y}_+^1 \in B$ can be defined as follows. For all $s \in [-1, 0]$,

$$\hat{y}_+^1(s) = \min\left\{ y_+^1(s), q(\underline{z}_1 + s), -\int_{s-1}^{-1} f_\alpha(q(\underline{z}_1 + u))\, du \right\}.$$

**6. A computer-assisted bounding scheme.** In an earlier paper [2], the first author investigated the problem with traditional verified differential equation solver algorithms. He found that a proof of the conjecture along these lines would require an enormous amount of computation time with the present technological conditions (compilers, algorithms and computer capacities). He was able to prove, e.g., that for all $\alpha$ values within the tiny interval $[1.5, 1.5 + 10^{-22}]$ the trajectories of the solutions will reach a phase when the absolute value of the solution remains below 0.075 for a time interval of a unit length. For wider parameter intervals, or for values closer to $\pi/2$ the required CPU times exploded. Thus traditional computer-assisted techniques for differential equations appear not suitable for settling the conjecture.

In this section we describe and prove the correctness of a new, computer-assisted bounding scheme that extends Wright's original reasoning and allows an efficient shrinking of the possible extreme values of a slowly oscillating periodic solution. The computational part of the proof of Theorem 1.1 will prove the following

THEOREM 6.1. *If* $\alpha \in [1.5, 1.5706]$ *and* $y : \mathbb{R} \to \mathbb{R}$ *is a slowly oscillating periodic solution of (1.2), then* $\max_{t\in\mathbb{R}} |y(t)| \le 1 - \frac{2\alpha}{\pi}$.

Now, a combination of Theorem 3.1, Corollary 4.2, and Theorem 6.1 proves Theorem 1.1.

The present approach follows another line of thought, still it is a kind of direct extension of that of Wright. Denote three subsequent zeroes of the trajectory by 0, $z_1$, and $z_2$. Let us define the following functions bounding the trajectories:

$y^{(upper)}_{(inc,1)}(t)$ : an upper bounding function for the time interval $0 \le t \le 1$,

$y^{(lower)}_{(inc,1)}(t)$ : a lower bounding function for the time interval $0 \le t \le 1$,

$y^{(upper)}_{(dec,n)}(t)$ : an upper bounding function for the time interval $1 \le t \le z_1$,

$y^{(lower)}_{(dec,1)}(t)$ : a lower bounding function for the time interval $z_1 \le t \le z_1 + 1$,

$y^{(upper)}_{(dec,1)}(t)$ : an upper bounding function for the time interval $z_1 \le t \le z_1 + 1$,

$y^{(lower)}_{(inc,n)}(t)$ : a lower bounding function for the time interval $z_1 + 1 \le t \le z_2$.

The trajectory bounding functions are illustrated by dashed lines on Figure 6.1. Here four consecutive time intervals will be considered defined by the zeros and by the extremal values of the trajectory denoted by $(inc, 1)$, $(dec, n)$, $(dec, 1)$, and $(inc, n)$, respectively. The length of the time intervals $(inc, 1)$ and $(dec, 1)$ are known to be one. On the other hand the length of $(dec, n)$, denoted as $p_M = z_1 - 1$ and that of $(inc, n)$, $p_m = z_1' - z_1 - 1$ are unknown, it is even unclear whether the these are larger than one.

The trajectory bounding functions will be sharpened sequentially, in an iterative way, i.e. the bounding functions of the time interval $(inc, 1)$ will be used to improve the bounding function on the interval $(dec, n)$, etc. Then, the bounding function of the last interval, $(inc, n)$ will be used to make the inequalities for the interval $(inc, 1)$ sharper, and so on. Those bounding function improvements that are based on a single bounding function of the earlier time interval are basically similar to the original technique used by Wright. The sharpening steps using two bounding functions on the argument interval apply a new, Taylor series based method to be described later in this paper. At start we set the upper bounding functions to constant $M$, the lower

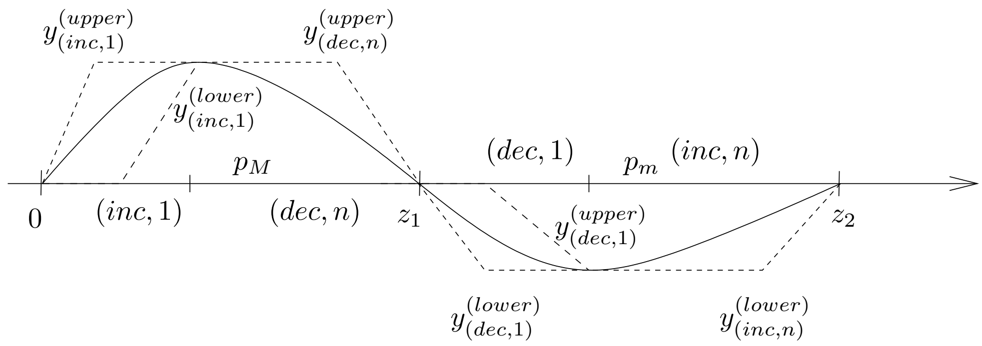


FIG. 6.1. *The trajectory bounding functions shown as dashed lines for a full period.*

bounding functions to $-m$ with the exceptions of $y^{(lower)}_{(inc,1)} = 0$ and $y^{(upper)}_{(dec,1)} = 0$.

We iterate only on such cases, when the conditions (1.4) to (1.6) and that of Corollary 4.2 are fulfilled. The conditions we check at the end of each iteration cycle of the bounding function sharpening procedure are

$$y^{(upper)}_{(inc,1)}(0+1) < M \quad \text{and} \quad -m < y^{(lower)}_{(dec,1)}(z_1+1). \tag{6.1}$$

In case at least one of these conditions are satisfied then the solution of the investigated delay differential equation cannot have a periodic solution with a maximal value of $M$ and the minimal value of $m$ as assumed for the given $\alpha$ parameter.

**6.1. Improved bounds for the unit width intervals.** We repeat the derivation of bounding functions in Section 5 with slightly different notation that will be applied within our computational procedure. First we show how to obtain an upper bound on the periodic trajectory on the interval $(inc,1)$ based on the $y^{(lower)}_{(inc,n)}(t)$ function. Since $y^{(lower)}_{(inc,n)}(t)$ is a lower bounding function, so $y^{(lower)}_{(inc,n)}(t) \le y(t)$ holds for all $t \le 0$. Now integrate $y'$ from 0 to $t$ $(0 \le t \le 1)$: $y(t) = y(t) - y(0) =$

$$-\alpha \int_0^t e^{y(x-1)} - 1\, dx = -\alpha \int_{0-1}^{t-1} e^{y(x)} - 1\, dx \le -\alpha \int_{0-1}^{t-1} e^{y^{(lower)}_{(inc,n)}(x)} - 1\, dx.$$

We can obtain a new, stronger bounding function from this bound and from the old one for the $t \ge 0$ case:

$$y^{(upper)}_{(inc,1)}(t) = \min \left\{ \begin{array}{c} y^{(upper)}_{(inc,1)}(t) \\ \\ -\alpha \int\limits_{0-1}^{t-1} e^{y^{(lower)}_{(inc,n)}(x)} - 1\, dx \end{array} \right\},\ t \in [0,1], \tag{6.2}$$

We suppress the iteration number in the bounding function, the new one on the left hand side of the defining equation is calculated from the old function on the right hand side as it is usual in computer programs. We can get a new bounding function for the lower bounding function in $(dec, 1)$ in a similar way:

$$y^{(lower)}_{(dec,1)}(t) = \max \left\{ \begin{array}{c} y^{(lower)}_{(dec,1)}(t) \\ \\ -\alpha \int\limits_{z_1-1}^{t-1} e^{y^{(upper)}_{(dec,n)}(x)} - 1 \; dx \end{array} \right\}, \; t \in [z_1, z_1+1]. \tag{6.3}$$

We can obtain an improved lower bound for the trajectory on the interval $(inc, 1)$ by $y(1) - y(t) = M - y(t) =$

$$-\alpha \int\limits_t^1 e^{y(x-1)} - 1 \; dx = -\alpha \int\limits_{t-1}^0 e^{y(x)} - 1 \; dx \leq -\alpha \int\limits_{t-1}^0 e^{y^{(lower)}_{(inc,n)}(x)} - 1 \; dx.$$

The new lower bounding function is then

$$y^{(lower)}_{(inc,1)}(t) = \max \left\{ \begin{array}{c} y^{(lower)}_{(inc,1)}(t) \\ \\ M + \alpha \int\limits_{t-1}^{0} e^{y^{(lower)}_{(inc,n)}(x)} - 1 \; dx \end{array} \right\}, \; \text{if } t \in [0, 1]. \tag{6.4}$$

We can build an improved upper bound also for the time interval $(dec, 1)$ in a similar way:

$$y^{(upper)}_{(dec,1)}(t) = \min \left\{ \begin{array}{c} y^{(upper)}_{(dec,1)}(t) \\ \\ -m + \alpha \int\limits_{t-1}^{0} e^{y^{(upper)}_{(dec,n)}(x)} - 1 \; dx \end{array} \right\}, \; \text{if } t \in [0, 1]. \tag{6.5}$$

By that we have completed the description of the improved bounding functions for the unit width time intervals.

**6.2. Bounds for the period length.** A sharp enclosure of the period length is very important for the success of the proof for the conjecture, especially for $\alpha$ values close to $\pi/2$. To calculate bounds on the period length and as a part of that bounds for the not unit length time intervals we apply an Euler type differential equation solution method

$$Y(x) = Y(x_0) + Y^{(1)}([x_0, x])(x - x_0),$$

$$Y([x_0, x]) = Y(x_0) + Y^{(1)}([x_0, x])([0, x - x_0]).$$

customized for delay equations. In these equations we used the notions of interval calculations [30], i.e. capitals denote interval values. The implementation details will be discussed in the next section. To use this method we need an enclosure $Y(x_0)$ of the trajectory in the start point, and bounds on a given number of time intervals covering together unit length time intervals.

For these calculations we need lower and upper bounds for the trajectory on the unit length time intervals before the investigated $(dec, n)$ and $(inc, n)$ phases. These are available due to the previous subsection. The lower and upper bounds for the

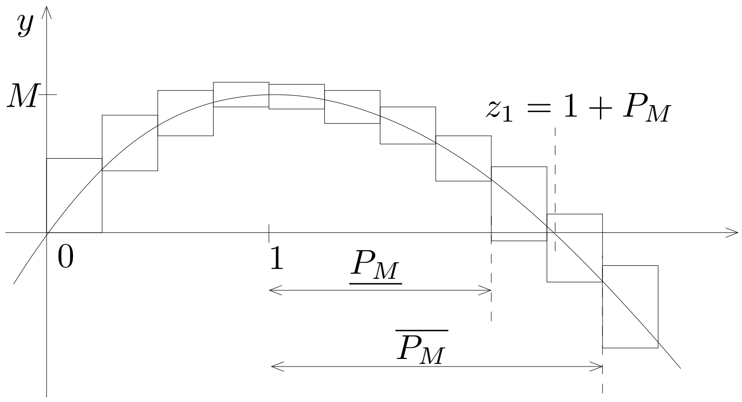


FIG. 6.2. *Illustration of the bounding procedure for the $z_1$ zero of the trajectory.*

zeros $z_1$ and $z_2$ of the trajectory will be determined using the interval enclosures obtained on time intervals for the trajectory. Consider first the case when we follow the trajectory from 1 to find $z_1$, i.e. we want to find bounds for $p_M$. Assume that as a part of the verified integration the first interval that contains zero is $Y(t_i, t_i + h)$, where $h$ is the step size of the numerical integration. Then there may follow some integration steps for which the respective $Y$ enclosures contain zero. Let the last such be $Y(t_j, t_j + h)$ (in some cases it is possible that $i = j$). Then $[t_i, t_j + h]$ is obviously a verified enclosing interval for $z_1$. The same technique that is illustrated on Figure 6.2 is also applicable for the bounding of $p_m$.

Denote the enclosures of $p_M$ and $p_m$ to be calculated from the above bounds of the zeros by $P_M$ and $P_m$, respectively. The lower and upper bounds of these intervals are denoted as usual in interval calculation, with underline and overline, e.g. $P_M = [\underline{P}_M, \overline{P}_M]$.

**6.3. Improved bounds for the not unit width intervals.** As we could see in the previous subsection, it is not easy to determine $z_1$, as the zero of the investigated trajectory. In the present subsection we build a valid upper bound for the trajectory on the $(inc, 1)$ and $(dec, n)$ intervals that can be applied as needed also until the point $z_1$ for calculating further improving bounds on the interval $(dec, 1)$.

Consider the trajectory on $[0, 1 + \overline{P}_M]$, i.e. on the intervals $(inc, 1)$ and $(dec, n)$. The bounds on the trajectory are at this point obtained by the new bounds of (6.4) and (6.5) on $(inc, 1)$, and by the verified solution of the differential equation, as described in Subsection 6.2 on $(dec, n)$. Let us call this complete bounding function as $Y$, and its upper bound as $\overline{Y}$. For a monotonically increasing $y(t)$ function we have

$$y(t) \geq y(t - \Delta t), \text{ if } \Delta t \geq 0$$

and for a monotonically decreasing $y(t)$ function

$$y(t) \geq y(t - \Delta t), \text{ ha } \Delta t \leq 0.$$

The trajectory is known to be strictly monotonically increasing on $(inc, 1)$, while strictly monotonically decreasing on $(dec, n)$.

Consider first the $(inc, 1)$ time interval, here the $y^{(upper)}_{(inc,1)}$ gives an upper bounding function, $\overline{Y}$ for the periodic trajectory. Since $p_M \leq \overline{P}_M$, the relation

$$\Delta t = \left(1 + \overline{P}_M\right) - z_1 = \overline{P}_M - p_M \geq 0$$

holds. Now these imply

$$\overline{Y}(t) \geq y(t) \geq y(t-\Delta t) = y\left(t - \left((1+\overline{P}_M) - z_1\right)\right).$$

These relations can be interpreted as $\overline{Y}$ is an upper bounding function also for $y(t-\Delta t)$, i.e. for the trajectory shifted by $\Delta t$ on the interval

$$\left[-\left((1+\overline{P}_M) - z_1\right), 1 - \left((1+\overline{P}_M) - z_1\right)\right] =$$

$$\left[-(\overline{P}_M - p_M), 1 - (\overline{P}_M - p_M)\right] = \left[z_1 - \overline{P}_M - 1, z_1 - \overline{P}_M\right].$$

Consider now the $(dec, n)$ phase, the verified solution will give an upper bound for $y(t)$ on $[1, 1+\underline{P}_M]$. Here $y(t)$ is strictly monotonically decreasing, thus due to $\underline{P}_M \leq p_M$ the relations

$$\overline{Y}(t) \geq y(t) \geq y(t-\Delta t) = y\left(t - ((1+\underline{P}_M) - z_1)\right)$$

hold with $\Delta t = \underline{P}_M - p_M \leq 0$. Here again $\overline{Y}$ is an upper bounding function also for $y(t-\Delta t)$, i.e. for the trajectory shifted by $\Delta t$ on the interval

$$[1 - (\underline{P}_M - p_M), 1 + \underline{P}_M - (\underline{P}_M - p_M)] =$$

$$[z_1 - \underline{P}_M, z_1].$$

The explanation for the above bounding technique is illustrated on Figure 6.3. The first case can be understood as if the original periodic solution would be shifted in such a way that the original $z_1$ zero coincides with $1+\overline{P}_M$. Since $y(t)$ is monotonically increasing on the interval $(inc, 1)$, thus the upper bounding function $\overline{Y}(t)$ remains an upper bound of the shifted function too (upper picture of Figure 6.3). The highlighted upper bounding functions parts are presented as bounds of the $y(t)$ trajectory.

In the second case the original trajectory is shifted in such a way that the zero $z_1$ coincides with $(1+\underline{P}_M)$. The monotonically decreasing $y(t)$ will then remain below $\overline{Y}(t)$ on the given time interval (see the second picture of Figure 6.3). As it can be seen on this figure, in the gap between the two highlighted function we consider the constant $M$ value. With the above considerations we have provided a bounding function that can be used also until the unknown $z_1$ time point.

The same technique can be applied to establish such a valid lower bound for the trajectory on the intervals $(dec, 1)$ and $(inc, n)$, that can be applied for further bound improvements even in the case when the necessary integration should start from the $z_2$ zero.

Let us see now how can we produce stronger bounds on the intervals $(dec, n)$ and $(inc, n)$ before the $z_1 - 1$, and $z_2 - 1$ time points, respectively – on the basis of the bounds discussed earlier in the present subsection. Consider first the $(dec, n)$ case, then for the present upper bounding function

$$y_{(dec,n)}^{(upper)} \geq y(t).$$

Integrate the derivative function $y'$ from $t$ to $z_1$, where $z_1 - 1 \leq t \leq z_1$:

$$-y(t) = y(z_1) - y(t) = -\alpha \int_t^{z_1} e^{y(x-1)} - 1\, dx = -\alpha \int_{t-1}^{z_1-1} e^{y(x)} - 1\, dx.$$

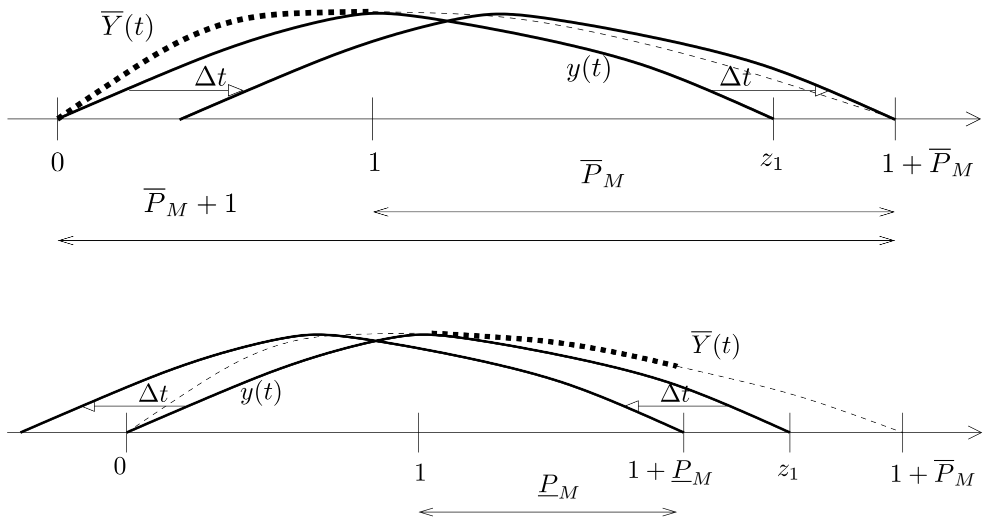


FIG. 6.3. *Illustrations of how the bounds can be obtained for the cases when the shifted* $z_1$ *coincides with* $1+\overline{P}_M$ *and with* $1+\underline{P}_M$*, respectively.*

In other terms

$$y(t) \leq \alpha \int_{t-1}^{z_1-1} e^{y^{(upper)}_{(dec,n)}(x)} - 1\, dx.$$

This bounding function can be use to update the old one:

$$y^{(upper)}_{(dec,n)}(t) = \min \left\{ \begin{array}{c} y^{(upper)}_{(dec,n)}(t) \\ \\ \alpha \int_{t-1}^{z_1-1} e^{y^{(upper)}_{(dec,n)}(x)} - 1\, dx \end{array} \right\}, \text{ if } t \in [z_1-1, z_1]. \tag{6.6}$$

In a similar way we can calculate a new lower bounding function on the interval $(inc, n)$:

$$y(t) \geq \alpha \int_{t-1}^{z_2-1} e^{y^{(lower)}_{(inc,n)}} - 1\, dx,$$

that implies the update

$$y^{(lower)}_{(inc,n)}(t) = \min \left\{ \begin{array}{c} y^{(lower)}_{(inc,n)}(t) \\ \\ \alpha \int_{t-1}^{z_2-1} e^{y^{(lower)}_{(inc,n)}(x)} - 1\, dx \end{array} \right\}, \text{ if } t \in [z_2-1, z_2]. \tag{6.7}$$

Notice that in both cases the new, improved bound utilizes earlier bound values also from more than 1 time unit distance to the actual right end zero of the trajectory. This gives an explanation how improvements made at the first part of the present subsection can improve our bounds at a much later time point.

**6.4. The iterative improvement of the bounding functions.** The lower and upper bounds derived in the earlier subsections will be applied in an iterative procedure to make them even sharper that possibly allows to conclude that for a given pair of $M$ and $m$ values the (1.1) delay differential equation with the investigated interval of $\alpha$ parameter leads to a contradiction. The iteration cycle begins with the time interval $(inc, 1)$, and with the integration of the right hand side of the differential equation we update the earlier upper bound on $(dec, n)$. This new upper bound will then be used to improve the lower and upper bounding functions on $(dec, 1)$, and finally the latter help us to make $y^{(lower)}_{(inc,n)}$ sharper.

Now the bounding functions $y^{(lower)}_{(inc,1)}$, $y^{(upper)}_{(dec,1)}$, $y^{(upper)}_{(inc,1)}$, and $y^{(lower)}_{(dec,1)}$ are defined on unit length time intervals, on $[0, 1]$ and $[z_1, z_1 + 1]$, respectively. In contrast to these, in the case of $y^{(lower)}_{(inc,n)}$ and $y^{(upper)}_{(dec,n)}$ we must also calculate with their values over wider time intervals. To be able to handle the delayed terms, we have to save bounding function values for a unit length interval in the first case, and for two width intervals otherwise (this later figure proved to be satisfactory for our investigation).

Due to the computer representation of reals, it is advantageous to subdivide these time intervals into $2^l$, and $2^{l+1}$ subintervals for a natural number $l$, respectively. Denote these subintervals by $t_i$, where $i \in (1, \dots, 2^l)$, and for the $(dec, n)$ and $(inc, n)$ time intervals $i \in (1, \dots, 2^{l+1})$ in increasing order as they depart from the zero. It is intentional that the order of the numeration for the unit length intervals is the opposite of that for $(dec, n)$ and $(inc, n)$. Within such a subinterval, the respective bounding function will be represented by a real number, i.e. we use a bounding step function for the saved bounding functions. This step function is denoted by $Y$, as also in Subsection 6.2. The right hand side of the differential equation can then easily be bounded using the step functions both at $t_j$ and at the same time at $t_j - 1$. The updated value of $Y^{(upper)}_{(inc,1)}(t_i)$ $(i = 1, \dots, 2^l)$ can be calculated applying $Y^{(lower)}_{(inc,n)}$ according to (6.2):

$$Y^{(upper)}_{(inc,1)}(t_i) = \min\left\{ -\alpha \sum_{j=1}^{i} \left( e^{Y^{(lower)}_{(inc,n)}(t_{2^l-j+1})} - 1 \right) / 2^l \ ; \ Y^{(upper)}_{(inc,1)}(t_i) \right\}. \tag{6.8}$$

In a similar way, we can obtain the other bounding functions updated using the stronger bounds given as (6.3) to (6.5):

$$Y^{(lower)}_{(dec,1)}(t_i) = \max\left\{ -\alpha \sum_{j=1}^{i} \left( e^{Y^{(upper)}_{(dec,n)}(t_{2^l-j+1})} - 1 \right) / 2^l \ ; \ Y^{(lower)}_{(dec,1)}(t_i) \right\}, \tag{6.9}$$

$$Y^{(lower)}_{(inc,1)}(t_i) = \max\left\{ M + \alpha \sum_{j=i}^{2^l} \left( e^{Y^{(lower)}_{(inc,n)}(t_{2^l-j+1})} - 1 \right) / 2^l \ ; \ Y^{(lower)}_{(inc,1)}(t_i) \right\}, \tag{6.10}$$

$$Y^{(upper)}_{(dec,1)}(t_i) = \min\left\{ -m + \alpha \sum_{j=i}^{2^l} \left( e^{Y^{(upper)}_{(dec,n)}(t_{2^l-j+1})} - 1 \right) / 2^l \ ; \ Y^{(upper)}_{(dec,1)}(t_i) \right\}. \tag{6.11}$$

On the basis of these bounding functions, we can calculate bounds on the trajectory for the next, not unit length time intervals. The bounds on the trajectory will provide lower and upper bounds on the next zero, as discussed in Subsection

**Algorithm 1** Determination of $\underline{P}_M$ and $\overline{P}_M$ for the bounds for the period length

*Input:*
- $s$: $M$ or $-m$ as an extremal value of the periodic trajectory,
- $\alpha$: a parameter of the studied delay differential equation,
- $2^l$: the number of equal width subintervals in the unit length time interval,
- $L$, $U$: lower and upper bound functions on the unit length time interval.

*Output:*
- An enclosure of the length for the not unit width interval, bounding of the trajectory from 1 and $z_1 + 1$, respectively.

**Step 1.** Compute $Y(t_i)$ $(i = 1, \dots, 2^l)$ as the enclosures of the periodic solution on subintervals of the unit length time period by using the $U$ and $L$ functions on the $(inc, 1)$ and $(dec, 1)$ intervals.
**Step 2.** Set $j = (2^l + 1)$ and $Y_{last} = [s, s]$.
**Step 3.** Enclose $Y(t_j)$ with the expression $\left(Y_{last} + \left(-\alpha\left(e^{Y(t_{j-2^l})} - 1\right)\right) \cdot [0, 1/2^l]\right)$.
**Step 4.** Set $Y_{last} = Y_{last} + \left(-\alpha\left(e^{Y(t_{j-2^l})} - 1\right)\right)/2^l$.
**Step 5.** If $0 \notin Y(t_{j-1})$ and $0 \in Y(t_j)$, then calculate the new lower bound for the length of the not unit width interval: $\underline{P}_M = (j-1)/2^l$.
**Step 6.** If $0 \in Y(t_{j-1})$ and $0 \notin Y(t_j)$, then calculate the new upper bound for the length of the not unit width interval: $\overline{P}_M = (j-1)/2^l$ and STOP.
**Step 7.** Set $j = j + 1$.
**Step 8.** If $j < 2^{l+2}$, then continue with Step 3, otherwise STOP.

6.2. Thus we obtain lower and upper bounds on the trajectory on the time intervals $[0, 1+\underline{P}_M]$, and $[0, 1+\overline{P}_M]$, respectively. The formal description of the algorithm for the determination of the bounds of zeros is given as Algorithm 1. Here we bound the trajectory after the time 1, or $z_1 + 1$, and check whether the respective $Y(t_j)$ interval contains zero. The algorithm is able to identify lower and upper bounds within length 2 intervals, this was satisfactory for our investigation. The reordering of the $2^{-l}$ size subintervals mentioned in Subsection 6.4 must be made after Algorithm 1 was run.

Consider now how these bounding functions can be used to improve $y^{(upper)}_{(dec,n)}$. The integration of the step function $Y(t_i)$, $i \in \left(1, \dots, 2^l\right)$ gives with (6.6) and (6.7) the updated upper and lower bounding functions

$$Y^{(upper)}_{(dec,n)}(t_i) = \max\left\{\alpha \sum_{j=i}^{2^l} \left(e^{Y^{(upper)}_{(dec,n)}(t_{j-2^l})} - 1\right)/2^l \ ; \ Y^{(upper)}_{(dec,n)}(t_i)\right\}, \tag{6.12}$$

and

$$Y^{(lower)}_{(inc,n)}(t_i) = \min\left\{\alpha \sum_{j=i}^{2^l} \left(e^{Y^{(lower)}_{(inc,n)}(t_{j-2^l})} - 1\right)/2^l \ ; \ Y^{(lower)}_{(inc,n)}(t_i)\right\}. \tag{6.13}$$

This completes the description of the iterative procedure to improve bounding functions on the periodic solutions of the delay differential equation (1.1). The periodic solution should reach at the time point 1 the maximal value of $M$, while at the end of $(dec, 1)$ the value $-m$. We can use this fact as a condition to be checked, whether to the given $M, m$ pair a periodic solution belongs for the actual $\alpha$ differential equation parameter. The corresponding inequalities are (cf. (6.1)):

$$Y^{(upper)}_{(inc,1)}(t_{2^n}) \geq M \quad \text{and} \quad Y^{(lower)}_{(dec,1)}(t_{2^n}) \leq -m.$$

The checking algorithm is also able to decide on these conditions when the $M$ values are given as intervals. To exclude such possible intervals of $M$ we apply the above conditions for the upper bounds of the respective intervals:

$$Y^{(upper)}_{(inc,1)}(t_{2^n}) < \underline{M}. \tag{6.14}$$

By this condition we can delete all points of the respective subintervals. The checking algorithm is given as Algorithm 2.

**7. A verified computational bounding procedure and numerical results.** We composed a computer program, a verified numerical algorithm that is able to check reliably whether $\alpha$ values in an interval allow a periodic solution with given maximal value ($M$). To check the condition (6.14), the interval version of (6.1) we used an adaptive branch-and-bound technique. The investigated interval was ($[m, M] \in [0.0, 6.0] \times [0.0, 6.0]$) since values for $m$ and $M$ beyond these bounds contradict conditions (1.3). This procedure generates such a subdivision of the starting interval that for all subintervals either:

- one of the conditions (1.4) to (1.6) are hurt, or
- $M < 1 - 2\alpha/\pi$, or
- one of the conditions in (6.14) holds, or
- it is shown that a (user set size) small subinterval exists that contradicts at least one of the relations that ensure the existence of the specified periodic solution.

The mentioned branch-and-bound algorithm was introduced in [6], and the correctness proof for it was given there too. This technique was applied to prove the chaotic behaviour of some iterated nonlinear mappings [3]. To achieve the reliability of numerical calculations necessary for computer aided proofs, we applied interval arithmetic based verified algorithms [1] as also in the solution of other mathematical problems [3, 5, 6, 25]. The computational environment for the computer aided proof was C-XSC [14] and PROFIL/BIAS [15]. These will provide support for the interval arithmetic, for the outward rounding, and for the interval versions of the standard functions. The runs were executed on a 2 processor, 4 core SUN Fire V490 workstation. The parallelization of the branch-and-bound algorithm was described in the paper [29].

The source code of the algorithm is available at the internet address of

`http://www.inf.u-szeged.hu/~csendes/Wright/WrightNM.cpp`

The numerical tests made for the narrow parameter interval $\alpha = [1.500, 1.568]$ are summarized in Table 7.1. The bounding propagation cycle was applied at most 5 times – four rounds were not always enough. We stored the bounding functions on 1/1024 to 1/4096 wide time intervals. The minimum interval size (for the intervals of both $m$ and $M$) used in the branch-and-bound technique was set to $10^{-4}$, $10^{-5}$, $10^{-6}$, and $10^{-7}$. The shortest calculation required about 6.6 hours in fully successful case. The algorithm parameter settings in the last two rows of Table 7.1 allowed to prove that for the $\alpha$ values within the interval [1.500, 1.568] the solution trajectories of (1.1)

**Algorithm 2** Check the existence of a periodic trajectory

*Input:*
- $M$ and $-m$: the extreme values of the periodic trajectory,
- $\alpha$: a parameter of the studied delay differential equation,
- $2^l$: the number of equal width subintervals in the unit length time interval,
- $cikl$: the maximal number of iterations.

*Output:*
- a statement whether a periodic solution can exist with the given extreme values.

**Step 0.** Check the conditions (1.4)-(1.6) and that of Corollary 4.2 for the given $m$ and $M$ values. If any of these is false then the answer is that the given periodic solution does not exists, and STOP.

**Step 1.** Set $c = 1$ and for all $i = 1, \dots, 2^l$:
$Y^{(upper)}_{(inc,1)}(t_i) = M$, $Y^{(lower)}_{(dec,1)}(t_i) = -m$,
$Y^{(lower)}_{(inc,1)}(t_i) = 0$, and $Y^{(upper)}_{(dec,1)}(t_i) = 0$,
furthermore for all $i = 1, \dots, 2^{l+1}$:
az $Y^{(upper)}_{(dec,n)}(t_i) = M$, and $Y^{(lower)}_{(inc,n)}(t_i) = -m$.

**Step 2.** Calculate stronger bounding functions for $Y^{(upper)}_{(inc,1)}$ and $Y^{(lower)}_{(inc,1)}$ from $Y^{(lower)}_{(inc,n)}$ by the expressions (6.8) and (6.10).

**Step 3.** Calculate stronger bounding functions for $Y^{(lower)}_{(dec,1)}$ and $Y^{(upper)}_{(dec,1)}$ from $Y^{(upper)}_{(dec,n)}$ by the expressions (6.9) and (6.11).

**Step 4.** If at least one of $M \leq Y^{(upper)}_{(inc,1)}(t_{2^l})$ and $Y^{(lower)}_{(dec,1)}(t_{2^l}) \leq -m$ is false then the answer is that the given periodic solution does not exists, and STOP.

**Step 5.** Apply Algorithm 1 to calculate $\underline{P}_M$ and $\overline{P}_M$ and a new bounding function for the trajectory on $(dec, n)$.

**Step 6.** Based on the new bounding functions of Step 5 improve the bounding function $Y^{(upper)}_{(dec,n)}$.

**Step 7.** Apply the distant points of $Y^{(upper)}_{(dec,n)}$ to calculate a stronger bounding function with the help of (6.12).

**Step 8.** Calculate bounds for the trajectory on the interval $(dec, 1)$.

**Step 9.** Apply Algorithm 1 to calculate $\underline{P}_m$ and $\overline{P}_m$ and a new bounding function for the trajectory on $(inc, n)$.

**Step 10.** Based on the new bounding functions of Step 9 improve the bounding function $Y^{(lower)}_{(inc,n)}$.

**Step 11.** Apply the distant points of $Y^{(lower)}_{(inc,n)}$ to calculate a stronger bounding function with the help of (6.13).

**Step 12.** If $c \geq cikl$, then answer that the existence of a periodic solution could not be excluded, and STOP.

**Step 13.** Set $c = c + 1$, and continue at Step 2.

converge to zero as conjectured by E.M. Wright. The parallelization was successful regarding the acceleration rates close to 4. This fact raises hope that more demanding problem instances can be solved by similar architecture computers with more cores and threads.

| Parameters $Mm_{eps}$ | resolution | proven interval | elapsed time (s) | CPU time (s) | acceleration rate |
|---|---|---|---|---|---|
| | 1,024 | [1.500, 1.557] | 428 | 1,504 | 3.51 |
| $10^{-4}$ | 2,048 | [1.500, 1.558] | 1,150 | 4,040 | 3.51 |
| | 4,096 | [1.500, 1.558] | 2,222 | 7,817 | 3.51 |
| | 1,024 | [1.500, 1.565] | 2,755 | 10,310 | 3.74 |
| $10^{-5}$ | 2,048 | [1.500, 1.565] | 4,046 | 15,099 | 3.73 |
| | 4,096 | [1.500, 1.565] | 7,243 | 26,730 | 3.69 |
| | 1,024 | [1.500, 1.566] | 7,009 | 26,422 | 3.76 |
| $10^{-6}$ | 2,048 | [1.500, 1.568] | 23,956 | 90,442 | 3.77 |
| | 4,096 | [1.500, 1.568] | 34,000 | 130,156 | 3.82 |
| | 1,024 | [1.500, 1.566] | 6,972 | 26,449 | 3.79 |
| $10^{-7}$ | 2,048 | [1.500, 1.568] | 82,366 | 319,629 | 3.88 |
| | 4,096 | [1.500, 1.568] | 34,041 | 130,233 | 3.82 |

Table 7.1

*Numerical results of the computer aided proof. Here $Mm_{eps}$ stands for the stopping criterion parameter: when subintervals of this width are reached by the B&B algorithm, then it is terminated. Resolution gives the number of subintervals checked within the time unit. Elapsed time denotes the length of time interval between the start and the halt of the algorithm, while CPU time provides the total amount of CPU time used by the four processors.*

The numerical results for the original problem are summarized on Figure 7.1. The first part of the interval we are interested in, was $\alpha \in [1.500, 1.542]$. To prove that the solutions converge to zero as stated we used 12.93 seconds on the hardware described above (and 47.01 seconds cumulative time regarding the four cores). We needed 4 iteration cycles, $2^l = 128$, and $Mm_{eps} = 10^{-5}$ was the stopping criterion parameter in the B&B algorithm. This part corresponds to $\alpha$ values for which Wright stated to have a proof (but was too long to include in his article [35]).

The second part proved the statement for $\alpha \in [1.500, 1.568]$. The algorithm parameters and the CPU times used are given in the last two rows of Table 7.1. With this we have 96% of the conjectured interval that was not proven yet, and 90% of that said to be open by Wright.

The conjecture was also proven for the intervals $[1.5000, 1.5702]$, $[1.5000, 1.5705]$, and $[1.5705, 1.5706]$ for $\alpha$ with the same technique using 12 days, 37 days, and 78 days of CPU time, respectively. In the last case the hardware we used was a Hewlett Packard ProLiant DL980 Generation 7 computer with 64 cores applying hyperthreading. The computation time was converted back according to the Sun Fire V490 workstation to have comparable computation times. This result shows that slightly larger parts of the conjectured interval of $[1.5, \pi/2]$ could be proved with the presented technique in reasonable time.

Finally, we run our program also for the last part that completes the whole parameter interval in question, and we obtained a computational proof of Theorem 1.2. This phase is based on the still unproven statement that if the $M$ value is below 0.04 then the trajectories always converge to zero. In this way, this part of our computational results is just an indication how long a computational proof would be if the mentioned statement would be proven (three days of CPU time). One order of magnitude of decrease in the bounding constant 0.04 corresponds to ca. 1000 times more computation time. In other words, substantial improvement of the theoretical part of the present proof is needed to prove Wright's conjecture fully.

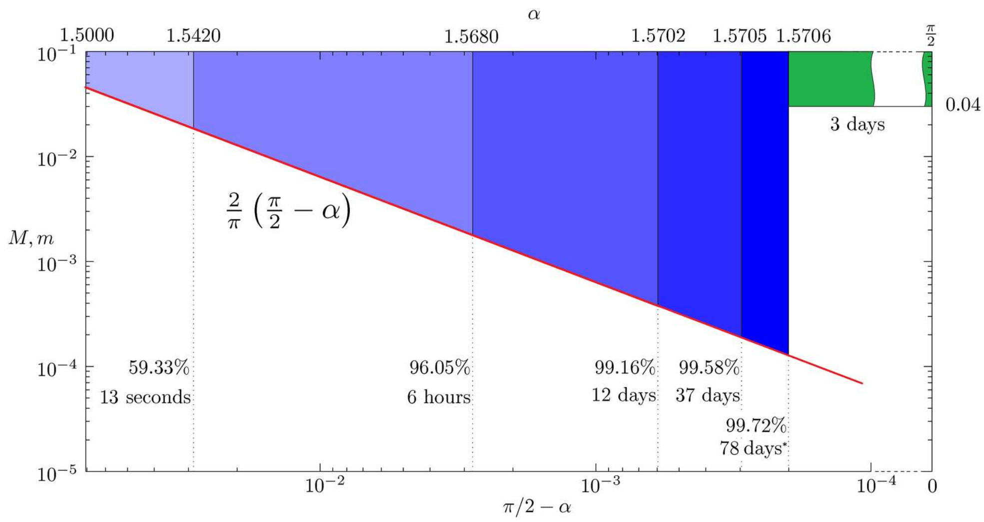


FIG. 7.1. *Illustration of the proved part of the conjecture, indicated by shades of blue (top left). The convergence of the solutions to zero for $\alpha \in [1.5, 1.5420]$ was stated to be proven by E.M. Wright, but the proof was not given. Wright thought that his technique can be successful until $\alpha = 1.567\ldots$ The green area in the top right corner demonstrates the virtual subproblem, in the case somebody could prove (e.g. by theoretical tools) that the slowly oscillating solutions cannot have an amplitude smaller than 0.04. The computation times are measured on a 4 core Sun Fire V490 workstation. (The result marked with * is calculated on a 64 core HP ProLiant DL980 Generation 7 computer with hyper-threading. We have converted the CPU time for the earlier mentioned 4 core SUN hardware, based on the number of threads.)*

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