

On the expected behaviour of the *NF* algorithm for a dual bin-packing problem

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Introduction

The following version of dual bin-packing problems was first studied by Assmann et al [1]: there is given a list $L = \{a_1, a_2, \dots, a_n\}$ of items (elements) and a size $s(a_i)$ for each item. Let us denote C as a positive constant, $C \cong \max_{1 \leq i \leq n} s(a_i)$. The aim is to pack the elements into a *maximum* number of bins so that the sum of the sizes in any given bin is *at least* C . (The name "dual" originates from the "classical" bin-packing problem, where the elements have to be packed into the *minimum* number of bins in such a way that the sum of the sizes of the elements in any given bin is *at most* C). This problem is *NP-hard* and hence the investigation of the performance of approximation algorithms is important. Without loss of generality, we may assume that $C=1$ and $0 < s(a_i) \leq 1$; if a_i is real, then $s(a_i) = a_i$.

The worst-case behaviour of the well-known heuristic algorithms Next-Fit (*NF*) and Next-Fit Decreasing (*NFD*) was analysed in [1]. The *NF* algorithm places a_i into the first bin (B_1). Let us suppose that a_i , $i > 1$, is to be packed, and let B_j ($j \geq 1$) be the highest indexed non-empty bin. The algorithm places a_i into B_j if the sum of elements in this bin (so far) is smaller than 1; otherwise it closes the bin B_j , opens a new bin (B_{j+1}) and places the element a_i into this newly-opened bin. The *NFD* algorithm differs from *NF* only in preordering the elements. The worst-case behaviour of an approximation algorithm may be characterized by means of the asymptotic worst-case ratio. To define this, let $OPT(L)$ be the maximum possible number of bins for a given instance L . For a given approximation algorithm A , let $A(L)$ denote the number of bins used by A to pack L . Let

$$R_A^N = \min \{A(L)/OPT(L) : L \text{ is an instance with } OPT(L) = N\}.$$

The asymptotic worst-case ratio for A is then defined as

$$R_A^\infty = \liminf_{N \rightarrow \infty} R_A^N$$

Assmann et al proved that

$$R_{NF}^\infty = R_{NFD}^\infty = 1/2$$

In the last part of that paper the average-case behaviour of these (and other) algo-

rithms was investigated. The behaviours of the algorithms were compared in randomly generated instances, where the elements of L were drawn from different distributions. In the conclusion of the paper it was suggested that the expected performance of these algorithms should be examined analytically as well. In [3] this analysis has been carried out for both algorithms. Csirik et al [3] showed that if the elements of $L = (a_1, a_2, \dots, a_n)$ are identically distributed and drawn independently from a uniform distribution on $(0, 1/k]$ (k is a positive integer), then

	$k = 1$	$k = 2$	$k = 3$	$k = 4$
R_{NF}^∞	0.735	0.8564	0.900	0.923
and R_{NFD}^∞	0.710	0.840	0.891	0.918

On the other hand, Knödel [4] showed that the first-fit (FF) algorithm is asymptotically optimal for the “classical” bin-packing problem if the elements of the input list are drawn independently from the following distribution:

$$a_i = \begin{cases} 1/3 & \text{with probability } 1/3, \\ 2/3 & \text{with probability } 1/3, \\ 1 & \text{with probability } 1/3. \end{cases}$$

(In the FF algorithm we try to pack the element a_i into all opened bins, i.e. into B_1, B_2, \dots, B_i , and open a new bin only if none of them has enough room for it.)

Csirik [2] generalized this result for the input sequence:

$$a_i = \begin{cases} b & \text{with probability } 1/2, \\ 1-b & \text{with probability } 1/2, \end{cases} \tag{I}$$

where $0 < b < 1/2$ and it was proved that the FF is asymptotically optimal for these sequences, too.

In this note we investigate the expected behaviour of the NF algorithm at sequence (I) for the dual version of the bin-packing problem.

Results

First we present our method for special lists. Let the elements of $L = (a_1, a_2, \dots, a_n)$ be chosen independently of the following distribution:

$$a_i = \begin{cases} 1/3 & \text{with probability } 1/2, \\ 2/3 & \text{with probability } 1/2. \end{cases} \tag{1}$$

Let us denote by E_n the expected number of full bins for the lists $L = (a_1, a_2, \dots, a_n)$ (the elements are drawn independently from (1)), and by $E_{n,k}$ the expected number of bins for lists with a number k of $2/3$ elements (and so a number $(n-k)$ of $1/3$ elements) if we pack L by NF . Then

$$E_n = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} E_{n,k} \tag{2}$$

On the other hand, in the packing of L by NF the first bin will be full after packing a_1 and a_2 if at least one of them is a $2/3$ element. If both of them are $1/3$ elements, then the first bin is full after the packing of a_3 . Thus, we have the following recursion:

$$E_{n,k} = \frac{k(k-1)}{n(n-1)} (E_{n-2,k-2} + 1) + 2 \frac{k(n-k)}{n(n-1)} (E_{n-2,k-1} + 1) + \frac{(n-k)(n-k-1)}{n(n-1)} \left(\frac{n-k-2}{n-2} E_{n-3,k} + \frac{k}{n-2} E_{n-3,k-1} + 1 \right), \text{ if } n \geq 3 \text{ and } k \geq 1. \quad (3)$$

It is easy to see that $E_{n,0} = \lfloor n/3 \rfloor$, $E_{1,1} = 0$, $E_{2,1} = 1$, $E_{2,2} = 1$.

Using (3), from (2) we get:

$$E_n = \frac{3}{4} E_{n-2} + \frac{1}{4} E_{n-3} + 1 \quad (4)$$

and we know that $E_0 = 0$, $E_1 = 0$, $E_2 = 3/4$.

Let us search E_n in the following form:

$$E_n = \frac{n}{2} (1-A) - B_n \quad (5)$$

Then from (4) we have

$$B_n = \frac{3}{4} B_{n-2} + \frac{1}{4} B_{n-3} + \frac{1}{8} (1-9A) \quad (6)$$

Our aim is to give the asymptotic behaviour of E_n . From (5) it would be enough to choose an appropriate A so that $|B_n| < T$, where T is a constant. If now $A = 1/9$, then from (6)

$$B_n = \frac{3}{4} B_{n-2} + \frac{1}{4} B_{n-3} \quad (7)$$

and from (4) and (5)

$$B_1 = \frac{4}{9}, \quad B_2 = \frac{5}{36}, \quad B_3 = \frac{1}{3}$$

By induction on i , we can prove from (7) that for all $i \geq 4$

$$5/36 \leq B_i \leq 4/9$$

and hence B_i is bounded. But then from (5) we have the following

Lemma.

$$\lim_{n \rightarrow \infty} \frac{E_n}{n/2} = 8/9.$$

We now generalize our result for the following input sequences: let the elements of $L = (a_1, a_2, \dots, a_n)$ be independent, identically distributed, random variables with distribution (I). Let $l_1 = \lceil 1/b \rceil$. We use the notation E_n in the above sense, and let $E_{n,k}$ denote the expected number of bins for the lists $L = (a_1, a_2, \dots, a_n)$ with

a number k of elements $1-b$. Then (2) is true for these sequences as well, and our lemma is valid for $l_1=3$.

In the packing \bar{L} by NF we have two cases:

1. If $a_1=1-b$, then the first bin is always full after the packing of a_2 .

2. If $a_1=b$, then the first bin is full with the first element $1-b$ in the sequel $a_2, a_3, \dots, a_{l_1-1}$. If all of a_2, \dots, a_{l_1-1} are equal b , then the first bin is full after packing of the element a_{l_1} .

Similarly to (3), from these two cases we have

$$\begin{aligned}
 E_{n,k} &= \frac{k(k-1)}{n(n-1)} (E_{n-2,k-2}+1) + 2 \frac{k(n-k)}{n(n-1)} (E_{n-2,k-1}+1) + \\
 &+ \frac{(n-k)(n-k-1)k}{n(n-1)(n-2)} (E_{n-3,k-1}+1) + \dots + \frac{(n-k)(n-k-1) \dots (n-k-l_1+3)k}{n(n-1) \dots (n-l_1+2)} \times \\
 &\times (E_{n-l_1+1,k-1}+1) + \frac{(n-k)(n-k+1) \dots (n-k-l_2+2)}{n(n-1) \dots (n-l_1+2)} \times \\
 &\times \left(\frac{n-k-l_1+1}{n-l_1+1} E_{n-l_1,k} + \frac{k}{n-l_1+1} E_{n-l_1,k-1} + 1 \right) \quad (8)
 \end{aligned}$$

and hence from (2)

$$E_n = \frac{3}{4} E_{n-2} + \frac{1}{2^3} E_{n-3} + \dots + \frac{1}{2^{l_1-1}} E_{n-l_1+1} + \frac{1}{2^{l_1-1}} E_{n-l_1} + 1 \quad (9)$$

We look for E_n again in the form given in (5). Then

$$\begin{aligned}
 B_n &= \frac{3}{4} B_{n-2} + \frac{1}{2^3} B_{n-3} + \dots + \frac{1}{2^{l_1-1}} B_{n-l_1+1} + \frac{1}{2^{l_1-1}} B_{n-l_1} + \\
 &+ (1-A) \left(\frac{n}{2} \left(1 - \frac{3}{4} - \frac{1}{2^3} - \dots - \frac{1}{2^{l_1-1}} - \frac{1}{2^{l_1-1}} \right) \right) + \\
 &+ (1-A) \left(\frac{3}{4} \cdot \frac{2}{2} + \frac{1}{2^3} \cdot \frac{3}{2} + \frac{1}{2^4} \cdot \frac{4}{2} + \dots + \frac{1}{2^{l_1-1}} \frac{l_1-1}{2} + \frac{1}{2^{l_1-1}} \frac{1}{2} \right) - 1 = \\
 &= \frac{3}{4} B_{n-2} + \frac{1}{2^3} B_{n-3} + \dots + \frac{1}{2^{l_1-1}} B_{n-l_1+1} + \frac{1}{2^{l_1-1}} B_{n-l_1} + \frac{2^{l_1-2}-1}{2^{l_1}} - A \frac{2^{l_1}+2^{l_1-2}-1}{2^{l_1}}. \quad (10)
 \end{aligned}$$

If we now choose

$$A = \frac{2^{l_1-2}-1}{2^{l_1}+2^{l_1-2}-1}$$

then

$$B_n = \frac{3}{4} B_{n-2} + \frac{1}{2^3} B_{n-3} + \dots + \frac{1}{2^{l_1-1}} B_{n-l_1+1} + \frac{1}{2^{l_1-1}} B_{n-l_1}$$

and thus B_n is again a bounded sequence. Accordingly, we have proved our main result:

Theorem. Let the elements of $L=(a_1, a_2, \dots, a_n)$ be independent, identically distributed, random variables with distribution

$$a_i = \begin{cases} b & \text{with probability } 1/2, \\ 1-b & \text{with probability } 1/2, \end{cases}$$

where $0 < b < 1/2$. Let $l_1 = \lceil 1/b \rceil$. If we pack the list L by the NF algorithm and if E_n denotes the expected number of filled bins, then

$$\lim_{n \rightarrow \infty} \frac{E_n}{n/2} = \frac{2^{l_1}}{2^{l_1} + 2^{l_1-2} - 1}$$

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