On the randomized complexity of monotone graph properties

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1 Introduction

Let \( C^R(P) \) be the number of questions of the form 'Does the graph \( G \) contain the edge \( e(i,j) \)?' that have to be asked in the worst case by any randomized decision tree algorithm for computing an \( n \)-vertex graph property \( P \). For non-trivial, monotone graph properties it is known, that the deterministic complexity is \( \Omega(n^2) \) (see [4]). R. Karp [5] conjectured, that this bound holds for randomized algorithms as well. As far as this conjecture we know the following results. The best uniform lower bound for all non-trivial, monotone graph properties is \( \Omega(n^{4/3}) \) due to P. Hajnal [1].

No non-trivial, monotone graph property is known having a randomized complexity of less than \( n^2/4 \). Some properties have been proven to have complexity of \( \Omega(n^2) \) (see A. Yao [6]).

In this paper we refine the idea of Yao. This leads to a further improvement in the reductions of arbitrary graph properties to bipartite graph properties. (see [1], [3]) and yields a uniform lower bound for the subgraph isomorphism properties of \( \Omega(n^{3/2}) \). Furthermore we show, that a large variety of isomorphism properties as well as \( k \)-colourability require \( \Omega(n^2) \) questions.

2 Preliminaries, notations

A decision tree is a rooted binary tree with labels on each node and edge. Each inner node is labeled by a variable symbol and the two edges leaving the node are labeled by 0 and 1. Each leaf is also labeled by 0 or 1. Obviously, any truth-assignment of the variables determines a unique path from the root to a leaf.

A decision tree \( A \) computes a boolean function \( f \) if for all input \( \bar{x} \) the corresponding path in \( A \) leads to a leaf labeled by \( f(\bar{x}) \).

Let \( \text{cost}(A, \bar{x}) \) be the number of questions asked when the decision tree \( A \) is executed on input \( \bar{x} \). This is the length of the path induced by \( \bar{x} \). The deterministic decision tree complexity of a boolean function \( f \) is \( C(f) = \min_A \max_{\bar{x}} \text{cost}(A, \bar{x}) \), where the minimum is taken over all decision trees \( A \) computing the function \( f \).

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In a randomized decision tree the question asked next not only depends on the answers it got so far but also on the outcome of a trial. Since all trials can be done in advance we can view a randomized decision tree as a probability distribution on the set of deterministic trees. A randomized decision tree computes a boolean function $f$ iff the distribution is non-zero only on deterministic trees computing $f$.

**Definition 2.1** Let $\{A_1, \ldots, A_N\}$ be the set of all deterministic decision trees computing $f$. Let $R = \{p_1, \ldots, p_N\}$ be a randomized decision tree, where $p_i$ denotes the probability of $A_i$. The cost of $R$ on input $\vec{x}$ is $\text{cost}(R, \vec{x}) = \sum_i p_i \cdot \text{cost}(A_i, \vec{x})$. The randomized decision tree complexity of a function $f$ is

$$C^R(f) = \min_R \max_{\vec{x}} \text{cost}(R, \vec{x}),$$

where the minimum is taken over all randomized decision trees computing the function $f$. The following lemma yields the base of all lower bound proofs for randomized decision tree complexity so far.

**Lemma 2.2** (A. Yao [6]) Let $d$ be a probability distribution on the set of all possible inputs and let $d(\vec{x})$ be the probability of input $\vec{x}$. We define the average case performance of a deterministic tree $A$ computing $f$ as $\text{av}(A, d) = \sum_{\vec{x}} d(\vec{x}) \text{cost}(A, \vec{x})$.

Then for any boolean function $f$

$$C^R(f) = \max_d \min_A \text{av}(A, d),$$

where the minimum is taken over all deterministic decision trees computing $f$.

A boolean function $f$ is called non-trivial, monotone iff $f(\vec{0}) = f(\vec{1}) = 1$ and $f(\vec{z_1}) \leq f(\vec{z_2})$ for all $\vec{z_1} \leq \vec{z_2}$. Here we mean component wise less or equal. In this paper we deal only with graph properties and bipartite graph properties. Since a graph on $n$ vertices can be identified with a $(0,1)$-string of length $\binom{n}{2}$, a graph property can be given by a boolean function which takes equal values on isomorphic graphs. So, by graph property we mean a suitable boolean function and sometimes instead of the function we give the property by the set of all graphs having this property. A graph property is called non-trivial, monotone iff the corresponding boolean function is non-trivial, monotone.

Let us denote the set of all $n$-vertex by $\mathcal{G}_n$ and the set of all non-trivial, monotone graph properties defined on $\mathcal{G}_n$, by $\mathcal{P}_n$. Clearly, a property $P \in \mathcal{P}_n$ can be characterized by the set of minimal graphs having that property. Let $\min(P)$ be the list of minimal graphs for $P$. If $\min(P)$ contains up to isomorphism only one graph $G$, we call $P$ a subgraph isomorphism property and denote it by $P_G$.

Let us denote by $d_G(x)$ the degree of a node $x$ in $G$, by $D(G)$ the maximal degree of $G$, by $\delta(G)$ the minimal degree of $G$ and by $\bar{d}(G)$ the average degree of $G$. Furthermore, denote $V(G)$ the set of vertices with non-zero degree of $G$, $E(G)$ the set of edges of $G$ and $K_{n_1}E_n$ the complete and the empty graph on $n$ nodes, respectively. Sometimes we use the disjoint union of $K_{n-r}$ and $\bar{E_r}$, and this graph is denoted by $K_{n-r}^\ast$.

Let $0 < m < n$ and $P \in \mathcal{P}_n$. Using the property $P$, we can define two (not necessarily non-trivial) monotone graph properties on $\mathcal{G}_m$. For this reason, divide the set of nodes, $P$ is defined on, into disjoint sets $V_1$ and $V_2$ so that $|V_1| = m$ and $|V_2| = n - m$. Let $\text{ind}(P|m)$ and $\text{red}(P|m)$ denote the following $m$-vertex properties:
On the randomized complexity of monotone graph properties

\[ G \in \text{ind}(P|m) \text{ iff adding all nodes in } V_2 \text{ to } G \text{ and keeping the original edge-set, we obtain a graph having property } P. \]

\[ G \in \text{red}(P|m) \text{ iff adding all nodes in } V_2 \text{ together with all possible edges incident to them to } G, \text{ we get a graph having property } P. \]

Obviously \( C^R(\text{ind}(P|m)) \leq C^R(P) \) and \( C^R(\text{red}(P|m)) \leq C^R(P) \).

We have to build up the same system of notions for the universe of labeled bipartite graphs with colour classes \( V = \{1, 2, \ldots, n\} \) and \( W = \{1, 2, \ldots, m\} \) denoted by \( \mathcal{G}_{n,m} \). The set of all non-trivial, monotone bipartite graph properties on \( \mathcal{G}_{n,m} \) is denoted by \( \mathcal{P}_{n,m} \). We also use the other corresponding notions \( C^R(P), \min(P) \) and \( E(G) \).

If \( G \in \mathcal{G}_{n,m} \) and \( U \) is a subset of the vertices then let us denote by \( d_{\text{max},U}(G) \) and \( d_{\text{av},U}(G) \) the maximal and average degree in the set \( U \), and by \( K_{n,m}, E_{n,m} \) the complete bipartite graph and the empty bipartite graph, respectively.

Let \( 0 < r < n \) and \( P \in \mathcal{P}_{n,m} \). Divide \( V \) into disjoint sets \( V_1 \) and \( V_2 \) so that \(|V_1| = r\) and \(|V_2| = n - r\). Let \( \text{ind}_V(P|r) \) and \( \text{red}_V(P|r) \) denote the following bipartite graph properties defined on \( \mathcal{G}_{n,m} \):

\[ G \in \text{ind}_V(P|r) \text{ iff adding all nodes of } V_2 \text{ to } G, \text{ we obtain a bipartite graph having property } P. \]

\[ G \in \text{red}_V(P|r) \text{ iff adding all nodes of } V_2 \text{ together with all possible edges between } V_2 \text{ and } V_1 \text{ to } G, \text{ we get a bipartite graph having property } P. \]

\[ C^R(\text{ind}_V(P|r)) \leq C^R(P) \text{ and } C^R(\text{red}_V(P|r)) \leq C^R(P). \]

Finally let

\[ C^R(n, m) = \min\{C^R(P)|P \in \mathcal{P}_{n,m}\}. \]

In lower bound proofs for the complexity of monotone graph properties the following reduction to bipartite graph properties plays an important role.

Let \( P \in \mathcal{P}_n \) and \( 0 < r < n \). Furthermore, let \( \text{bipart}(P|r, n - r) \) be the following bipartite graph property defined on \( \mathcal{G}_{r,n-r} \):

\[ G \in \text{bipart}(P|r, n - r) \text{ iff adding all edges between nodes in } W, \text{ we obtain a graph having property } P. \]

\[ C^R(\text{bipart}(P|r, n - r)) \leq C^R(P) \text{ and so if } \text{bipart}(P|r, n - r) \text{ is non-trivial, then } C^R(r, n - r) \leq C^R(P). \]

A good survey of previous techniques can be found in [1]. We only mention those, we will apply.

**Theorem 2.3 (Basic Method [6])** (i) Let \( P \in \mathcal{P}_n \) and \( G \in \min(P) \) be any minimal graph for \( P \). Then

\[ C^R(P) \geq |E(G)|. \]

(ii) Let \( P \in \mathcal{P}_{n,m} \) and \( G \in \min(P) \) be any minimal graph for \( P \). Then

\[ C^R(P) \geq |E(G)|. \]

**Definition 2.4** Let \( L \) be a list of graphs from \( \mathcal{G}_{n,m} \). For each \( G \in L \) let us consider the sequence of degrees in colour class \( V \). Let \( d_1 \geq d_2 \geq \ldots \geq d_n \) be the ordered list of degrees. If \( \{d_1, d_2, \ldots, d_n\} \) is the lexicographically minimal sequence considering all the ordered lists then we refer to \( G \) as the \( V \)-lexicographically first element of \( L \).

**Theorem 2.5 (Yao's Method [7])** Let \( P \in \mathcal{P}_{n,m} \) and \( G \) be the \( V \)-lexicographically first graph of \( \min(P) \). Then

\[ C^R(P) = \Omega\left(\frac{d_{\text{max},V}(G)}{d_{\text{av},V}(G)} \cdot |V|\right). \]
A very useful tool for proving lower bounds is duality. For every non-trivial, monotone boolean function \( f \) we can define the dual function \( f^D \) as follows:

\[
f^D(\overline{z}) = \neg f(\overline{z}).
\]

It is easy to see that \( f^D \) is also non-trivial, monotone and \( C^R(f^D) = C^R(f) \).

**Definition 2.6** (i) Let \( G, H \in \mathcal{G}_n \) with vertex sets \( V \) and \( V' \), respectively. A packing is an identification between \( V \) and \( V' \) such that no edge of \( G \) is identified with any edge of \( H \).

(ii) Let \( G, H \in \mathcal{G}_{n,m} \) with colour classes \( V, W \) and \( V', W' \), respectively. A bipartite packing is an identification between \( V \) and \( V' \) and between \( W \) and \( W' \) such that no edge of \( G \) is identified with any edge of \( H \).

**Lemma 2.7** (Yao [6]) (i) If \( P \in \mathcal{P}_n \) and \( G \in \text{min}(P) \) and \( H \in \text{min}(P^D) \) then \( G \) and \( H \) can't be packed. (ii) If \( P \in \mathcal{P}_{n,m} \) and \( G \in \text{min}(P) \) and \( H \in \text{min}(P^D) \) then \( G \) and \( H \) can't be packed as bipartite graphs.

3 Results

By a covering of a graph \( G \) we mean a subset \( K \) of \( V \) such that any edge of \( G \) is adjacent to at least one vertex in \( K \). A covering \( K \) is minimal if \( G \) has no covering \( K' \) with \( |K'| < |K| \).

The width of a graph \( G \) denoted by \( \text{width}(G) \) is the size of a minimal covering of \( G \). The trace of a graph \( G \) denoted by \( \text{trace}(G) \) is the minimal number of edges we have to remove from \( G \) in order to decrease its width.

Now we extend these notions to monotone graph properties. The width of a monotone graph property \( P \) is defined as follows:

\[
\text{width}(P) = \min\{\text{width}(G) | G \in \text{min}(P)\}
\]

The trace of a monotone graph property \( P \) is defined by

\[
\text{trace}(P) = \min\{\text{trace}(G) | G \in \text{min}(P)\}, \text{width}(P) = \text{width}(G)
\]

The following assertions show some fundamental properties of these notions.

**Lemma 3.1** If \( P \in \mathcal{P}_n \) and \( 1 \leq r < n \) is a fixed integer then

(i) \( \text{width}(P) \geq r \) iff \( K_{n+1-r} \in P^D \)

(ii) If \( \text{width}(P) > r \) then

\( \text{red}(P|n-r) \in \mathcal{P}_n-r, \text{width} (\text{red}(P|n-r)) = \text{width}(P) - r, \text{trace} (\text{red}(P|n-r)) = \text{trace}(P) \).

**Lemma 3.2** If \( P \in \text{calP}_n \) and \( \text{width}(P) = 1 \) then for any \( G \in P^D \), \( G \) has at least \( \frac{1}{2}n \cdot (n - \text{trace}(P)) \) edges.

**Proof.** Since \( P^D \) is a non-trivial, monotone graph property, it is sufficient to prove the statement for \( G \in \text{min}(P^D) \). Indeed, let \( G \in \text{min}(P^D) \) be arbitrary and let \( H \in \text{min}(P) \) such a graph for which \( \text{width}(H) = \text{width}(P) \) and \( \text{trace}(H) = \text{trace}(P) \) holds. With other words, \( H \) is a star with \( \text{trace}(P) \) many edges. According to Lemma 2.7, \( G \) and \( H \) can't be packed. This implies \( \delta(G) \geq |V(G)| - \text{trace}(H) = n - \text{trace}(H) \), therefore \( |E(G)| \geq \frac{1}{2}n \cdot (n - \text{trace}(H)) \).
Lemma 3.3 For any \( P \in P_n \) the following assertions hold:

(i) \( C^R(P) \geq \text{width}(P) \cdot \text{trace}(P) \).

(ii) \( C^R(P) \geq \frac{1}{2}(n + 1 - \text{width}(P)) \cdot (n + 1 - (\text{width}(P) + \text{trace}(P))) \).

(iii) For any \( 0 < \varepsilon < 1 \), if \( \text{width}(P) \leq (1 - \varepsilon) \cdot n \) then \( C^R(P) \geq \frac{\varepsilon^2}{2-\varepsilon} n \cdot \text{width}(P) \).

Proof. Assertion (i) is a straightforward consequence of Theorem 2.3. To prove (ii) choose in Lemma 3.1. \( r = \text{width}(P) - 1 \) and apply Lemma 3.2. to the reduced property \( \alpha(P|n - (\text{width}(P) - 1)) \). Finally Theorem 2.3. yields the result. If \( \text{trace}(P) \geq \frac{\varepsilon^2}{2-\varepsilon} \cdot n \), then assertion (iii) follows from (i), else it can be proved, using (ii) and assumption \( \text{width}(P) \leq (1 - \varepsilon) \cdot n \).

Before we state our main results we apply this method to some special graph properties. For this reason let us denote the property that an \( n \)-vertex graph contains a Hamiltonian cycle by \( PH_n \) and the property that an \( n \)-vertex graph has a vertex colouring with \( k \) colours by \( PC_{k,n} \).

Theorem 3.4

\[
C^R(PH_n) \geq \frac{1}{8} \cdot (n^2 - 1)
\]

\[
C^R(PC_{n,k}) \geq \binom{n + 1 - k}{2}.
\]

Proof. We have only to determine the width and trace of the given properties. The required values are:

\[
\text{width}(PH_n) = \left\lfloor \frac{n}{2} \right\rfloor,
\]

\[
\text{trace}(PH_n) = \begin{cases} 
1, & \text{if } n \text{ is odd} \\
2, & \text{if } n \text{ is even}
\end{cases}
\]

Since \( PC_{k,n} \) itself is not monotone, we consider instead of \( PC_{k,n} \) the property \( \neg PC_{k,n} \) which is non-trivial, monotone and obviously, \( C^R(\neg PC_{k,n}) = C^R(PC_{k,n}) \).

It can be seen that the corresponding values are:

\[
\text{width}(\neg PC_{k,n}) = k
\]

\[
\text{trace}(\neg PC_{k,n}) = 1.
\]

The following theorem improves the known reductions of non-trivial, monotone graph properties to bipartite graph properties. Although King [3] has already stated a similar result, the new approach can help to prove better uniform lower bounds, since the reduction is to colour classes both of size \( \Omega(n) \).

Theorem 3.5 The randomized decision tree complexity of any non-trivial, monotone graph property \( P \in P_n \) is

\[
C^R(P) \geq \min \{ \frac{1}{40} \cdot n^{3/2}, C^R(\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor) \}.
\]
Proof. We have only to consider the case that the property \( P \) can't be reduced to a non-trivial bipartite graph property bipart\( P\left(\left\lfloor \frac{3}{4} \right\rfloor, \left\lfloor \frac{3}{4} \right\rfloor \right) \). This implies \( K_{\left\lfloor \frac{3}{4} \right\rfloor} \in \mathcal{P} \) or \( K_{\left\lceil \frac{3}{4} \right\rceil} \in \mathcal{P}^D \). Therefor, it remains only to prove, that for any \( P \in \mathcal{P}_n \), if \( K_{\left\lceil \frac{3}{4} \right\rceil} \in \mathcal{P} \) then \( C^R(P) \geq \frac{1}{40} \cdot n^{3/2} \) holds.

Let us suppose that we found a property \( P \in \mathcal{P}_n \) with \( K_{\left\lceil \frac{3}{4} \right\rceil} \in \mathcal{P} \) and \( C^R(P) < \frac{1}{40} \cdot n^{3/2} \). Let us construct the following sequence of induced graph properties

\[
\{ P_i | 0 \leq i \leq \left\lfloor \frac{1}{2} n^{1/2} \right\rfloor \}, P_i = \text{ind}(P \left( \left\lfloor \frac{3}{4} n + \frac{1}{2} i \cdot n^{1/2} \right\rfloor \right)).
\]

Since \( K_{\left\lceil \frac{3}{4} \right\rceil} \in \mathcal{P} \) and for any \( i \) \( P_i \) is an induced property of \( P \) on at least \( \left\lfloor \frac{3}{4} n \right\rfloor \) vertices, \( P_i \) is non-trivial and

\[
C^R(P_i) \leq C^R(P) < \frac{1}{40} n^{3/2} \tag{1}
\]

\( K_{\left\lceil \frac{3}{4} \right\rceil} \in \mathcal{P}_0 \) implies \( \text{width}(P_0) \leq \left\lfloor \frac{3}{4} \right\rfloor - 1 \leq \frac{1}{2} n \). Assertion (iii) of Lemma 3.3. yields \( C^R(P_0) \geq \frac{1}{20} n \cdot \text{width}(P_0) \). Hence

\[
\text{width}(P_0) \leq \left\lfloor \frac{1}{2} n^{1/2} \right\rfloor \tag{2}
\]

Obviously \( G \in P_i \) implies \( G \in P_{i+1} \). Therefore

\[
\text{width}(P_{i+1}) \leq \text{width}(P_i), i \geq 0. \tag{3}
\]

Let us suppose, that for some \( i \geq 0 \) \( \text{width}(P_{i+1}) = \text{width}(P_i) \) holds. Then \( \text{trace}(P_{i+1}) \leq \text{trace}(P_i) \). Now Lemma 3.3. yields

\[
C^R(P_{i+1}) \geq \frac{1}{2} \left( \left\lfloor \frac{3}{4} n + \frac{1}{2} (i+1) \cdot n^{1/2} \right\rfloor + 1 - \text{width}(P_{i+1}) \right) \cdot
\]

\[
\left( \left\lfloor \frac{3}{4} n + \frac{1}{2} (i+1) \cdot n^{1/2} \right\rfloor + 1 - (\text{width}(P_{i+1}) + \text{trace}(P_{i+1})) \right)
\]

\[
\geq \frac{1}{2} \cdot \left( \frac{3}{4} n + \frac{1}{2} n^{1/2} - \text{width}(P_0) \right) \cdot
\]

\[
\left( \frac{3}{4} n + \frac{1}{2} i \cdot n^{1/2} + 1 - (\text{width}(P_i) + \text{trace}(P_i)) + \frac{1}{2} n^{1/2} \right)
\]

\[
\geq \frac{3}{16} n^{3/2} > \frac{1}{40} n^{3/2},
\]

which contradicts (1).

The sequence of positive integers \( \{\text{width}(P_i) | 0 \leq i \leq \left\lfloor \frac{1}{2} n^{1/2} \right\rfloor \} \) therefore decreases strictly monotone, and so

\[
\text{width}(P_0) \geq \left\lfloor \frac{1}{2} n^{1/2} \right\rfloor + 1,
\]

which contradicts (2).

Since our assumption \( C^R(P) < \frac{1}{40} n^{3/2} \) led to a contradiction we have completed the proof.

A straightforward consequence of the improved reduction is the following result.
Theorem 3.6 For the randomized decision tree complexity of any subgraph isomorphism property \( P_G \in P_n \)

\[
C^R(P_G) = \Omega(n^{3/2}).
\]

Proof. According to Theorem 3.5, we have only to settle the case that
bipart\((P_G||\lfloor \frac{n}{4} \rfloor, \lfloor \frac{n}{2} \rfloor)\) is nontrivial. Depending on width\((P_G)\) and trace\((P_G)\) we shall distinguish three cases.

Case 1. Assume that width\((P_G) \geq \frac{1}{4} n\). Since bipart\((P_G||\lfloor \frac{n}{4} \rfloor, \lfloor \frac{n}{2} \rfloor)\) is non-trivial, we get width\((P_G) \leq \frac{1}{2} n\) and assertion (iii) of Lemma 3.3. implies a lower bound of \( \Omega(n^2) \).

Case 2. If width\((P_G) < \frac{1}{4} n\) and trace\((P_G) < \frac{2}{3} n\), then we can apply assertion (ii) of Lemma 2.5. and get also a lower bound of \( \Omega(n^2) \).

Case 3. Suppose that width\((P_G) < \frac{1}{4} n\) and trace\((P_G) \geq \frac{2}{3} n\). Since trace\((P_G) \geq \frac{2}{3} n\) the corresponding bipartite graph property bipart\((P_G||\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor)\) has only such minimal graphs \( H \) for that \( D_V(H) \geq \frac{1}{4} n \) holds. If bipart\((P_G||\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor)\) has a minimal graph with at least \( n^{3/2} \) edges we can apply Theorem 2.3. Otherwise we can apply Theorem 2.5. In both cases we get a lower bound of \( \Omega(n^{3/2}) \) which completes the proof.

Before we prove the sharper version of Theorem 2.6. we consider some special bipartite graph properties. Let us denote by \( S_{n,m} \) the graph which has one vertex of positive degree in \( V \) and \( m \) edges.

Lemma 3.7 Let \( P_S \in P_{n,m} \) denote the property of containing a subgraph isomorph to \( S_{n,m} \). Then

\[
C^R(P_S) \geq \frac{1}{2} m \cdot n.
\]

Proof. (analogue to Yao [7]). We consider the dual property \( P_S^D \), which is easy to see to contain exactly those graphs, that have no isolated nodes in colour class \( W \). According to Lemma 2.2. we choose as a “hard” input distribution the uniform distribution over all minimal graphs. Let be \( A \) an optimal deterministic decision tree, that computes our \( P_S^D \). We denote by \( X_i(G) \) the number of edges incident to \( w_i \) asked by \( A \). Then

\[
C^R(P_S^D) \geq E \left( \sum_{i=1}^{m} X_i(G) \right)
\]

\[
= \sum_{i=1}^{m} E(X_i(G))
\]

Since for any value of \( i \) we have to find one edge out of \( n \) edges, we get

\[
E(X_i(G)) \geq \frac{1}{2} n
\]

and finally

\[
C^R(P_S) \geq \frac{1}{2} m \cdot n.
\]
Lemma 3.8 Let $P \in \mathcal{P}_{n,m}$ such a property, that every $G \in \min(P)$ has exactly $k \leq \frac{1}{2} n$ vertices of positive degree in colour class $V$. Then

$$C^R(P) \geq \frac{1}{6} m \cdot n.$$ 

Proof. We consider the reduced graph property $P' = \text{red}_V(P|n+1-k)$. Obviously, $P'$ is non-trivial and $\min(P')$ contains up to isomorphy exactly one graph. This graph has exactly one vertex with positive degree $(d)$ in the colour class $V'$. We distinguish two cases.

Case 1. Assume that $d \leq \frac{3}{2} m$. Since the minimal graphs of $P'$ and $P'^D$ can't be packed as bipartite graphs, any $G \in \min(P'^D)$ has at least $(n+1-k) \cdot (m+1-d) \geq \frac{1}{6} m \cdot n$ edges. Hence Theorem 2.1. implies the required lower bound.

Case 2. If $d > \frac{3}{2} m$ then let us consider the induced property $\text{ind}_W(P'|d)$ on colour classes of size $n+1-k$ and $d$, respectively. Since $\text{ind}_W(P'|d) = S_{n+1-k,d}$, Lemma 3.7. yields the statement.

Lemma 3.9 The randomized decision tree complexity of any subgraph isomorphism property $P_G \in \mathcal{P}_n$ with width at most $\frac{2}{3} n$ fulfills

$$C^R(P_G) \geq \frac{1}{24} (n^2 - 1).$$

Proof. Depending on width($P_G$) and trace($P_G$) we distinguish six cases.

Case 1. If $\frac{2}{3} n \leq \text{width}(P_G) \leq \frac{2}{3} n$ and trace($P_G$) $\geq \frac{1}{12} n$, then assertion (i) of Lemma 3.3. implies the lower bound.

Case 2. If $\frac{2}{3} n \leq \text{width}(P_G) \leq \frac{2}{3} n$ and trace($P_G$) $< \frac{1}{12} n$, then assertion (ii) of Lemma 3.3. implies the lower bound.

Case 3. If $K_{\lfloor n/2 \rfloor} \in P_G$, then $\text{width}(P_G) + \text{trace}(P_G) \leq \frac{1}{n}$. Therefore, by assertion (ii) of Lemma 3.3., we obtain $C^R(P_G) \geq \frac{1}{6} n^2 \geq \frac{1}{24} n^2$.

So far we have considered all the cases, when $P_G$ can't be reduced to a non-trivial bipartite graph property bipart($P_G|\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor$).

Case 4. If $\frac{n}{2} \leq \text{width}(P_G) < \frac{n}{2}$, then we can apply assertion (iii) of Lemma 3.3. for $\varepsilon = \frac{1}{2}$ and get the required lower bound.

Case 5. If width($P_G$) $< \frac{n}{4}$ and trace($P_G$) $\geq \frac{5}{6} n$, then $G$ contains width($P_G$) vertices with degree at least $\frac{5}{6} n$. In our reduction to the bipartite graph property bipart($P_G|\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor$) we have to put them all into $V$. On the other hand, these vertices build a covering of the graph $G$. Hence $G$ contains no edge independent of this vertex set. Therefore any minimal graph of the property bipart($P_G|\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor$) has exactly width($P_G$) vertices of positive degree in $V$ and Lemma 3.8. implies

$$C^R(P_G) \geq \text{bipart}(P_G|\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor) \geq \frac{1}{24} \cdot (n^2 - 1).$$

Case 6. If width($P_G$) $< \frac{n}{4}$ and trace($P_G$) $< \frac{5}{6} n$, then by assertion (ii) of Lemma 3.3., we get that

$$C^R(P_G) \geq \frac{1}{2} \cdot \frac{3}{4} n \cdot \left( \frac{3}{4} n - \frac{5}{9} n \right) \geq \frac{1}{24} n^2,$$
which completes the proof.

The following statement is an immediate consequence of this theorem and generalizes the results of Yao [6].

**Assertion 3.10** For every \( \varepsilon > 0 \) we can find a \( \lambda > 0 \) which depends only on \( \varepsilon \), such that the randomized decision tree complexity of any subgraph isomorphism property \( P_G \in P_n \) with \( d(G) \leq \varepsilon \) fulfills:

\[
C^R(P_G) \geq \lambda(\varepsilon) \cdot n^2.
\]

After finishing this manuscript the author has learnt that M. Karpinski et al [2] independently proved Theorem 3.5.

**References**


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