8. Optical Flow

Computer Vision

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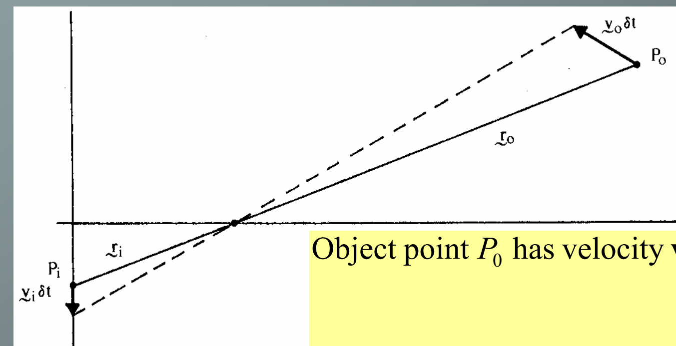
Motion Field and Optical Flow Field

- **Motion field**: projection of 3D motion vectors on image plane
- **Optical flow field**: apparent motion of brightness patterns
- In practice, we equate motion field with optical flow field



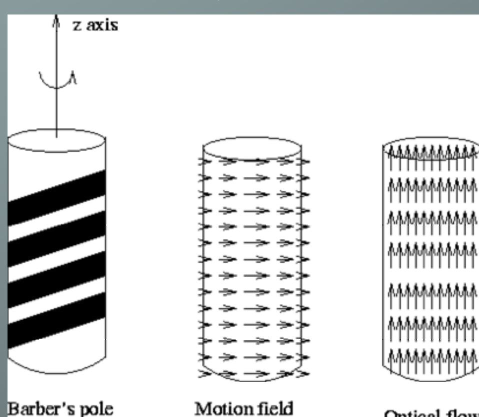
Object point P_0 has velocity $\mathbf{v}_0$, induces $\mathbf{v}_i$ in image

$$\mathbf{v}_0 = \frac{d\mathbf{r}_0}{dt} \qquad \mathbf{v}_1 = \frac{d\mathbf{r}_i}{dt}$$

$\mathbf{r}_0$ related to $\mathbf{r}_i$ by $\dfrac{\mathbf{r}_i}{f} = \dfrac{\mathbf{r}_0}{\mathbf{r}_0 \cdot \hat{\mathbf{z}}_0}$

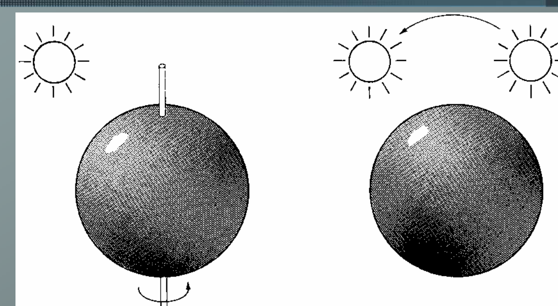
Optical Flow Vs. Motion Field

- Optical flow does not always correspond to motion field



- Optical flow is an approximation of the motion field. The error is small at points with high spatial gradient under some simplifying assumptions

When Optical Flow≠Motion Field

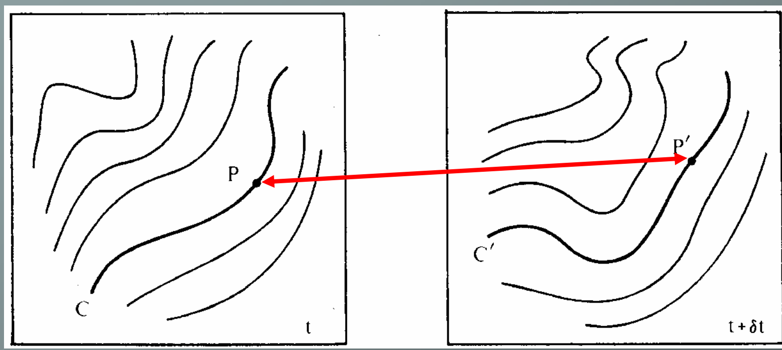


(a) (b)

(a) A smooth sphere is rotating under constant illumination. Thus the optical flow field is zero, but the motion field is not (no visible brightness change)

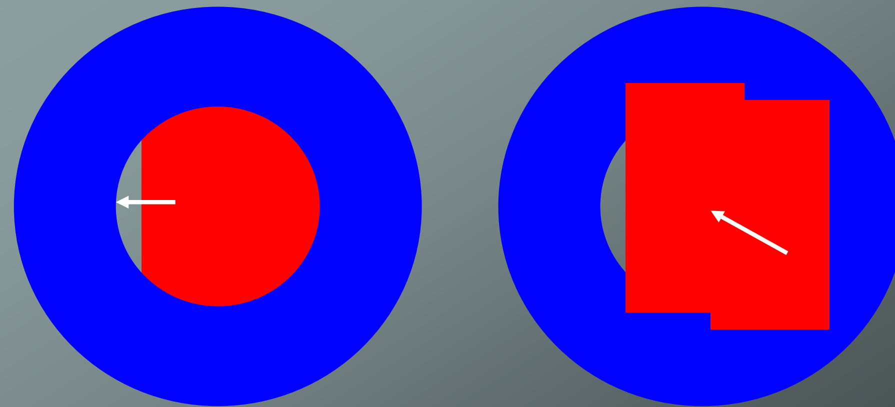
(b) A fixed sphere is illuminated by a moving source ➔ the shading of the image changes. Thus the motion field is zero, but the optical flow field is not (brightness change due to other factors than motion).

What is Meant by Apparent Motion of Brightness Pattern?



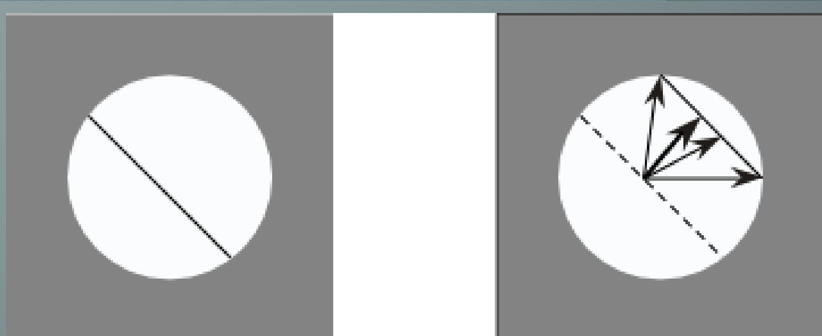
- The apparent motion of brightness patterns is an awkward concept.
  - It is not easy to decide which point P' on a contour C' of constant brightness in the second image corresponds to a particular point P on the corresponding contour C in the first image.

The Aperture Problem



- Only the flow component perpendicular to the line feature can be computed.

Aperture Problem



(a) (b)

(a) Line feature observed through a small aperture at time *t*.

(b) At time *t+δt* the feature has moved to a new position. It is not possible to determine exactly where each point has moved. From local image measurements only the flow component perpendicular to the line feature can be computed.

Normal flow: Component of flow perpendicular to line feature.

Image Brightness Constancy Equation (IBCE)

- Let **P** be a moving point in 3D:
  - At time **t**, **P** has coords **(X(t),Y(t),Z(t))**
  - Let **p=(x(t),y(t))** be the coords. of its image at time **t**.
  - Let **E(x(t),y(t),t)** be the brightness of **p** at time **t**.
- **Brightness Constancy Assumption**:
  - As **P** moves over time, **E(x(t),y(t),t)** remains constant.
  - Issues:
    - Lighting may change
    - Objects may reflect differently at different angles

Brightness Constancy Equation

$$E(x(t), y(t), t) = Constant$$

Taking derivative wrt time:

$$\frac{dE(x(t), y(t), t)}{dt} = 0$$

$$\frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} + \frac{\partial E}{\partial t} = 0$$

Brightness Constancy Equation

$$\frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} + \frac{\partial E}{\partial t} = 0$$

Let

$$\nabla E = \begin{bmatrix} \frac{\partial E}{\partial x} \\ \frac{\partial E}{\partial y} \end{bmatrix}$$ (Frame spatial gradient)

$$v = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}$$ (optical flow)

and

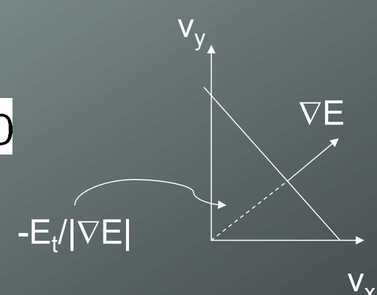
$$E_t = \frac{\partial E}{\partial t}$$ (derivative across frames)

Brightness Constancy Equation

$$\frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} + \frac{\partial E}{\partial t} = 0$$

Becomes:

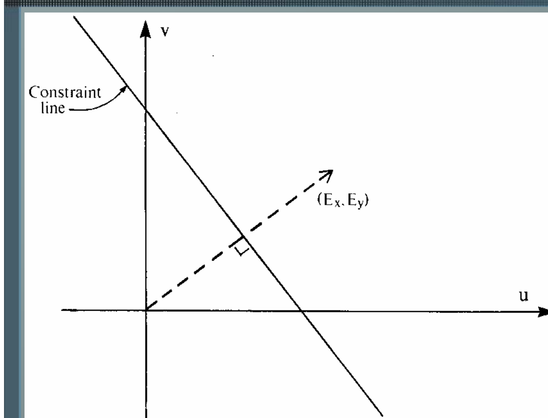
$$(\nabla E)^T . v + E_t = 0$$



- Also known as the *Horn and Schunck optical flow equation*
- Relation of the apparent motion with the spatial and temporal derivatives of the image brightness

The OF is CONSTRAINED to be on a line !

Interpretation



- Values of **(u,v)** satisfying the constraint equation lie on a straight line in velocity space.
- A local measurement only provides this constraint line (aperture problem).

Normal flow $\mathbf{v}_n$

$$(E_x, E_y)\cdot(u,v) = -E_t$$

$$\text{Let } \mathbf{n} = \frac{(E_x, E_y)^T}{\left\|(E_x, E_y)^T\right\|}$$

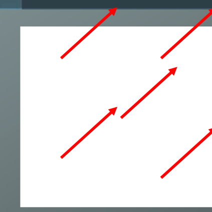
$$\mathbf{v}_n = (\mathbf{v}\cdot\mathbf{n})\mathbf{n} = \left(\frac{-E_x E_t}{E_x^2 + E_y^2}, \frac{-E_y E_t}{E_x^2 + E_y^2}\right)^T$$

Estimating Optical Flow

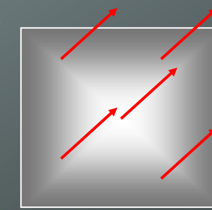
- **Differential techniques**: based on spatial & temporal variations of the image at *all* pixels
- **Matching (feature-based) techniques**: rely on special image points (features) and track them through frames

Estimating Optical Flow

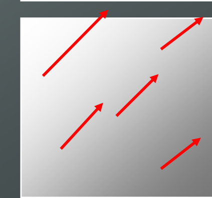
- Constant Flow Method
  - Assumption: the motion field is well approximated by a constant vector within any small region of the image plane
  - Solution: Least square of two variables **(u,v)** from NxN Equations – NxN (=5x5) planar patch
  - **Condition**: A^TA is NOT singular (null or parallel gradients)



- Weighted Least Square Method
  - Assumption: the motion field is approximated by a constant vector within any small region, and the error made by the approximation increases with the distance from the center where optical flow is to be computed
  - Solution: Weighted least square of two variables **(u,v)** from NxN Equations – NxN patch



- Assuming a Motion Model (eg. Affine Flow)
  - Assumption: the motion field is well approximated by a affine parametric model $v^T = Ap^T + b$ (a plane patch with arbitrary orientation)
  - Solution: Least square of 6 variables **(A,b)** from NxN Equations – NxN planar patch



Spatial Motion Models

- Assume a parametric model of the image velocities **v=(u,v)**:
  - Translational model: $\mathbf{v}(\mathbf{p}) = (a_1, a_2)^T$
  - Affine model:

$$\mathbf{v}(\mathbf{p}) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} a_3 & a_4 \\ a_5 & a_6 \end{pmatrix} \mathbf{p}$$

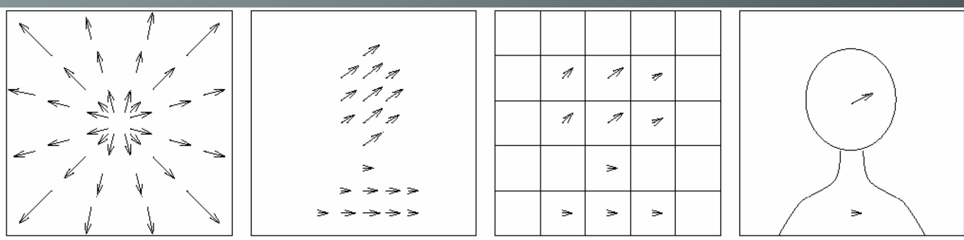
  - Projective model, ...

Temporal Motion Model

- Linear trajectory (2-parameter) models:
  - assume a constant velocity $\mathbf{v}_t(\mathbf{p})$ at the time interval **(t,T); T>t**:
    - $\mathbf{p}(T) = \mathbf{p}(t) + \mathbf{v}_t(\mathbf{p})(T-t) = \mathbf{p}(t) + \mathbf{d}(t,T(\mathbf{p}))$
    - $\mathbf{d}(t,T(\mathbf{p}))$: displacement vector
- Quadratic trajectory (4-parameter) models:
  - account for acceleration:
    - $\mathbf{p}(T) = \mathbf{p}(t) + \mathbf{v}_t(\mathbf{p})(T-t) + 0.5\,\mathbf{a}_t(\mathbf{p})(T-t)^2$
    - $\mathbf{a} = (a_1, a_2)^T$ is the acceleration component

Motion Models

- Region of support **R**:
  - The set of points **p** to which a spatial and temporal motion model applies.
  - smaller regions ➔ better approximations
    - Whole image
    - Single pixel
    - Rectangular block of pixels (H.26x, MPEG-1 and 2)
    - Irregularly shaped region (MPEG-4) –– requires a good segmentation



(a) (b) (c) (d)

Observation Models

- **$E_t(p) = E_{t-1}(p-d) + q(p)$**
- Along motion trajectory **s**: **dE/ds=0**
- Motion constraint equation:

$$\frac{\partial E}{\partial x}u + \frac{\partial E}{\partial y}v + \frac{\partial E}{\partial t} = 0$$

- Motion compensated error measure: **$E_t(p)-E^*_t(p)$**
  - **$E^*_t(p) = E_{t-1}(p-d)+q(p)$** is the motion-compensated prediction of **$E_t(p)$**.
  - **q(p)** is known changes in image brightness

Differential Techniques

- For each pixel **p**, must satisfy IBCE: **$(\nabla E)\mathbf{v}+E_t=0$**
  - Additional constraints:
    - IBCE holds in the neighborhood of **p** ***with constant*** **v**
      - Write this equation for a small (typically 5x5) patch centered at **p**
      - Then we find the LSE fit of **v** ➔ this is the calculated optical flow at pixel **p**
    - In case of rigid motion, the motion field of a moving plane is a quadratic polynomial in the coordinates **(x,y,f)** of the image points.
      - Therefore, if the object is **smooth & rigid**, we can assume the motion field varies smoothly

Constant flow assumption

- N=5 ➔ 25 equations:

$$\underbrace{\begin{bmatrix} E_x(p_1) & E_y(p_1) \\ E_x(p_2) & E_y(p_2) \\ \vdots & \vdots \\ E_x(p_{N^2}) & E_y(p_{N^2}) \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\mathbf{v}} = - \underbrace{\begin{bmatrix} E_t(p_1) \\ E_t(p_2) \\ \vdots \\ E_t(p_{N^2}) \end{bmatrix}}_{\mathbf{b}}$$

- Solve as a standard LSE problem:

$$\mathbf{Av} = \mathbf{b} \quad \text{LSE solution}: \quad (\mathbf{A}^T\mathbf{A})\mathbf{v} = \mathbf{A}^T\mathbf{b}$$

$$\underbrace{\begin{bmatrix} \sum E_x^2 & \sum E_x E_y \\ \sum E_x E_y & \sum E_y^2 \end{bmatrix}}_{\mathbf{A}^T\mathbf{A}} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\mathbf{v}} = - \underbrace{\begin{bmatrix} \sum E_x E_t \\ \sum E_y E_t \end{bmatrix}}_{\mathbf{A}^T\mathbf{b}}$$

What is (A^TA)?

- It is the matrix for corner detection (Harris):

$$\begin{bmatrix} \sum E_x^2 & \sum E_x E_y \\ \sum E_x E_y & \sum E_y^2 \end{bmatrix}$$

- Singular when **$\det(A^TA)=\lambda_1\lambda_2=0$**
  - → one or both eigenvalues are 0
  - → aperture problem:
    - One is 0 → no corner, just an edge
    - Both are 0 → no corner, homogeneous region
- Additional constraints are needed in order to *regularize* the problem.

Differential Techniques: Horn-Schunck Algorithm

- Optical flow constraint equation gives the component in direction of brightness gradient :

$$\frac{E_t}{\sqrt{E_x^2 + E_y^2}}$$

- Additional Constraint: *smoothness of optical flow*
  - Neighboring surface points of a rigid object have approximately same local displacement vectors

Horn-Schunck Algorithm

- Two criteria:
  - Optical flow is smooth:

$$F_s(u,v) = \iint_D \left(u_x^{\ 2} + u_y^{\ 2}\right) + \left(v_x^{\ 2} + v_y^{\ 2}\right) dx\, dy$$

  - Small error in optical flow constraint equation:

$$F_h(u,v) = \iint_D \left(E_x u + E_y v + E_t\right)^2 dx\, dy$$

- Minimize a combined error functional:

$$F(u,v) = \iint (\nabla E \cdot \mathbf{v} + E_t)^2 + \lambda\left(\|\nabla u\|_2^{\ 2} + \|\nabla v\|_2^{\ 2}\right) dx\, dy \to \min$$

λ is a weighting parameter

Horn-Schunck Algorithm

- Variation calculus gives a pair of second order differential equations that can be solved iteratively
  - Derivatives (and error functionals) are approximated by difference operators:

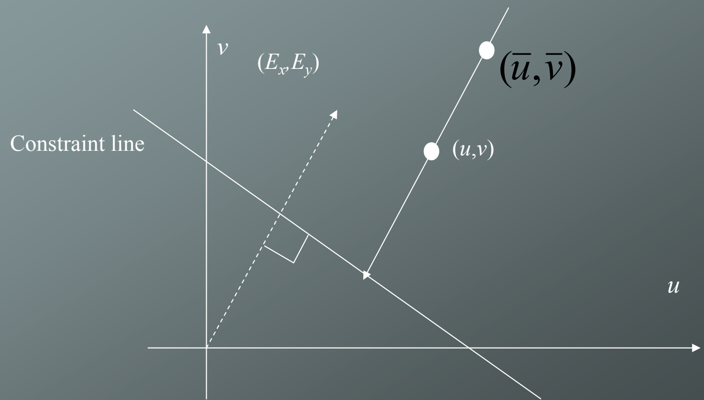
$$u_{ij}^{n+1} = \bar{u}_{ij}^{n} - \alpha E_x \qquad \text{where } \alpha = \frac{E_x \bar{u}_{ij}^n + E_y \bar{v}_{ij}^n + E_t}{1 + \lambda(E_x^2 + E_y^2)}$$

$$v_{ij}^{n+1} = \bar{v}_{ij}^{n} - \alpha E_y$$

$\bar{u}, \bar{v}$ is the average of values of neighbors

Iterative Scheme

- The new value of **(u,v)** at a point is equal to the average of surrounding values minus an adjustment in the direction of the brightness gradient



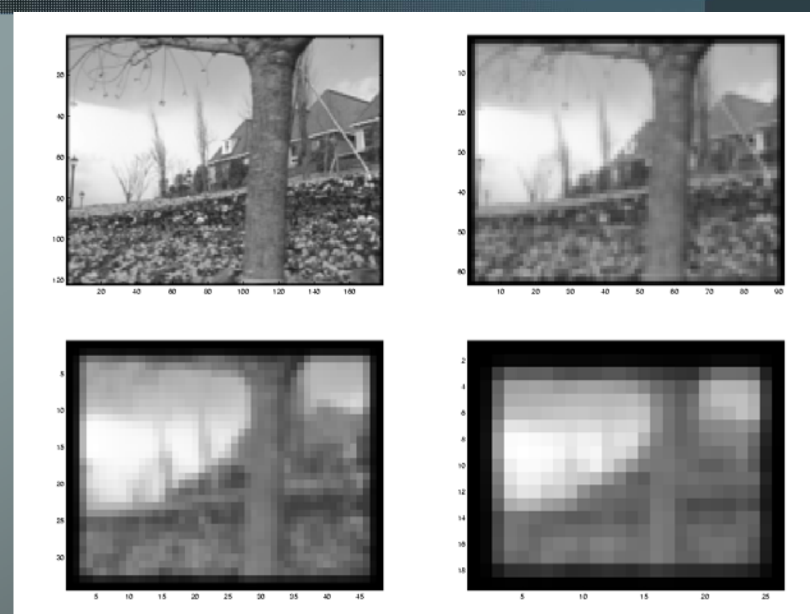
Horn-Schunck Algorithm

begin
    for *j* := 1 to *N* **do**   **for** *I*:= 1 to *M* **do**   **begin**
        calculate the values $E_x(i,j,t)$, $E_y(i,j,t)$ and $E_t(i,j,t)$ using a selected approx formula
        initialize the values *u(I,j)* and *v(i,j)* to zero
    end {for}
    choose a suitable weighting value λ
    choose a suitable number $n_0 \geq 1$ of iterations
    $n := 1$
    while $n \leq n_0$ **do begin**
        for *j* := 1 to *N* **do**   **for** *i* := 1 to *M* **do**   **begin**
            compute $\underline{u}$, $\underline{v}$, α
            update *u(i,j)*, *v(i,j)*
        end {for}
        $n := n + 1$
    end {while}
end

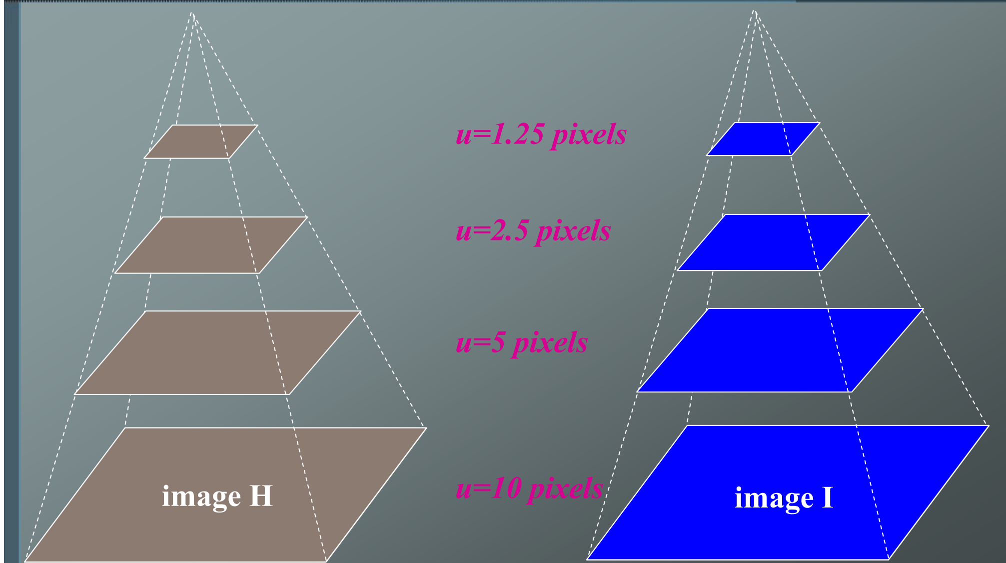
What about larger motions?

- Is this motion small enough?
  - Probably not—it's much larger than one pixel (2nd order terms dominate)
  - How might we solve this problem?

Reduce the resolution!



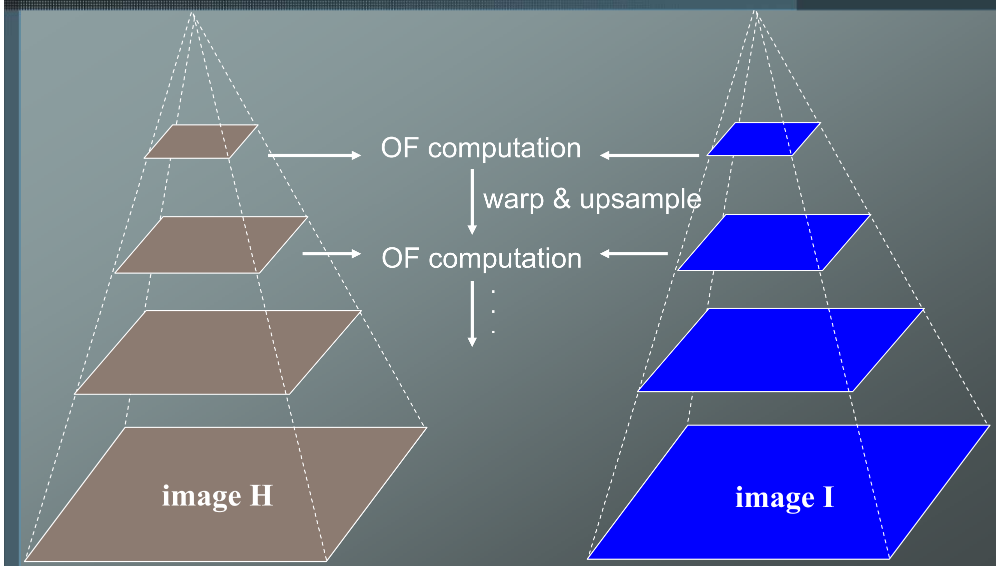
Coarse-to-fine optical flow estimation



Gaussian pyramid of image H

Gaussian pyramid of image I

Coarse-to-fine optical flow estimation



Gaussian pyramid of image H

Gaussian pyramid of image I

Optical Flow Examples