

Metric representations by v_i -products

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The purpose of this paper is to compare the metric representation powers of the product and v_i -products introduced in [1]. It is shown that a class of automata is metrically complete with respect to the product if and only if it is metrically complete regarding the v_1 -product. It is also proved that the v_3 -product is metrically equivalent to the product.

We start with some basic notions and notations.

An *alphabet* is a nonvoid finite set. The free monoid generated by an alphabet X will be denoted by X^* . An element $p = x_1 \dots x_n \in X^*$ ($x_i \in X$, $i = 1, \dots, n$) is a *word* over X , and n is the *length* of p , in notation, $|p| = n$. If $n = 0$ then p is the *empty word*, which will be denoted by e . For arbitrary integer $n (\cong 0)$, $X^{(n)}$ will stand for the subset of X^* consisting of all words with length less than or equal to n .

An *automaton* is a system $\mathfrak{A} = (X, A, \delta)$, where X is the *input alphabet*, A is a nonvoid finite set of *states* and the mapping $\delta: A \times X \rightarrow A$ is the *transition function* of \mathfrak{A} . We extend δ to a mapping $\delta: A \times X^* \rightarrow A$ in the following way: for arbitrary $a \in A$, $\delta(a, e) = a$ and $\delta(a, px) = \delta(\delta(a, p), x)$ ($p \in X^*$, $x \in X$).

Take an automaton $\mathfrak{A} = (X, A, \delta)$, a state $a \in A$ and an integer $n (\cong 0)$. We say that the system (\mathfrak{A}, a) is *n-free* if $\delta(a, p) \neq \delta(a, q)$ for arbitrary $p, q \in X^{(n)}$ with $p \neq q$.

If we add an output to an automaton then we get the concept of a sequential machine. More precisely, a system $\mathfrak{M} = (X, A, Y, \delta, \lambda)$ is a *Mealy machine*, where (X, A, δ) is an automaton, Y is the *output alphabet* and the mapping $\lambda: A \times X \rightarrow Y$ is the *output function* of \mathfrak{M} . We can extend λ to a mapping $\lambda: A \times X^* \rightarrow Y^*$ in the following way: for every $a \in A$, $\lambda(a, e) = e$ and $\lambda(a, px) = \lambda(a, p)\lambda(\delta(a, p), x)$. A mapping $\mu: X^* \rightarrow Y^*$ is called an *automaton mapping* if there exist a Mealy machine $\mathfrak{M} = (X, A, Y, \delta, \lambda)$ and an $a \in A$ such that $\mu(p) = \lambda(a, p)$ ($p \in X^*$). If this is the case then we say that μ can be *induced* by \mathfrak{M} in the state a .

Take a Mealy machine $\mathfrak{M} = (X, A, Y, \delta, \lambda)$, an automaton mapping $\mu: X^* \rightarrow Y^*$ and an integer $n (\cong 0)$. It is said that \mathfrak{M} *induces* μ in *length* n if for some $a \in A$, $\mu(p) = \lambda(a, p)$ ($p \in X^{(n)}$).

Let $\mathfrak{A}_j = (X_j, A_j, \delta_j)$ ($j = 1, \dots, t$) be automata, X and Y alphabets, and

$$\varphi: A_1 \times \dots \times A_t \times X \rightarrow X_1 \times \dots \times X_t,$$

$$\psi: A_1 \times \dots \times A_t \times X \rightarrow Y$$

mappings. Then the Mealy machine $\mathfrak{A}=(X, A, Y, \delta, \lambda)$ is the *product* (α_t -product, v_t -product) of \mathfrak{A}_j ($j=1, \dots, t$) with respect to X, Y and φ, ψ if the automaton (X, A, δ) is the product (α_t -product, v_t -product) of \mathfrak{A}_j ($j=1, \dots, t$) with respect to X and φ , and for arbitrary $\mathbf{a}=(a_1, \dots, a_t) \in A$ and $x \in X, \lambda(\mathbf{a}, x)=\psi(a_1, \dots, a_t, x)$.

A class K of automata is *metrically complete* with respect to the product (α_t -product, v_t -product) if for arbitrary automaton mapping $\mu: X^* \rightarrow Y^*$ and integer $n(\cong 0)$ there exists a product (α_t -product, v_t -product) $\mathfrak{A}=(X, A, Y, \delta, \lambda)$ of automata from K inducing μ in length n . Moreover, the v_t -product is *metrically equivalent* to the product provided that for every class K of automata and non-negative integer n an automaton mapping $\mu: X^* \rightarrow Y^*$ can be induced in length n by a v_t -product $\mathfrak{A}=(X, A, Y, \delta, \lambda)$ of automata from K if and only if it can be induced in length n by a product $\mathfrak{B}=(X, B, Y, \delta', \lambda')$ of automata from K .

Let $\mathfrak{A}_i=(X_i, A_i, \delta_i)$ ($i=1, \dots, t$) be automata, and take a product

$$\mathfrak{A} = (X, A, \delta) = \prod_{i=1}^t A_i[X, \varphi].$$

Then for arbitrary $\mathbf{a}=(a_1, \dots, a_t) \in A, p \in X^*$ and i ($1 \leq i \leq t$) define $\varphi_i(\mathbf{a}, p)$ in the following way: $\varphi_i(\mathbf{a}, e)=e$ and $\varphi_i(\mathbf{a}, qx)=\varphi_i(\mathbf{a}, q)\varphi_i(\delta(\mathbf{a}, q), x)$ ($q \in X^*, x \in X$).

For notions and notations not defined here, see [3] and [4].

Now we are ready to state and prove

Theorem 1. A class K of automata is metrically complete with respect to the product if and only if K is metrically complete with respect to the v_1 -product.

Proof. The condition is obviously sufficient.

To show the necessity assume that K is metrically complete with respect to the product. We prove that for every alphabet Y and integer $k(\cong 0)$ there exist a v_1 -product $\mathfrak{D}=(Y, D, \delta'')$ of automata from K and a state $d \in D$ such that the system (\mathfrak{D}, d) is k -free. This obviously implies that K is metrically complete with respect to the v_1 -product.

It is shown in [2] that K is metrically complete with respect to the product if and only if for arbitrary integer $k(\cong 0)$ there exist an $\mathfrak{A}=(X, A, \delta)$ in K a state $a_0 \in A$ and a word $p \in X^*$ with $|p|=k$ such that $\delta(a_0, p)$ is ambiguous, that is $\delta(a_0, px) \neq \delta(a_0, px')$ for some $x, x' \in X$. Let us distinguish the following two cases.

Case 1. K contains an $\mathfrak{A}=(X, A, \delta)$ such that for certain pairwise distinct states $a_0, a_1, \dots, a_{n-1}, a'_1$ and inputs $x_0, x_1, \dots, x_{n-1}, x'_1$ we have

$$\delta(a_0, x_1) = a_1, \delta(a_1, x_2) = a_2, \dots, \delta(a_{n-2}, x_{n-1}) = a_{n-1}, \delta(a_{n-1}, x_0) = a_0$$

$$\text{and } \delta(a_0, x'_1) = a'_1.$$

Let $k(>0)$ be an integer, and take two words $p=y_1 \dots y_r y_{r+1} \dots y_s, q=y_1 \dots y_r z_{r+1} \dots z_t \in Y^{(k)}$ ($y_1, \dots, y_s, z_{r+1}, \dots, z_t \in Y$) with $t \leq s$, and $y_{r+1} \neq z_{r+1}$ if $t \neq r$, where Y is an arbitrarily fixed alphabet. Consider the v_1 -product

$$\mathfrak{B} = (Y, B, \delta') = \prod_{i=1}^{s+1} \mathfrak{B}_i[Y, \varphi, v]$$

given as follows.

$$\mathfrak{B}_i = \mathfrak{A} \quad (i = 1, \dots, s+1).$$

$$v(1) = \emptyset \quad \text{and} \quad v(i) = i-1 \quad (i = 2, \dots, s+1).$$

$$\varphi_i(a_j, y) = x_j \quad (i=2, \dots, s+1; j=0, \dots, n-1; y \in Y).$$

$$\varphi_i(a'_1, y) = \begin{cases} x_1 & \text{if } i = r+2 \text{ and } y = z_{r+1}, \\ x'_1 & \text{otherwise} \end{cases} \quad (i = 2, \dots, s+1; y \in Y).$$

In all other cases φ is given arbitrarily such that the resulting product is a v_1 -product.

Take the state $\mathbf{b} = (b_1, b_2, \dots, b_{s+1}) \in B$ with $b_1 = a'_1$, $b_i = a_{n-(i-2)}$ ($i = 2, \dots, s+1$), where the indices of a 's are taken modulo n . One can easily show by induction on j that for every $j (= 1, \dots, s)$

$$\delta'(\mathbf{b}, y_1 \dots y_j) = (c_1, \dots, c_{j+1}, c_{j+2}, \dots, c_{s+1})$$

where $c_{j+1} = a'_1$ and $c_i = a_{n-(i-2)+j}$ ($i = j+2, \dots, s+1$). Moreover, for every $j (= r+1, \dots, t)$

$$\delta'(\mathbf{b}, y_1 \dots y_r z_{r+1} \dots z_j) = (c_1, \dots, c_{j+1}, \dots, c_{s+1})$$

where $c_i = a_{n-(i-2)+j}$ ($i = j+1, \dots, s+1$). (The indices of a 's are considered modulo n in the latter two cases, too.)

Therefore, the last component of $\delta'(\mathbf{b}, p)$ is a'_1 , and the last component of $\delta'(\mathbf{b}, q)$ is in the set $\{a_0, a_1, \dots, a_{n-1}\}$. Thus $\delta'(\mathbf{b}, p) \neq \delta'(\mathbf{b}, q)$.

Case 2. K does not satisfy the conditions of Case 1. Then for every integer $k (\geq 0)$ there is an $\mathfrak{A} = (X, A, \delta)$ in K with pairwise distinct states $a_0, a_1, \dots, a_k, a_{k+1}, a'_{k+1}$ and inputs $x_1, x_2, \dots, x_k, x_{k+1}, x'_{k+1}$ such that $\delta(a_i, x_{i+1}) = a_{i+1}$ ($i = 0, \dots, k$) and $\delta(a_k, x'_{k+1}) = a'_{k+1}$. Again take the alphabet Y and the words p, q of Case 1. Consider the v_1 -product

$$\mathfrak{B} = (Y, B, \delta') = \prod_{i=1}^s \mathfrak{B}_i[Y, \varphi, v]$$

given in the following way.

$$\mathfrak{B}_i = \mathfrak{A} \quad (i = 1, \dots, s).$$

$$v(1) = \emptyset \quad \text{and} \quad v_i = i-1 \quad (i = 2, \dots, s).$$

$$\varphi_1(y_1) = x_{k+1} \quad (\text{and } \varphi_1(z_1) = x'_{k+1} \text{ if } r = 0 \text{ and } t \neq 0).$$

$$\varphi_i(a_j, y) = \begin{cases} x'_{k+1} & \text{if } i = r+1, j = k+1 \text{ and } y = z_{r+1}, \\ x_j & \text{otherwise} \end{cases}$$

$$(i = 2, \dots, s; j = 1, \dots, k+1).$$

$$\varphi_i(a'_{k+1}, y) = x'_{k+1} \quad (i = 2, \dots, s).$$

In all other cases φ is given arbitrarily in accordance with the definition of the v_1 -product.

Take the state $\mathbf{b} = (a_k, a_{k-1}, \dots, a_{k-s+1}) \in B$. Again it is easy to show that for every $j (= 1, \dots, s)$

$$\delta'(\mathbf{b}, y_1 \dots y_j) = (c_1, \dots, c_j, \dots, c_s),$$

where $c_i = a_{k-(i-1)+j}$ ($i=j, \dots, s$). Moreover, for every $j (=r+1, \dots, s)$

$$\delta'(\mathbf{b}, y_1 \dots y_r z_{r+1} \dots z_j) = (c_1, \dots, c_j, \dots, c_s)$$

with $c_j = a'_{k+1}$ and $c_i = a_{k-(i-1)+j}$ ($i=j+1, \dots, s$).

Therefore, the last component of $\delta'(\mathbf{b}, p)$ is a_{k+1} . If $s=t$ then the last component of $\delta'(\mathbf{b}, q)$ is a'_{k+1} . Moreover, if $t < s$ then the last component of $\delta'(\mathbf{b}, q)$ is $a_{k-(s-1)+t}$. In both cases we have $\delta'(\mathbf{b}, p) \neq \delta'(\mathbf{b}, q)$.

To end the proof of Theorem 1 take an integer $k (\cong 1)$ and an alphabet Y . Moreover, set $I = \{(p, q) | p, q \in Y^{(k)}, p \neq q\}$. As it has been shown for every pair $(p, q) \in I$ there exist a v_1 -product $\mathfrak{D}_{(p,q)} = (Y, D_{(p,q)}, \delta_{(p,q)})$ of automata from K and a state $d_{(p,q)} \in D_{(p,q)}$ such that $\delta_{(p,q)}(d_{(p,q)}, p) \neq \delta_{(p,q)}(d_{(p,q)}, q)$. Form the direct product $\mathfrak{D} = \prod (\mathfrak{D}_{(p,q)} | (p, q) \in I)$, and take the state $\mathbf{d} \in D$ with $pr_{(p,q)}(\mathbf{d}) = d_{(p,q)}$, where $pr_{(p,q)}$ denotes the $(p, q)^{\text{th}}$ projection. Obviously, $(\mathfrak{D}, \mathbf{d})$ is a k -free system. Since the direct product of v_1 -products of automata is isomorphic to a v_1 -product of the same automata this completes the proof of Theorem 1.

Let us note that the v_1 -product used in the proof of Theorem 1 is also an α_0 -product.

Next we prove

Theorem 2. The product is metrically equivalent to the v_3 -product.

Proof. Let K be a class of automata. If K is metrically complete with respect to the product then, by Theorem 1, for arbitrary integer $k (\cong 0)$ every automaton mapping $\mu: X^* \rightarrow Y^*$ can be induced in length $k+1$ by a v_1 -product $\mathfrak{A} = (X, A, Y, \delta, \lambda)$ of automata from K . Thus we assume that K is not metrically complete with respect to the product. Therefore, none of Case 1 and Case 2 holds for K . This implies that either there is no ambiguous state in any of the automata from K or there is a maximal positive integer k such that for some $\mathfrak{A} = (X, A, \delta) \in K$, $a \in A$ and $p \in X^*$ with $|p| = k-1$, $\delta(a, p)$ is ambiguous. In the first case every product of automata from K can be given as a quasi-direct product of the same automata. Thus we suppose the existence of the above k .

Let

$$\mathfrak{A} = (X, A, \delta) = \prod_{i=1}^s \mathfrak{A}_i[X, \varphi] \quad (\mathfrak{A}_i = (X_i, A_i, \delta_i) \in K, i = 1, \dots, s)$$

be a product and $\mathbf{a} = (a_1, \dots, a_s) \in A$ a state. We shall prove the existence of a v_3 -product

$$\mathfrak{B} = (X, B, \delta') = \prod_{i=1}^t \mathfrak{B}_i[X, \varphi', \nu] \quad (\mathfrak{B}_i = (X'_i, B_i, \delta'_i), i = 1, \dots, t)$$

with a state $\mathbf{b} = (b_1, \dots, b_t) \in B$ such that the following conditions are satisfied.

(i) (\mathfrak{B}_1, b_1) is k -free, $X'_1 = X$, φ'_1 is the identity mapping on X and \mathfrak{B}_1 is a v_1 -product of automata from K .

(ii) \mathfrak{B}_2 is a v_1 -product of automata from K , $X'_2 = X$ and for any two words $p, q \in X^*$ with $|p| < k$ and $|q| \cong k$, $\delta'_2(b_2, \varphi'_2(\mathbf{b}, p)) \neq \delta'_2(b_2, \varphi'_2(\mathbf{b}, q))$.

(iii) $\mathfrak{B}_i \in K$ ($i=3, \dots, t$).

(iv) For arbitrary two words $p, q \in X^*$ with $|p| = |q| = k$ and integer i ($1 \leq i \leq s$) there is a j ($1 \leq j \leq t$) with $\mathfrak{B}_j = \mathfrak{A}_i$, $b_j = a_i$, $\delta'_j(b_j, \varphi'_j(\mathbf{b}, p)) = \delta_i(a_i, \varphi_i(\mathbf{a}, p))$ and $\delta'_j(b_j, \varphi'_j(\mathbf{b}, q)) = \delta_i(a_i, \varphi_i(\mathbf{a}, q))$.

This will imply that the subautomaton of \mathfrak{A} generated by \mathbf{a} is a homomorphic image of the subautomaton of \mathfrak{B} generated by \mathbf{b} . Indeed, take two words $p, q \in X^*$ with $\delta(\mathbf{a}, p) \neq \delta(\mathbf{a}, q)$. It is enough to show that $\delta'(\mathbf{b}, p) \neq \delta'(\mathbf{b}, q)$. Let us distinguish the following cases.

(I) $|p|, |q| \leq k$. Then $\delta'(\mathbf{b}, p) \neq \delta'(\mathbf{b}, q)$ since they differ at least in their first components.

(II) $|p| < k$ and $|q| > k$. Then $\delta'(\mathbf{b}, p)$ and $\delta'(\mathbf{b}, q)$ are different at least in their 2nd components.

(III) $|p|, |q| \geq k$. First of all observe that, by the maximality of k , for arbitrary automaton $\mathfrak{C} = (Y, C, \delta'') \in K$, state $c \in C$ and words $r, r_1, r_2 \in Y^*$ with $|r| = k$ and $|r_j| = |r_2|$, $\delta''(c, rr_1) = \delta''(c, rr_2)$. Let $p = p_1 p_2$ and $q = q_1 q_2$ ($|p_1| = |q_1| = k$). Moreover, let i ($1 \leq i \leq s$) be an index for which $\delta_i(a_i, \varphi_i(\mathbf{a}, p)) \neq \delta_i(a_i, \varphi_i(\mathbf{a}, q))$. Take the index j given by (iv) to this i and p_1, q_1 . Then by our remark above $\delta'_j(b_j, \varphi'_j(\mathbf{b}, p_1 p_2)) = \delta'_j(b_j, \varphi'_j(\mathbf{b}, p_1) p_2) = \delta_i(a_i, \varphi_i(\mathbf{a}, p_1) p_2) = \delta_i(a_i, \varphi_i(\mathbf{a}, p_1 p_2))$ where $p_2 \in X_i^*$ is a word with $|p_2| = |p_2|$. Similarly, $\delta'_j(b_j, \varphi'_j(\mathbf{b}, q_1 q_2)) = \delta_i(a_i, \varphi_i(\mathbf{a}, q_1 q_2))$. Therefore, $\delta'(\mathbf{b}, p) \neq \delta'(\mathbf{b}, q)$ since they differ at least in their j th components.

The k -free automaton in (i) can be constructed by using the same method as in the proof of Theorem 1 (according to Case 2).

To give \mathfrak{B}_2 take an automaton $\mathfrak{C} = (Y, C, \delta'') \in K$ with pairwise distinct states $c_0, c_1, \dots, c_{k-1}, c_k, c'_k$ and inputs $y_1, \dots, y_{k-1}, y_k, y'_k$ such that $\delta''(c_0, y_1) = c_1, \dots, \delta''(c_{k-2}, y_{k-1}) = c_{k-1}, \delta''(c_{k-1}, y_k) = c_k$ and $\delta''(c_{k-1}, y'_k) = c'_k$. Form the single factor v_1 -product

$$\mathfrak{B}_2 = \mathfrak{C}[X, \varphi'', v']$$

where $v'(1) = 1$ and $\varphi''(c_i, x) = y_{i+1}$ ($i = 0, \dots, k-1; x \in X$). Moreover, in all other cases φ'' is given arbitrarily. Since K is not metrically complete \mathfrak{B}_2 satisfies (ii).

Next we show that for arbitrary words $p, q \in X^*$ with $|p| = |q| = k$ and integer i ($1 \leq i \leq s$) there are a v_3 -product

$$\mathfrak{D} = (X, D, \delta'') = \prod_{i=1}^r \mathfrak{C}_i[X, \varphi'', v']$$

($\mathfrak{C}_i = (Y_i, C_i, \delta'_i) \in K, i = 1, \dots, r$) and a state $\mathbf{d} = (d_1, \dots, d_r) \in D$ such that $\mathfrak{C}_r = \mathfrak{A}_i, d_r = a_i, \delta'_r(d_r, \varphi_r''(\mathbf{d}, p)) = \delta_i(a_i, \varphi_i(\mathbf{a}, p))$ and $\delta'_r(d_r, \varphi_r''(\mathbf{d}, q)) = \delta_i(a_i, \varphi_i(\mathbf{a}, q))$. Then taking the direct product of $\mathfrak{B}_1, \mathfrak{B}_2$ and these automata \mathfrak{D} the resulting automaton \mathfrak{B} with a suitable $\mathbf{b} \in B$ will obviously satisfy (i)–(iv).

Since the case

$$(*) \quad \delta_i(a_i, \varphi_i(\mathbf{a}, p)) = \delta_i(a_i, \varphi_i(\mathbf{a}, q))$$

is trivial we may assume that $(*)$ does not hold. Then $p \neq q$. Let $p = x_1 \dots x_m x_{m+1} \dots x_k, q = x_1 \dots x_m y_{m+1} \dots y_k, x_{m+1} \neq y_{m+1}, \varphi_i(\mathbf{a}, p) = \bar{p} = u_1 \dots u_m u_{m+1} \dots u_k$ and $\varphi_i(\mathbf{a}, q) = \bar{q} = u_1 \dots u_m v_{m+1} \dots v_k$. Moreover, set $p_j = x_1 \dots x_j, \bar{p}_j = u_1 \dots u_j$ ($j = 0, 1, \dots, k$) and

$$q_j = \begin{cases} x_1 \dots x_j & \text{if } 0 \leq j \leq m, \\ x_1 \dots x_m y_{m+1} \dots y_j & \text{if } m < j \leq k, \end{cases}$$

$$\bar{q}_j = \begin{cases} u_1 \dots u_j & \text{if } 0 \leq j \leq m, \\ u_1 \dots u_m v_{m+1} \dots v_j & \text{if } m < j \leq k. \end{cases}$$

Denote a_i by c_0 . Let l_1 be the smallest integer u for which there is a v with $u < v \leq k$ such that $\delta_i(c_0, \bar{p}_u) = \delta_i(c_0, \bar{p}_v)$. If there are no such u and v then let $l_1 = k$. Sim-

ilarly, let l_2 be the least integer u such that for some v ($u < v \leq k$), $\delta_i(c_0, \bar{q}_u) = \delta_i(c_0, \bar{q}_v)$. Again if there are no such u and v then let $l_2 = k$. Assume that $l_1 \cong l_2$. Finally, denote by w the maximal number with $\delta_i(c_0, \bar{p}_w) = \delta_i(c_0, \bar{q}_w)$ ($0 \leq w \leq k$). Since $\delta_i(c_0, \bar{p}) \neq \delta_i(c_0, \bar{q})$ the inequality $l_2 > w$ holds. Moreover, $w \cong m$. Let us introduce the notations $\delta_i(c_0, \bar{p}_j) = c_j$ ($j = 0, \dots, l_1$) and $\delta_i(c_0, \bar{q}_j) = c'_j$ ($j = 0, \dots, l_2$). Then the elements $c_0, \dots, c_w, c_{w+1}, c'_{w+1}$ are pairwise distinct, and so are the elements of the sets $\{c_0, \dots, c_{l_1}\}$ and $\{c'_0, \dots, c'_{l_2}\}$. We continue the proof by distinguishing the following two cases.

Case 1. $w = m$. Then let $r = 2$ and $\mathfrak{C}_1 = \mathfrak{C}_2 = \mathfrak{A}_i$. Moreover, $v'(1) = 1$, $v'(2) = \{1, 2\}$ and

$$\varphi_1''(c_j, x) = u_{j+1} \quad (j = 0, \dots, l_1 - 1; x \in X),$$

$$\varphi_2''(c_j, c_j, x_{j+1}) = u_{j+1} \quad (j = 0, \dots, l_1 - 1),$$

$$\varphi_2''(c_j, c'_j, y_{j+1}) = v_{j+1} \quad (j = m, \dots, l_2 - 1).$$

In all other cases φ'' is given arbitrarily. φ'' is well defined. It is obvious that φ_1'' is a function. Assume that $(c_j, c_j, x_{j+1}) = (c'_j, y_{j+1})$ holds for some j ($m < j < l_2$). But this would imply $w > m$.

It is seen immediately that by taking $\mathbf{d} = (c_0, c_0)$ the equalities

$$\delta''(\mathbf{d}, p_j) = (c_j, c_j) \quad (j = 0, \dots, l_1)$$

and

$$\delta''(\mathbf{d}, q_j) = (c_j, c'_j) \quad (j = 0, \dots, l_2)$$

hold. Since K is not metrically complete with respect to the product, by the choice of l_1 and l_2 , this implies

$$\delta''(\mathbf{d}, p) = (c, \delta_i(c_0, \bar{p})) \quad (c \in A_i)$$

and

$$\delta''(\mathbf{d}, q) = (c', \delta_i(c_0, \bar{q})) \quad (c' \in A_i).$$

Case 2. $w > m$. Let $r = w - m + 2$ and $\mathfrak{C}_1 = \dots = \mathfrak{C}_r = \mathfrak{A}_i$. Moreover, $v'(1) = 1$, $v'(j) = j - 1$ ($j = 2, \dots, r - 2$), $v'(r - 1) = r - 1$ and $v'(r) = \{r - 2, r - 1, r\}$. Furthermore,

$$\varphi_1''(c_{w-m+l}, x_{l+1}) = u_{w-m+l+1} \quad (l = 0, \dots, m),$$

$$\varphi_1''(c_w, y_{m+1}) = v_{w+1},$$

$$\varphi_j''(c_{w-m-j+2+l}, x_{l+1}) = u_{w-m-j+2+l} \quad (j = 2, \dots, r - 2; l = 0, \dots, m + j - 1),$$

$$\varphi_j''(c_{w-m-j+2+l}, y_{l+1}) = u_{w-m-j+2+l} \quad (j = 2, \dots, r - 2; l = m, \dots, m + j - 2),$$

$$\varphi_j''(c'_{w+1}, y_{m+j}) = v_{w+1} \quad (j = 2, \dots, r - 2),$$

$$\varphi_{r-1}''(c_l, x_{l+1}) = u_{l+1} \quad (l = 0, \dots, l_1 - 1),$$

$$\varphi_{r-1}''(c_l, y_{l+1}) = u_{l+1} \quad (l = m, \dots, l_2 - 1),$$

$$\varphi_r''(c_{l+1}, c_l, c_l, x_{l+1}) = u_{l+1} \quad (l = 0, \dots, w),$$

$$\varphi_r''(c_{l+1}, c_l, c_l, y_{l+1}) = u_{l+1} \quad (l = m, \dots, w - 1),$$

$$\varphi_r''(c'_{w+1}, c_w, c_w, y_{w+1}) = v_{w+1},$$

$$\varphi_r''(c, c_{w+l}, c_{w+l}, x_{w+l+1}) = u_{w+l+1} \quad (c \in A_i, l = 1, \dots, l_1 - (w + 1)),$$

$$\varphi_r''(c, c_{w+l}, c'_{w+l}, y_{w+l+1}) = v_{w+l+1} \quad (c \in A_i, l = 1, \dots, l_2 - (w + 1)).$$

In all other cases φ'' is given arbitrarily in accordance with the definition of the ν_3 -product. φ'' is well defined. This is clear in all cases except when

$$(c, c_{w+l}, c_{w+l}, x_{w+l+1}) = (c', c_{w+l}, c'_{w+l}, y_{w+l+1})$$

for an l ($1 \leq l \leq l_2 - (w+1)$). But this would contradict the choice of w .

One can easily show by induction on l that for $\mathbf{d} = (c_{w-m}, c_{w-m-1}, \dots, c_1, c_0, c_0)$ the following equalities hold.

$$\delta''(\mathbf{d}, p_l) = (c_{w-m+l}, c_{w-m-1+l}, \dots, c_{1+l}, c_l, c_l) \quad (l = 0, \dots, m),$$

$$\delta''(\mathbf{d}, p_{m+l}) = (c''_1, \dots, c''_{l-1}, c_{w+1}, c_w, \dots, c_{m+l+1}, c_{m+l}, c_{m+l})$$

$$(c''_1, \dots, c''_{l-1} \in A_i; l = 1, \dots, w-m),$$

$$\delta''(\mathbf{d}, q_{m+l}) = (c''_1, \dots, c''_{l-1}, c'_{w+1}, c_w, \dots, c_{m+l+1}, c_{m+l}, c_{m+l})$$

$$(c''_1, \dots, c''_{l-1} \in A_i; l = 1, \dots, w-m),$$

$$\delta''(\mathbf{d}, p_l) = (c''_1, \dots, c''_{r-2}, c_l, c_l) \quad (c''_1, \dots, c''_{r-2} \in A_i; l = w+1, \dots, l_1),$$

$$\delta''(\mathbf{d}, q_l) = (c''_1, \dots, c''_{r-2}, c_l, c'_l) \quad (c''_1, \dots, c''_{r-2} \in A_i; l = w+1, \dots, l_2).$$

Since K is not metrically complete with respect to the product, by the choice of l_1 and l_2 , the last two equalities imply

$$\delta''(\mathbf{d}, p) = (c''_1, \dots, c''_{r-1}, \delta_i(c_0, \bar{p})) \quad \text{and} \quad \delta''(\mathbf{d}, q) = (\bar{c}_1, \dots, \bar{c}_{r-1}, \delta_i(c_0, \bar{q}))$$

$$(c''_1, \dots, c''_{r-1}, \bar{c}_1, \dots, \bar{c}_{r-1} \in A_i)$$

which ends the proof of Theorem 2.

Let us note that the ν_3 -product \mathfrak{B} in the proof of Theorem 2 is also an α_1 -product.

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