On star-products of automata

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The study of complete systems of automata was initiated by V. M. Gluškov in [3]. In this work he characterized isomorphically complete systems with respect to the Gluškov-type product. Further characterizations of isomoprhically complete systems with respect to different kinds of products were presented in the works [1], [2] and [5]. In this paper we deal with star-products which have been deeply investigated in [6] and [7], and study isomorphic completeness for this kind of products. It will turn out that there exists no finite isomorphically complete system, however, as shown in [6], there are finite isomorphically S-complete systems with respect to it.

1. Definitions

By an *automaton* we mean a system $\mathbf{A}=(X, A, \delta)$, where A and X are finite nonvoid sets, and $\delta\colon A\times X^*\to A$ is the transition function. (Here and in the sequel X^* denotes the free monoid generated by X.) The concepts of *subautomaton* and *isomorphism* will be used in the usual sense.

Let $A_t=(X_t, A_t, \delta_t)$ $(t=1, \dots, k)$ be a system of automata. Moreover, let X be a finite nonvoid set and φ a mapping of $A_1\times\dots\times A_k\times X$ into $X_1\times\dots\times X_k$ such that φ can be given in the form

$$\varphi(a_1, \dots, a_k, x) = (\varphi_1(a_1, \dots, a_k, x), \varphi_2(a_1, a_2, x), \dots, \varphi_k(a_1, a_k, x)).$$

We say that

$$\mathbf{A} = (X, A, \delta)$$

is a *star-product* of $\mathbf{A}_t$ $(t=1, \dots, k)$ with respect to X and φ if $A=A_1\times\dots\times A_k$ and for arbitrary $(a_1, \dots, a_k)\in A$ and $x\in X$

$$\delta((a_1, \dots, a_k), x) = (\delta_1(a_1, \varphi_1(a_1, \dots, a_k, x)), \delta_2(a_2, \varphi_2(a_1, a_2, x)), \dots,$$
$$\dots, \delta_k(a_k, \varphi_k(a_1, a_k, x))).$$

For this product we use the notation

$$\prod_{t=1}^{k} \mathbf{A}_t(X, \varphi).$$

As regards the introduced composition, let us observe the following: if the product-automaton is in the state $(a_1, \ldots, a_k)$ and receives an input sign x, then the automaton $\mathbf{A}_1$ receives the input sign $x_1 = \varphi_1(a_1, \ldots, a_k, x)$ which depends on x and all the actual states, and for every index $2 \leqq j \leqq k$ the automaton $\mathbf{A}_j$ receives the input sign $x_j = \varphi_j(a_1, a_j, x)$ which depends on the actual states a_1, a_j and x. Therefore, at a given moment the working of $\mathbf{A}_1$ depends on all component automata, while the working of $\mathbf{A}_j$ $(2 \leqq j \leqq k)$ depends on $\mathbf{A}_1$ and $\mathbf{A}_j$ only. This connection can be realized if the automaton $\mathbf{A}_1$ is placed in the centre and it is connected directly to each $\mathbf{A}_j$ $(2 \leqq j \leqq k)$, as illustrated in Fig. 1. This network of automata corresponds to the simplest computer network.

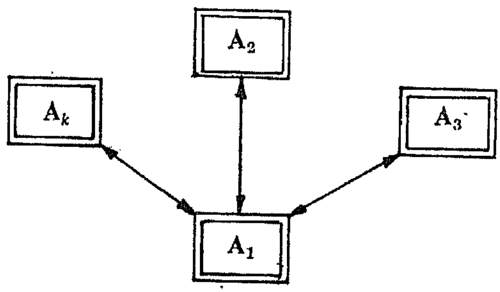

Fig. 1. Schematic diagram for the star-product of $\mathbf{A}_1, \ldots, \mathbf{A}_k$

2. Isomorphic realization

Let Σ be a system of automata. Σ is called *isomorphically comptete* with respect to the star-product if every automaton can be embedded isomorphically into a star-product of automata from Σ. Furthermore, Σ is a *minimal* isomorphically complete system if Σ is isomorphically complete and for arbitrary $\mathbf{A} \in \Sigma$, the system $\Sigma \setminus \{\mathbf{A}\}$ is not isomorphically complete.

For arbitrary positive integer n, let us denote by

$$\mathbf{D}_n = (\{x_{rs} \colon 1 \leqq r, s \leqq n\}, \{1, \ldots, n\}, \delta_n)$$

the automaton, where δ_n is determined in the following way: for arbitrary $i \in \{1, \ldots, n\}$ and input sign x_{rs} $(1 \leqq r, s \leqq n)$,

$$\delta_n(i, x_{rs}) = \begin{cases} s & \text{if } \quad i = r, \\ i & \text{otherwise.} \end{cases}$$

Now we present a necessary and sufficient condition for the isomorphic completeness.

Theorem 1. A system Σ of automata is isomorphically complete with respect to the star-product if and only if for every positive integer n, there exists an automaton $\mathbf{A}\in\Sigma$ such that $\mathbf{D}_n$ can be embedded isomorphically into a star-product of $\mathbf{A}$ with a single factor.

Proof. First we show that $\mathbf{D}_n$ $(n>1)$ can be embedded isomorphically into a star-product of automata from Σ with at most two factors if $\mathbf{D}_n$ can be embedded isomorphically into a star-product of automata from Σ. For this, suppose that $\mathbf{D}_n$ can be embedded isomorphically into the star-product

$$\prod_{t=1}^{k} \mathbf{A}_t(\{x_{rs}\colon\ 1\leqq r, s\leqq n\}, \varphi),$$

where $\mathbf{A}_t\in\Sigma$ $(t=1, \ldots, k)$ and $k>2$. Let us denote by μ such an isomorphism, and for arbitrary $i\in\{1, \ldots, n\}$ let $(a_{i1}, \ldots, a_{ik})$ be the image of i under μ. Now take an m $(2\leqq m\leqq k)$, and assume that $a_{i1}=a_{j1}$ and $a_{im}=a_{jm}$ hold for some indices $i\neq j$ $(1\leqq i, j\leqq n)$. Moreover, let $v\in\{1, \ldots, n\}$ be arbitrary. Then $\delta_n(i, x_{iv})=v$, $\delta_n(j, x_{iv})=j$, and since μ is an isomorphism, we obtain

$$\delta_m(a_{im}, \varphi_m(a_{i1}, a_{im}, x_{iv}))=a_{vm},$$
$$\delta_m(a_{jm}, \varphi_m(a_{j1}, a_{jm}, x_{iv}))=a_{jm}.$$

From this, by our assumption $a_{i1}=a_{j1}$ and $a_{im}=a_{jm}$, it follows that $a_{vm}=a_{jm}$. Since v is arbitrary, $a_{jm}=a_{vm}$ $(v=1, \ldots, n)$. Therefore, there is an index $(2\leqq m\leqq k)$ such that the pairs (a_{i1}, a_{im}) $(i=1, \ldots, n)$ are pairwise different. But then the automaton $\mathbf{D}_n$ can be embedded isomorphically into a star-product of $\mathbf{A}_1$ and $\mathbf{A}_m$, which yields the validity of our statement.

Now in order to prove the necessity, let us assume that Σ is isomoprhically complete with respect to the star-product. Let n be an arbitrary positive integer. The case $n=1$ being obvious, we may assume that $n>1$. Let $w=n^2$. Since Σ is isomorphically complete, $\mathbf{D}_w$ can be embedded isomorphically into a star-product

$$\prod_{t=1}^{k} \mathbf{A}_t(\{x_{rs}\colon\ 1\leqq r, s\leqq w\}, \varphi)$$

of automata from Σ. From this, by the above assertion, it follows that $\mathbf{D}_w$ can be embedded isomorphically into a star-product of $\mathbf{A}_1$ and $\mathbf{A}_m$ for some $2\leqq m\leqq k$. But in this case it is easy to see that $\mathbf{D}_n$ can be embedded isomorphically into a star-product of one of the automata $\mathbf{A}_1$ and $\mathbf{A}_m$ with a single factor, which results the necessity of the condition.

To prove the sufficiency, it is enough to show that arbitrary automaton with n states can be embedded isomorphically into a star-product of $\mathbf{D}_n$ with a single factor, which is obvious. This ends the proof of Theorem 1.

Corollary. There exists no system of automata which is isomorphically complete with respect to the star-product and minimal.

Proof. Let Σ be isomorphically complete with respect to the star-product, and take an $\mathbf{A}\in\Sigma$ with $|A|=n$. Let $m>n$ be a fixed positive integer. Then $\mathbf{A}$ can be embedded isomorphically into a star-product of $\mathbf{D}_m$ with a single factor. On the other hand, by Theorem 1, there exists an $\mathbf{A}^* \in \Sigma$ such that $\mathbf{D}_m$ can be embedded isomorphically into a star-product of $\mathbf{A}^*$ with a single factor. But then $\mathbf{A}$ can also be embedded into a star-product of $\mathbf{A}^*$ with a single factor. This results that $\Sigma \setminus \{\mathbf{A}\}$ is isomorphically complete with respect to the star-product. Therefore, Σ is not minimal.

3. Isomoprhic simulation

In [2] products are generalized in such a way that feedback functions take their values from the set of input words of the factors. Moreover, in homomorphic and isomorphic representations the words are permitted as counter images of input signs. It turned out that these new concepts are more powerful than the old ones. Under these new concepts completeness results for α_i-products are presented in [2], while [1] is dealing with the corresponding problems concerning ν_i-products. The representation of automata by isomorphic simulation and generalized products corresponds to the computation of functions on networks of automata. Going on this line, we introduce the concept of the generalized star-product, and study complete systems with respect to such products and isomorphic simulation.

We start with the definition of the generalized star-product. Let $\mathbf{A}_t=(X_t, A_t, \delta_t)$ $(t=1, ..., k)$ be a system of automata. Moreover, let X be a finite nonviod set and φ a mapping of $A_1 \times ... \times A_k \times X$ into $X_1^* \times ... \times X_k^*$ such that φ can be given in the form

$$\varphi(a_1, ..., a_k, x) = (\varphi_1(a_1, ..., a_k, x), \varphi_2(a_1, a_2, x), ..., \varphi_k(a_1, a_k, x)).$$

It is said that the automaton

$$\mathbf{A} = (X, \prod_{t=1}^{k} A_t, \delta)$$

is a *generalized star-product* of $\mathbf{A}_t$ $(t=1, ..., k)$ with respect to X and φ is for arbitrary $(a_1, ..., a_k) \in \prod_{t=1}^{k} A_t$, and $x \in X$,

$$\delta((a_1, ..., a_k), x) = (\delta_1(a_1, \varphi_1(a_1, ..., a_k, x)), \delta_2(a_2, \varphi_2(a_1, a_2, x)), ..., \delta_k(a_k, \varphi_k(a_1, a_k, x))).$$

Obviously, if for each automaton $\mathbf{A}_t$ its characteristic semigroup is equal to X_t, then the generalized star-product is simply the star-product.

Let $\mathbf{A}=(X, A, \delta)$ and $\mathbf{B}=(Y, B, \delta')$ be arbitrary automata. We say that $\mathbf{A}$ *isomorphically simulates* $\mathbf{B}$ if there exist one-to-one mappings $\mu: B \to A$ and $\tau: Y \to X^*$ such that $\mu(\delta'(b, y))=\delta(\mu(b), \tau(y))$ for arbitrary state $b \in B$ and input sign $y \in Y$.

As far as the isomorphic simulation is concerned, we have

Lemma 1. If $\mathbf{A}$ isomorphically simulates $\mathbf{B}$ and $\mathbf{B}$ isomorphically simulates $\mathbf{C}$, then $\mathbf{C}$ can be simulated isomorphically by $\mathbf{A}$, too.

Now we define isomorphic S-completeness.

A system Σ of automata is *isomorphically* S-*complete* with respect to the generalized star-product if every automaton can be simulated isomorphically by a generalized star-product of automata from Σ.

We shall use the following special automata. For arbitrary $n \geqq 1$, let us denote by $\mathbf{T}_n=(T_n, N, \delta_n)$ the automaton for which $N=\{1, \dots, n\}$, T_n is the set of all transformations of N and $\delta_n(i, t)=t(i)$ for all $i \in N$ and $t \in T_n$.

Now we are ready to prove the following result giving necessary and sufficient conditions for S-completeness.

Theorem 2. A system Σ of automata is isomorphically S-complete with respect to the generalized star-product if and only if Σ contains an automaton $\mathbf{A}=(X, A, \delta)$ which has two different states a, b and two (not necessarily different) words $p, q \in X^*$ with $\delta(a, p)=b$ and $\delta(b, q)=a$.

Proof. The necessity of the conditions is obvious. The sufficiency can be derived from Theorem 1 in [6]. Here, using a different approach, we present a constructive proof. For this let us suppose that the conditions are satisfied by $\mathbf{A} \in \Sigma$ under the states 0, 1 and words p, q. Let $s=qp$ and $r=pq$. Then $\delta(0, r)=0$ and $\delta(1, s)=1$.

From the definition of $\mathbf{T}_n$ it follows that every automaton $\mathbf{B}=(X, B, \delta)$ can be embedded isomorphically into $\mathbf{T}_n$ if $n \geqq |B|$. Therefore, by Lemma 1, it is enough to show that for arbitrary $n \geqq 1$, $\mathbf{T}_n$ can be simulated isomorphically by a generalized star-product of automata from Σ. On the other hand, in [4] it is proved that the mappings t_1, t_2, t_3 generate the full transformation semigroup over N, where t_1, t_2, t_3 are determined as follows:

$$t_1(i) = i+1 \quad \text{if} \quad 1 \leqq i < n \quad \text{and} \quad t_1(n) = 1,$$

$$t_2(1) = 2, \quad t_2(2) = 1 \quad \text{and} \quad t_2(i) = i \quad \text{if} \quad 3 \leqq i \leqq n,$$

$$t_3(1) = t_3(2) = 1, \quad \text{and} \quad t_3(i) = i \quad \text{if} \quad 3 \leqq i \leqq n.$$

Therefore, the automaton $\mathbf{T}_n$ can be simulated isomorphically by the subautomaton $\mathbf{T}'_n=(\{t_1, t_2, t_3\}, N, \delta'_n)$ of the automaton $\mathbf{T}_n$. Therefore, again by Lemma 1, we obtain that if for every n the automaton $\mathbf{T}'_n$ can be simulated isomorphically by a generalized star-product of automata from Σ, then Σ is isomorphically S-complete with respect to the generalized star-product.

Obviously, if $n \leqq 2$, then $\mathbf{T}'_n$ can be simulated isomorphically by a generalized star-product of $\mathbf{A}$ with a single factor. Thus, suppose that $n > 2$ is an arbitrarily fixed integer. To obtain a simulation of $\mathbf{T}'_n$ by a generalized star-product of automata from Σ, consider the generalized star-power $\mathbf{A}^n(Y, \varphi)$, where $Y=\{y_j \colon 1 \leqq j \leqq n\}$, and using a function $\psi \colon \{0, 1\} \to \{r, s\}$, the mappings φ_j are defined in the following way: for arbitrary $a, b, a_1, \dots, a_k \in \{0, 1\}$,

$$\psi(a) = \begin{cases} s, & \text{if} \quad a = 1, \\ r, & \text{if} \quad a = 0, \end{cases}$$

$$\varphi_1(a_1, \dots, a_n, y_1) = \begin{cases} p, & \text{if} \quad a_1 = 0,\ a_2 = 1, \\ \psi(a_1) & \text{otherwise}, \end{cases}$$

$$\varphi_2(a, b, y_1) = \begin{cases} q, & \text{if} \quad a = 0,\ b = 1, \\ \psi(b) & \text{otherwise}, \end{cases}$$

$$\varphi_j(a, b, y_1) = \psi(b)\ (j = 3, \dots, n),$$

$$\varphi_1(a_1, \ldots, a_n, y_i) = \begin{cases} q, & \text{if} \quad a_1 = 1, \\ p, & \text{if} \quad a_1 = 0, \; a_i = 1, \; (i = 2, \ldots, n) \\ \psi(a_1) & \text{otherwise}, \end{cases}$$

$$\varphi_j(a, b, y_j) = \begin{cases} p, & \text{if} \quad a = 1, \\ q, & \text{if} \quad a = 0, \; b = 1, \; (j = 2, \ldots, n) \\ \psi(b) & \text{otherwise}, \end{cases}$$

$$\varphi_j(a, b, y_i) = \psi(b) \quad (2 \leqq j \leqq n, \; 2 \leqq i \leqq n, \; i \neq j).$$

Take the mappings

$$\mu: \begin{cases} 1 \to (1, 0, \ldots, 0, 0), \\ 2 \to (0, 1, \ldots, 0, 0), \\ \vdots \\ n \to (0, 0, \ldots, 0, 1), \end{cases}$$

and

$$\tau: \begin{cases} t_1 \to y_2 \ldots y_n, \\ t_2 \to y_2, \\ t_3 \to y_1. \end{cases}$$

The validity of the equalities $\mu(\delta_n'(i, t_j)) = \delta_{\mathbf{A}^n}(\mu(i), \tau(t_j))$ $(j = 1, 2, 3)$ can be checked in a trivial way. This completes the proof of Theorem 2.

Remark. Let us consider the automaton $\mathbf{A}_2 = (\{x, y\}, \{0, 1\}, \delta)$ with the transition function $\delta(0, x) = \delta(1, y) = 1$, $\delta(1, x) = \delta(0, y) = 0$. From the above constructive proof it follows that $\Sigma = \{\mathbf{A}_2\}$ is isomorphically S-complete with respect to the star-product.

Acknowledgement. The authors are grateful to Z. Ésik for calling their attention to the papers [6] and [7], and for suggesting a simplification in the original proof of Theorem 2.

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(Received February 20, 1989)