

# On $\alpha_r$ -product of tree automata

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In the theory of finite automata it is a central problem to represent a given automaton by composition of — possibly simpler — automata. The composition of tree automata has received little attention. Namely, the cascade product of tree automata was studied in [4] and the work [5] contains the investigation of the general product of tree automata (see also [1]). In this paper generalizing the notion of  $\alpha_r$ -product (cf. [2]), we introduce the  $\alpha_r$ -product of tree automata, and using the idea in [3] give necessary and sufficient conditions for a system of tree automata to be isomorphically complete with respect to the  $\alpha_r$ -product. From the characterizations of complete systems we obtain the  $\alpha_r$ -products constitute a proper hierarchy.

## 1. Definitions

By a *set of operational symbols* we mean the nonempty union  $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \dots$  of pairwise disjoint sets of symbols, and for any nonnegative integer  $m$ ,  $\Sigma_m$  is called the *set of  $m$ -ary operational symbols*. It is said that the *rank* or *arity* of a symbol  $\sigma \in \Sigma$  is  $m$  if  $\sigma \in \Sigma_m$ . Now let a set  $\Sigma$  of operational symbols be given. A set  $R$  of nonnegative integers is called the *rank-type* of  $\Sigma$  if for any  $m$ ,  $\Sigma_m \neq \emptyset$  if and only if  $m \in R$ . Next we shall work always under a fixed rank-type  $R$ .

Let  $\Sigma$  be a set of operational symbols with rank-type  $R$ . Then by a  $\Sigma$ -algebra  $\mathcal{A}$  we mean a pair consisting of a nonempty set  $A$  (of elements of  $\mathcal{A}$ ) and a mapping that assigns to every operational symbol  $\sigma \in \Sigma$  an  $m$ -ary operation  $\sigma^{\mathcal{A}}: A^m \rightarrow A$ , where the arity of  $\sigma$  is  $m$ . The operation  $\sigma^{\mathcal{A}}$  is called the *realization of  $\sigma$  in  $\mathcal{A}$* . The mapping  $\sigma \rightarrow \sigma^{\mathcal{A}}$  will not be mentioned explicitly, but we write  $\mathcal{A} = (A, \Sigma)$ . The  $\Sigma$ -algebra  $\mathcal{A}$  is *finite* if  $A$  is finite, and it is of *finite type* if  $\Sigma$  is finite. By a *tree automaton* we mean a finite algebra of finite type. We say that the rank-type of a tree automaton  $\mathcal{A} = (A, \Sigma)$  is  $R$  if the rank-type of  $\Sigma$  is  $R$ . Let us denote by  $\mathfrak{A}_R$  the class of all tree automata with rank-type  $R$ .

Now let  $i$  be a fixed nonnegative integer, and let

$$\mathcal{A} = (A, \Sigma) \in \mathfrak{A}_R, \quad \mathcal{A}_j = (A_j, \Sigma^j) \in \mathfrak{A}_R \quad (j = 1, \dots, k).$$

Moreover, take a family  $\psi$  of mappings

$$\psi_{m,j}: (A_1 \times \dots \times A_k)^m \times \Sigma_m \rightarrow \Sigma_m^j, \quad m \in R, \quad 1 \leq j \leq k.$$

It is said that the tree automaton  $\mathcal{A}$  is the  $\alpha_r$ -product of  $\mathcal{A}_j$  ( $j=1, \dots, k$ ) with respect to  $\psi$  if the following conditions are satisfied:

$$(1) \quad A = \prod_{i=1}^k A_i,$$

$$(2) \quad \text{for any } m \in R, j \in \{1, \dots, k\},$$

$$((a_{11}, \dots, a_{1k}), \dots, (a_{m1}, \dots, a_{mk})) \in (A_1 \times \dots \times A_k)^m$$

the mapping  $\psi_{mj}$  is independent of elements  $a_{rs}$  ( $1 \leq r \leq m, j+i \leq s$ ),

$$(3) \quad \text{for any } m \in R, \sigma \in \Sigma_m, ((a_{11}, \dots, a_{1k}), \dots, (a_{m1}, \dots, a_{mk})) \in (A_1 \times \dots \times A_k)^m,$$

$$\sigma^{\mathcal{A}}((a_{11}, \dots, a_{1k}), \dots, (a_{m1}, \dots, a_{mk})) = (\sigma_1^{\mathcal{A}_1}(a_{11}, \dots, a_{m1}), \dots, \sigma_k^{\mathcal{A}_k}(a_{1k}, \dots, a_{mk})),$$

where

$$\sigma_j = \psi_{mj}((a_{11}, \dots, a_{1k}), \dots, (a_{m1}, \dots, a_{mk}), \sigma) \quad (j = 1, \dots, k).$$

For the above product we shall use the notation  $\prod_{j=1}^k \mathcal{A}_j(\Sigma, \psi)$  and sometimes we shall write only those variables of  $\psi_{mj}$  on which  $\psi_{mj}$  depends.

Finally, we shall denote by  $\lceil \sqrt[n]{n} \rceil$  the largest integer less than or equal to  $\sqrt[n]{n}$ .

## 2. Completeness

Let  $i$  be a fixed nonnegative integer and  $\mathfrak{B} \subseteq \mathfrak{A}_R$ .  $\mathfrak{B}$  is called *isomorphically complete* for  $\mathfrak{A}_R$  with respect to the  $\alpha_i$ -product if any tree automaton from  $\mathfrak{A}_R$  can be embedded isomorphically into an  $\alpha_i$ -product of tree automata from  $\mathfrak{B}$ . Furthermore,  $\mathfrak{B}$  is called *minimal isomorphically complete system* if  $\mathfrak{B}$  is isomorphically complete and for arbitrary  $\mathcal{A} \in \mathfrak{B}$ ,  $\mathfrak{B} \setminus \{\mathcal{A}\}$  is not isomorphically complete.

For any natural number  $n > 0$  let us denote by  $\mathcal{B}_n = (\{0, \dots, n-1\}, \theta^n)$  the tree automaton where for every  $m$ -ary operation  $q: \{0, \dots, n-1\}^m \rightarrow \{0, \dots, n-1\}$  there exists exactly one  $\sigma \in \theta_m^n$  with  $\sigma^{\mathcal{B}_n} = q$  provided that  $m \in R$ .

The following statement is obvious.

**Lemma.** *If  $\mathcal{A}_j \in \mathfrak{A}_R$  ( $j=1, 2, 3$ ) and  $\mathcal{A}_j$  can be embedded isomorphically into and  $\alpha_r$ -product of  $\mathcal{A}_{j+1}$  with a single factor ( $j=1, 2$ ) then  $\mathcal{A}_1$  can be embedded isomorphically into an  $\alpha_r$ -product of  $\mathcal{A}_3$  with a single factor.*

First we consider the special case  $R = \{0\}$ . Then the following statement is obvious.

**Theorem 1.**  $\mathfrak{B} \subseteq \mathfrak{A}_R$  is isomorphically complete for  $\mathfrak{A}_R$  with respect to the  $\alpha_r$ -product if and only if there exists an  $\mathcal{A} \in \mathfrak{B}$  such that  $\mathcal{B}_2$  can be embedded isomorphically into an  $\alpha_r$ -product of  $\mathcal{A}$  with a single factor.

Now let us suppose  $R \neq \{0\}$ . Then the results of completeness is based on the following Theorem.

**Theorem 2.** If the tree automaton  $\mathcal{B}_n$  ( $n > 1$ ) can be embedded isomorphically

into an  $\alpha_i$ -product  $\prod_{j=1}^k \mathcal{A}_j(\theta^n, \psi)$  of the tree automata  $\mathcal{A}_j \in \mathfrak{A}_R$  ( $j=1, \dots, k$ ) then  $\mathcal{B}_{[i^*, \sqrt{n}]}$  can be embedded isomorphically into an  $\alpha_i$ -product of  $\mathcal{A}_j$  with a single factor for some  $j \in \{1, \dots, k\}$ , where  $i^*=i$  if  $i>0$  and  $i^*=1$  else.

*Proof.* If  $k=1$  then the statement is obvious. Now let  $k>1$ . Assume that  $\mathcal{B}_n$  can be embedded isomorphically into the  $\alpha_i$ -product  $\mathcal{A} = \prod_{j=1}^k \mathcal{A}_j(\theta^n, \psi)$  and let  $\mu$  denote a suitable isomorphism. Let  $\mu(t) = (a_{t1}, \dots, a_{tk})$  ( $t=0, \dots, n-1$ ). We may suppose that there exist natural numbers  $u \neq v$  ( $0 \leq u, v \leq n-1$ ) such that  $a_{u1} \neq a_{v1}$  since otherwise  $\mathcal{B}_n$  can be embedded isomorphically into an  $\alpha_i$ -product of  $\mathcal{A}_j$  ( $j=2, \dots, k$ ). Now assume that there exist natural numbers  $p \neq q$  ( $0 \leq p, q \leq n-1$ ) with  $a_{ps} = a_{qs}$  ( $s=1, \dots, i^*$ ). For any  $t$  ( $0 \leq t \leq n-1$ ) let us denote by  $\sigma_{pt}^{\mathcal{B}_n}$  the  $m$ -ary operation of  $\mathcal{B}_n$  for which  $\sigma_{pt}^{\mathcal{B}_n}(0, \dots, 0, p) = t$  and  $\sigma_{pt}^{\mathcal{B}_n}(0, \dots, 0, q) = q$ , for some  $m \in R$ . Such operations exist since  $R \neq \{0\}$ . Then for any  $t \in \{0, \dots, n-1\}$

$$(a_{t1}, \dots, a_{tk}) = \mu(t) = \mu(\sigma_{pt}^{\mathcal{B}_n}(0, \dots, 0, p)) = \sigma_{pt}^{\mathcal{A}}(\mu(0), \dots, \mu(0), \mu(p)) =$$

$$= (\sigma_1^{\mathcal{A}^1}(a_{01}, \dots, a_{01}, a_{p1}), \sigma_2^{\mathcal{A}^2}(a_{02}, \dots, a_{02}, a_{p2}), \dots, \sigma_k^{\mathcal{A}^k}(a_{0k}, \dots, a_{0k}, a_{pk}))$$

holds, and so  $a_{t1} = \sigma_1^{\mathcal{A}^1}(a_{01}, \dots, a_{01}, a_{p1})$  where

$$\sigma_1 = \psi_{m1}((a_{01}, \dots, a_{0k}), \dots, (a_{01}, \dots, a_{0k}), (a_{p1}, \dots, a_{pk}), \sigma_{pt}) =$$

$$= \psi_{m1}(a_{01}, \dots, a_{0i^*}, a_{p1}, \dots, a_{pi^*}, \sigma_{pt}) \quad \text{if } i > 0$$

and  $\sigma_1 = \psi_{m1}(\sigma_{pt})$  if  $i=0$ . In the same way we obtain the equality

$$a_{q1} = \bar{\sigma}_1^{\mathcal{A}^1}(a_{01}, \dots, a_{01}, a_{q1})$$

where

$$\bar{\sigma}_1 = \psi_{m1}(a_{01}, \dots, a_{0i^*}, a_{q1}, \dots, a_{qi^*}, \sigma_{pt}) \quad \text{if } i > 0$$

and

$$\bar{\sigma}_1 = \psi_{m1}(\sigma_{pt}) \quad \text{if } i = 0.$$

Since  $a_{ps} = a_{qs}$  ( $s=1, \dots, i^*$ ) we obtain that  $\sigma_1 = \bar{\sigma}_1$  which implies the equality  $a_{t1} = a_{q1}$  for any  $t \in \{0, \dots, n-1\}$ . This contradicts our assumption  $a_{u1} \neq a_{v1}$ , therefore the elements  $(a_{t1}, \dots, a_{ti^*})$  ( $0 \leq t \leq n-1$ ) are pairwise different. Now we shall show that in this case  $\mathcal{B}_n$  can be embedded isomorphically into an  $\alpha_i$ -product

$\bar{\mathcal{A}} = \prod_{j=1}^{i^*} \mathcal{A}_j(\theta^n, \varphi)$ . Indeed, let us define the family  $\varphi$  of mappings as follows: for

any  $m \in R, j \in \{1, \dots, i^*\}, ((a_1^1, \dots, a_1^{i^*}), \dots, (a_m^1, \dots, a_m^{i^*})) \in \prod_{j=1}^{i^*} A_j, \sigma \in \theta^n$  elements

(1) if  $i>0$  then

$$\varphi_{mj}((a_1^1, \dots, a_1^{i^*}), \dots, (a_m^1, \dots, a_m^{i^*}), \sigma) = \begin{cases} \psi_{mj}((a_{u_1 1}, \dots, a_{u_1 k}), \dots, (a_{u_m 1}, \dots, a_{u_m k}), \sigma) \\ \text{if there exist } u_1, \dots, u_m \in \{0, \dots, n-1\} \\ \text{such that } a_t^s = a_{u_s t} \text{ (} t=1, \dots, i^*, s=1, \dots, m\text{),} \\ \text{arbitrary operational symbol from} \\ \Sigma_m^j \text{ otherwise,} \end{cases}$$

(2) if  $i=0$  then  $\varphi_{mj}(\sigma) = \psi_{mj}(\sigma)$ .

It is clear that  $\varphi_{mj}$  is well defined. On the other hand, it is easy to see that the mapping  $\nu(t) = (a_{t1}, \dots, a_{ti^*})$  ( $t=0, \dots, n-1$ ) is an isomorphism of  $\mathcal{B}_n$  into  $\overline{\mathcal{A}}$ . Using this isomorphism  $\nu$  we prove that  $\mathcal{B}_{\lceil i^* \sqrt{n} \rceil}$  can be embedded isomorphically into an  $\alpha_i$ -product of  $\mathcal{A}_j$  with a single factor for some  $j \in \{1, \dots, i^*\}$ . If  $i=0$  or  $i=1$  then this statement obviously holds. Now assume that  $i > 1$ . Since the elements  $(a_{t1}, \dots, a_{ti^*})$  ( $t=0, \dots, n-1$ ) are pairwise different, there exists an  $s \in \{1, \dots, i^*\}$  such that the number of pairwise different elements among  $a_{0s}, a_{1s}, \dots, a_{n-1s}$  is greater than or equal to  $v = \lceil i^* \sqrt{n} \rceil$ . Without loss of generality we may assume that  $a_{0s}, \dots, a_{v-1s}$  are pairwise different elements of  $\mathcal{A}_s$ . For any  $m \in R$ ,  $\sigma \in \theta_m^v$  let us denote by  $\bar{\sigma}$  an operational symbol from  $\theta_m^n$  for which  $\sigma^{\mathcal{B}_n|_{0, \dots, v-1}^m} = \bar{\sigma}^{\mathcal{B}_v}$ . Now let us define the  $\alpha_i$ -product  $\mathcal{A}_s(\theta^v, \bar{\varphi})$  as follows: for any  $m \in R$ ,  $\sigma \in \theta_m^v$ ,  $(a_{u_1s}, \dots, a_{u_ms}) \in A_m^n$

$$\bar{\varphi}_m(a_{u_1s}, \dots, a_{u_ms}, \sigma) = \begin{cases} \varphi_{ms}((a_{u_11}, \dots, a_{u_1i^*}), \dots, (a_{u_m1}, \dots, a_{u_mi^*}), \bar{\sigma}) & \text{if} \\ 0 \leq u_t \leq v-1 \ (t=1, \dots, m), & \\ \text{arbitrary operational symbol from } \Sigma_m^s \text{ otherwise.} & \end{cases}$$

It can be easily see that the correspondence  $\nu': t \rightarrow a_{ts}$  ( $t=0, \dots, v-1$ ) is an isomorphism of  $\mathcal{B}_v$  into  $\mathcal{A}_s(\theta^v, \bar{\varphi})$ , which completes the proof of Theorem 2.

**Theorem 3.**  $\mathcal{B} \subseteq \mathcal{U}_R$  is isomorphically complete for  $\mathcal{U}_R$  with respect to the  $\alpha_0$ -product if and only if for any natural number  $n > 1$  there exists an  $\mathcal{A} \in \mathcal{B}$  such that  $\mathcal{B}_n$  can be embedded isomorphically into an  $\alpha_0$ -product of  $\mathcal{A}$  with a single factor.

*Proof.* The necessity follows from Theorem 2. To prove the sufficiency let us observe that any tree automaton  $\mathcal{A} \in \mathcal{U}_R$  with  $|A|=n$  can be embedded isomorphically into an  $\alpha_0$ -product of  $\mathcal{B}_n$  with a single factor. From this fact, by our Lemma, we obtain the completeness of  $\mathcal{B}$ .

Now let  $i > 0$  be a fixed nonnegative integer. Then in a similar way as above we obtain the following result.

**Theorem 4.**  $\mathcal{B} \subseteq \mathcal{U}_R$  is isomorphically complete for  $\mathcal{U}_R$  with respect to the  $\alpha_i$ -product if and only if for any natural number  $n > 1$  there exists an  $\mathcal{A} \in \mathcal{B}$  such that  $\mathcal{B}_n$  can be embedded isomorphically into an  $\alpha_i$ -product of  $\mathcal{A}$  with a single factor.

Since an  $\alpha_i$ -product with a single factor is an  $\alpha_1$ -product with a single factor, by Theorem 4, we get the next corollary.

**Corollary 1.**  $\mathcal{B} \subseteq \mathcal{U}_R$  is isomorphically complete for  $\mathcal{U}_R$  with respect to the  $\alpha_1$ -product if and only if  $\mathcal{B}$  is isomorphically complete for  $\mathcal{U}_R$  with respect to the  $\alpha_i$ -product.

Now let  $i$  be a nonnegative integer. Then we have the following result for the minimal isomorphically complete systems in the case  $R \neq \{0\}$ .

**Theorem 5.** There exists no system  $\mathcal{B} \subseteq \mathcal{U}_R$  which is isomorphically complete for  $\mathcal{U}_R$  with respect to the  $\alpha_i$ -product and minimal.

*Proof.* Let  $\mathfrak{B} \subseteq \mathfrak{A}_R$  be isomorphically complete for  $\mathfrak{A}_R$  with respect to the  $\alpha_i$ -product. Moreover, let  $\mathcal{A} \in \mathfrak{B}$  with  $|\mathcal{A}|=n$ . It is obvious that  $\mathcal{A}$  can be embedded isomorphically into an  $\alpha_i$ -product of  $\mathfrak{B}_s$  with a single factor if  $s \geq n$ . Take a natural number  $s > n$ . By Theorem 3 and Theorem 4, there exists an  $\overline{\mathcal{A}} \in \mathfrak{B}$  such that  $\mathfrak{B}_s$  can be embedded isomorphically into an  $\alpha_i$ -product of  $\overline{\mathcal{A}}$  with a single factor. Therefore, by our Lemma,  $\mathcal{A}$  can be embedded isomorphically into an  $\alpha_i$ -product of  $\overline{\mathcal{A}}$  with a single factor. From this it follows that  $\mathfrak{B} \setminus \{\mathcal{A}\}$  is isomorphically complete for  $\mathfrak{A}_R$  with respect to the  $\alpha_i$ -product, showing that  $\mathfrak{B}$  is not minimal.

### 3. The hierarchy of $\alpha_i$ -products

Let  $R \neq \{0\}$  be a fixed rank-type. Take a nonempty set  $M \subseteq \mathfrak{A}_R$ , and let  $i$  be an arbitrary nonnegative integer. Let  $\alpha_i(M)$  denote the class of all tree automata from  $\mathfrak{A}_R$  which can be embedded isomorphically into an  $\alpha_i$ -product of tree automata from  $M$ . It is said that the  $\alpha_i$ -product is *isomorphically more general* than the  $\alpha_j$ -product if for any set  $M \subseteq \mathfrak{A}_R$  the relation  $\alpha_j(M) \subseteq \alpha_i(M)$  holds and there exists at least one set  $\overline{M} \subseteq \mathfrak{A}_R$  such that  $\alpha_j(\overline{M})$  is a proper subclass of  $\alpha_i(\overline{M})$ . This notion was introduced in [2].

As far as the hierarchy of the  $\alpha_i$ -products is concerned, we have the following Theorem.

**Theorem 6.** For any  $i, j$  ( $i, j \in \{0, 1, \dots\}$ ) the  $\alpha_i$ -product is isomorphically more general than the  $\alpha_j$  product if  $j < i$ .

*Proof.* We shall prove that the  $\alpha_1$ -product is isomorphically more general than the  $\alpha_0$ -product and the  $\alpha_{i+1}$ -product is isomorphically more general than the  $\alpha_i$ -product if  $i \geq 1$ .

First let  $M = \{\mathcal{A}_2\}$ , where  $\mathcal{A}_2 = (\{1, 2\}, \bigcup_{m \in R} \{\sigma_{m1}, \sigma_{m2}\})$  and the operations of  $\mathcal{A}_2$  are defined as follows: for any  $0 \neq m, m \in R, (a_1, \dots, a_m) \in \{1, 2\}^m$

$$\sigma_{m1}^{\mathcal{A}_2}(a_1, \dots, a_m) = \begin{cases} 1 & \text{if } a_m = 2, \\ 2 & \text{if } a_m = 1, \end{cases}$$

$$\sigma_{m2}^{\mathcal{A}_2}(a_1, \dots, a_m) = a_m,$$

and  $\sigma_{01}^{\mathcal{A}_2} = 1, \sigma_{02}^{\mathcal{A}_2} = 2$  if  $0 \in R$ .

Now let us denote by  $\mathcal{A}_3 = (\{1, 2, 3\}, \Sigma')$  the tree automaton where for any  $0 \neq m \in R, \sigma \in \Sigma'_m, (a_1, \dots, a_m) \in \{1, 2, 3\}^m$

$$\sigma^{\mathcal{A}_3}(a_1, \dots, a_m) = \begin{cases} a_m + 1 & \text{if } a_m < 3, \\ 3 & \text{if } a_m = 3, \end{cases}$$

and  $\bar{\sigma}^{\mathcal{A}_3} = 1$  if  $0 \in R$  and  $\bar{\sigma} \in \Sigma'_0$ .

It is easy to see that  $\mathcal{A}_3 \notin \alpha_0(M)$  and  $\mathcal{A}_2 \in \alpha_1(M)$  which yields the required inclusion  $\alpha_0(M) \subset \alpha_1(M)$ .

Now let  $i \geq 1$  and  $M = \{\mathcal{B}_2\}$ . Then, by the proof of Theorem 2, we obtain that  $\mathcal{B}_{2^{i+1}} \notin \alpha_i(M)$ . On the other hand, we shall show that  $\mathcal{B}_{2^{i+1}} \in \alpha_{i+1}(M)$  which yields the required inclusion  $\alpha_i(M) \subset \alpha_{i+1}(M)$ . To prove the above statement it is enough to show that  $\mathcal{B}_{2^i} \in \alpha_i(M)$  if  $i > 1$ . Indeed, let us take the  $\alpha_i$ -product

$\mathcal{A} = \prod_{j=1}^i \mathcal{B}_2(\theta^{2^j}, \psi)$  where the family  $\psi$  of mappings is defined as follows: for any  $0 \neq m, \sigma \in \theta_m^{2^i}$ ,

$$((a_{11}, \dots, a_{1i}), \dots, (a_{m1}, \dots, a_{mi})) \in (\{0, 1\}^i)^m$$

if

$$\sigma^{2^i} \left( \sum_{t=1}^i a_{1t} 2^{i-t}, \dots, \sum_{t=1}^i a_{mt} 2^{i-t} \right) = w = \sum_{t=1}^i a_{wt} 2^{i-t} \quad \text{and} \quad \bar{\sigma}^{2^i}(a_{1j}, \dots, a_{mj}) = a_{wj}$$

then

$$\psi_{mj}((a_{11}, \dots, a_{1i}), \dots, (a_{m1}, \dots, a_{mi}), \sigma) = \bar{\sigma}.$$

In the case  $\sigma \in \theta_0^{2^i}$  if  $\sigma^{2^i} = \sum_{t=1}^i a_{vt} \cdot 2^{i-t}$  and  $\bar{\sigma}^{2^i} = a_{vj}$  then  $\psi_{mj}(\sigma) = \bar{\sigma}$ .

It is easy to see that  $\mathcal{B}_{2^i}$  can be embedded isomorphically into  $\mathcal{A}$  under the isomorphism  $\mu$  defined as follows: if  $w = \sum_{t=1}^i a_t 2^{i-t}$  then  $\mu(w) = (a_1, \dots, a_i)$  ( $w = 0, \dots, 2^i - 1$ ).

#### 4. A decidability result

In this section we show that it is decidable if an algebra can be represented isomorphically by an  $\alpha_i$ -product of algebras from a given finite set.

**Theorem 7.** For any nonnegative integer  $i$ ,  $\mathcal{A} \in \mathfrak{M}_R$  and finite set  $M \subseteq \mathfrak{M}_R$  it can be decided whether or not  $\mathcal{A} \in \alpha_i(M)$ .

*Proof.* Let us suppose that  $\mathcal{A}$  with  $A = \{a_1, \dots, a_k\}$  can be embedded isomorphically into an  $\alpha_i$ -product  $\mathcal{B} = \prod_{j=1}^s \mathcal{A}_j(\Sigma, \varphi)$  of tree automata from  $M$ . Let  $V = \max \{|A_i| : \mathcal{A}_i \in M\}$  and let  $(a_{u1}, \dots, a_{us})$  denote the image of  $a_u$  under a suitable isomorphism  $\mu$  ( $u = 1, \dots, k$ ). We define an equivalence relation  $\pi$  on the set of indices of the  $\alpha_i$ -product  $\mathcal{B}$  as follows: for any  $l, n$  ( $1 \leq l, n \leq s$ ),  $l \pi n$  holds if and only if  $\mathcal{A}_l = \mathcal{A}_n$  and  $a_{il} = a_{in}$  for all  $i = 1, \dots, k$ .

It is easy to see that the partition corresponding to  $\pi$  has at most  $|M| \cdot V^k$  blocks. Since  $\mu(A)$  is a subalgebra of  $\mathcal{B}$ , if  $a_{il} = a_{in}$  ( $i = 1, \dots, k$ ) then the  $l$ -th and  $n$ -th components of  $\mu(\sigma(a^1, \dots, a^m))$  are equal, where  $m \in R$ ,  $\sigma \in \Sigma_m$ ,  $a^j \in A$  ( $j = 1, \dots, m$ ). From this it follows that  $\mathcal{A}$  can be embedded isomorphically into an  $\alpha_i$ -product of tree automata from  $M$  with at most  $|M| \cdot V^k$  factors.

## References

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