Acta Cybernetica, Vol. 10, Nr. 3, Szeged, 1992

# On the randomized complexity of monotone graph properties

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## 1 Introduction

Let $C^R(P)$ be the number of questions of the form 'Does the graph $G$ contain the edge $e(i,j)$ ?' that have to be asked in the worst case by any randomized decision tree algorithm for computing an $n$-vertex graph property $P$. For non-trivial, monotone graph properties it is known, that the deterministic complexity is $\Omega(n^2)$ (see [4]). R. Karp [5] conjectured, that this bound holds for randomized algorithms as well. As far as this conjecture we know the following results. The best uniform lower bound for all non-trivial, monotone graph properties is $\Omega(n^{4/3})$ due to P. Hajnal [1].

No non-trivial, monotone graph property is known having a randomized complexity of less than $n^2/4$. Some properties have been proven to have complexity of $\Omega(n^2)$ (see A. Yao [6]).

In this paper we refine the idea of Yao. This leads to a further improvement in the reductions of arbitrary graph properties to bipartite graph properties. (see [1], [3]) and yields a uniform lower bound for the subgraph isomorphism properties of $\Omega(n^{3/2})$. Furthermore we show, that a large variety of isomorphism properties as well as $k$-colourability require $\Omega(n^2)$ questions.

## 2 Preliminaries, notations

*A decision tree* is a rooted binary tree with labels on each node and edge. Each inner node is labeled by a variable symbol and the two edges leaving the node are labeled by 0 and 1. Each leaf is also labeled by 0 or 1. Obviously, any truth-assignment of the variables determines a unique path from the root to a leaf.

A decision tree $A$ *computes a boolean function* $f$ if for all input $\underline{x}$ the corresponding path in $A$ leads to a leaf labeled by $f(\underline{x})$.

Let $cost(A,\underline{x})$ be the number of questions asked when the decision tree $A$ is executed on input $\underline{x}$. This is the length of the path induced by $\underline{x}$. The *deterministic decision tree complexity* of a boolean function $f$ is $C(f) = \min_A \max_{\underline{x}} cost(A,\underline{x})$, where the minimum is taken over all decision trees $A$ computing the function $f$.

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In a *randomized decision tree* the question asked next not only depends on the answers it got so far but also on the outcome of a trial. Since all trials can be done in advance we can view a randomized decision tree as a probability distribution on the set of deterministic trees. *A randomized decision tree computes a boolean function $f$* iff the distribution is non-zero only on deterministic trees computing $f$.

**Definition 2.1** *Let $\{A_1, \ldots, A_N\}$ be the set of all deterministic decision trees computing $f$. Let $R = \{p_1, \ldots, p_N\}$ be a randomized decision tree, where $p_i$ denotes the probability of $A_i$. The* cost of $R$ on input $\underline{x}$ *is $cost(R, x) = \sum_i p_i \cdot cost(A_i, \underline{x})$. The* randomized decision tree complexity *of a function $f$ is*

$$C^R(f) = \min_R \max_{\underline{x}} cost(R, \underline{x}),$$

*where the minimum is taken over all randomized decision trees computing the function $f$. The following lemma yields the base of all lower bound proofs for randomized decision tree complexity so far.*

**Lemma 2.2** *(A. Yao [6]) Let $d$ be a probability distribution on the set of all possible inputs and let $d(\underline{x})$ be the probability of input $\underline{x}$. We define the* average case performance *of a deterministic tree $A$ computing $f$ as $av(A, d) = \sum_{\underline{x}}) d(\underline{x}) cost(A, \underline{x})$.*

Then for any boolean function $f$

$$C^R(f) = \max_d \min_A av(A, d),$$

where the minimum is taken over all deterministic decision trees computing $f$.

A boolean function $f$ is called *non-trivial, monotone* iff $f(\underline{0}) = f(\underline{1}) = 1$ and $f(\underline{x}_1) \leq f(\underline{x}_2)$ for all $\underline{x}_1 \leq \underline{x}_2$. Here we mean component wise less or equal. In this paper we deal only with graph properties and bipartite graph properties. Since a graph on $n$ vertices can be identified with a $(0,1)$-string of length $\binom{n}{2}$, a graph property can be given by a boolean function which takes equal values on isomorphic graphs. So, by *graph property* we mean a suitable boolean function and sometimes instead of the function we give the property by the set of all graphs having this property. A graph property is called *non-trivial, monotone* iff the corresponding boolean function is non-trivial, monotone.

Let us denote the set of all $n$-vertex by $\mathcal{G}_n$ and the set of all non-trivial, monotone graph properties defined on $\mathcal{G}_n$ by $\mathcal{P}_n$. Clearly, a property $P \in \mathcal{P}_n$ can be characterized by the set of minimal graphs having that property. Let $\min(P)$ be the list of minimal graphs for $P$. If $\min(P)$ contains up to isomorphism only one graph $G$, we call $P$ a *subgraph isomorphism property* and denote it by $P_G$.

Let us denote by $d_G(x)$ the degree of a node $x$ in $G$, by $D(G)$ the maximal degree of $G$, by $\delta(G)$ the minimal degree of $G$ and by $\overline{d}(G)$ the average degree of $G$. Furthermore, denote $V(G)$ the set of vertices with non-zero degree of $G, E(G)$ the set of edges of $G$ and $K_n, E_n$ the complete and the empty graph on $n$ nodes, respectively. Sometimes we use the disjoint union of $K_{n-r}$ and $E_r$, and this graph is denoted by $K^*_{n-r}$.

Let $0 < m < n$ and $P \in \mathcal{P}_n$. Using the property $P$, we can define two (not necessarily non-trivial) monotone graph properties on $\mathcal{G}_m$. For this reason, divide the set of nodes, $P$ is defined on, into disjoint sets $V_1$ and $V_2$ so that $|V_1| = m$ and $|V_2| = n - m$. Let $ind(P|m)$ and $red(P|m)$ denote the following $m$-vertex properties:

$G \in ind(P|m)$ iff adding all nodes in $V_2$ to $G$ and keeping the original edge-set, we obtain a graph having property $P$.

$G \in red(P|m)$ iff adding all nodes in $V_2$ together with all possible edges incident to them to $G$, we get a graph having property $P$.

Obviously $C^R(ind(P|m)) \leq C^R(P)$ and $C^R(red(P|m)) \leq C^R(P)$.

We have to build up the same system of notions for the universe of labeled bipartite graphs with colour classes $V = \{1, 2, \ldots, n\}$ and $W = \{\overline{1}, \overline{2}, \ldots, \overline{m}\}$ denoted by $\mathcal{G}_{n,m}$. The set of all non-trivial, monotone bipartite graph properties on $\mathcal{G}_{n,m}$ is denoted by $\mathcal{P}_{n,m}$. We also use the other corresponding notions $C^R(P)$, $\min(P)$ and $E(G)$.

If $G \in \mathcal{G}_{n,m}$ and $U$ is a subset of the vertices then let us denote by $d_{\max,U}(G)$ and $d_{av,U}(G)$ the maximal and average degree in the set $U$, and by $K_{n,m}$, $E_{n,m}$ the complete bipartite graph and the empty bipartite graph, respectively

Let $0 < r < n$ and $P \in \mathcal{P}_{n,m}$. Divide $V$ into disjoint sets $V_1$ and $V_2$ so that $|V_1| = r$ and $|V_2| = n - r$. Let $ind_V(P|r)$ and $red_V(P|r)$ denote the following bipartite graph properties defined on $\mathcal{G}_{n,m}$:

$G \in ind_V(P|r)$ iff adding all nodes of $V_2$ to $G$, we obtain a bipartite graph having property $P$.

$G \in red_V(P|r)$ iff adding all nodes of $V_2$ together with all possible edges between $V_2$ and $W$ to $G$, we get a bipartite graph having property $P$.

Obviously $C^R(ind_V(P|r)) \leq C^R(P)$ and $C^R(red_V(P|r)) \leq C^R(P)$.

Finally let

$$C^R(n, m) = \min\{C^R(P) | P \in \mathcal{P}_{n,m}\}.$$

In lower bound proofs for the complexity of monotone graph properties the following reduction to bipartite graph properties plays an important role.

Let $P \in \mathcal{P}_n$ and $0 < r < n$. Furthermore, let $bipart(P|r, n-r)$ be the following bipartite graph property defined on $\mathcal{G}_{r,n-r}$

$G \in bipart(P|r, n-r)$ iff adding all edges between nodes in $W$, we obtain a graph having property $P$.

Obviously $C^R(bipart(P|r, n-r)) \leq C^R(P)$ and so if $bipart(P|r, n-r)$ is non-trivial, then $C^R(r, n-r) \leq C^R(P)$.

*A* good survey of previous techniques can be found in [1]. We only mention those, we will apply.

**Theorem 2.3** *(Basic Method [6]) (i) Let $P \in \mathcal{P}_n$ and $G \in \min(P)$ be any minimal graph for $P$. Then*

$$C^R(P) \geq |E(G)|.$$

*(ii) Let $P \in \mathcal{P}_{n,m}$ and $G \in \min(P)$ be any minimal graph for $P$. Then*

$$C^R(P) \geq |E(G)|.$$

**Definition 2.4** *Let $\mathcal{L}$ be a list of graphs from $\mathcal{G}_{n,m}$. For each $G \in \mathcal{L}$ let us consider the sequence of degrees in colour clas $V$. Let $d_1 \geq d_2 \geq \ldots \geq d_n$ be the ordered list of degrees. If $(d_1, d_2, \ldots, d_n)$ is the lexicographically minimal sequence considering all the ordered lists then we refer to $G$ as the* $V$-lexicographically first element *of $\mathcal{L}$*

**Theorem 2.5** *(Yao's Method [7]) Let $P \in \mathcal{P}_{n,m}$ and $G$ be the $V$-lexicographically first graph of* $\min(P)$. *Then*

$$C^R(P) = \Omega(\frac{d_{\max,V}(G)}{d_{av,V}(G)} \cdot |V|).$$

*A very useful tool for proving lower bounds is dualiy. For every non-trivial, monotone boolean function $f$ we can define the* dual function $f^D$ *as follows:*

$$f^D(\underline{x}) = \neg f(\neg \underline{x}).$$

*It is easy to see that $f^D$ is also non-trivial, monotone and $C^R(f^D) = C^R(f)$.*

**Definition 2.6** *(i) Let $G, H \in \mathcal{G}_n$ with vertex sets $V$ and $V'$, respectively. A* packing *is an identification between $V$ and $V'$ such that no edge of $G$ is identified with any edge of $H$.*

*(ii) Let $G, H \in \mathcal{G}_{n,m}$ with colour classes $V, W$ and $V', W'$, respectively. A* bipartite packing *is an identification between $V$ and $V'$ and between $W$ and $W'$ such that no edge of $G$ is identified with any edge of $H$.*

**Lemma 2.7** *(Yao [6]) (i) If $P \in \mathcal{P}_n, G \in \min(P)$ and $H \in \min(P^D)$ then $G$ and $H$ can't be packed. (ii) If $P \in \mathcal{P}_{n,m}, G \in \min(P)$ and $H \in \min(P^D)$ then $G$ and $H$ can't be packed as bipartite graphs.*

# 3 Results

By a *covering* of a graph $G$ we mean a subset $K$ of $V$ such that any edge of $G$ is adjacent to at least one vertex in $K$. A covering $K$ is *minimal* if $G$ has no covering $K'$ with $|K'| < |K|$.

The *width* of a graph $G$ denoted by width$(G)$ is the size of a minimal covering of $G$. The *trace* of a graph $G$ denoted by trace$(G)$ is the minimal number of edges we have to remove from $G$ in order to decrease its width.

Now we extend these notions to monotone graph properties. The *width of a monotone graph property* $P$ is defined as follows:

$$width(P) = \min\{width(G) | G \in \min(P)\}$$

The *trace of a monotone graph property* $P$ is defined by

$$trace(P) = \min\{trace(G) | G \in \min(P) \text{ and } width(P) = width(G)\}$$

The following assertions show some fundamental properties of these notions.

**Lemma 3.1** *If $P \in \mathcal{P}_n$ and $1 \leq r < n$ is a fixed integer then*
*(i) $width(P) \geq r$ iff $K^*_{n+1-r} \in P^D$*
*(ii) If $width(P) > r$ then*

$$red(P|n-r) \in \mathcal{P}_{n-r}, width(red(P|n-r)) = width(P)-r, trace(red(P|n-r)) = trace(P).$$

**Lemma 3.2** *If $P \in calP_n$ and $width(P) = 1$ then for any $G \in P^D, G$ has at least $\frac{1}{2}n \cdot (n -\ trace(P))$ edges.*

**Proof.** Since $P^D$ is a non-trivial, monoton graph property, it is sufficient to prove the statement for $G \in \min(P^D)$. Indeed, let $G \in \min(P^D)$ be arbitrary and let $H \in \min(P)$ such a graph for which width$(H)$ = width$(P)$ and trace$(H)$ = trace$(P)$ holds. With other words, $H$ is a star with trace$(P)$ mary edges. According to Lemma 2.7. $G$ and $H$ can't be packed. This implies $\delta(G) \geq |V(G)| - trace(H) = n - trace(H)$, therefore $|E(G)| \geq \frac{1}{2}n \cdot (n - trace(H))$.

**Lemma 3.3** *For any $P \in \mathcal{P}_n$ the following assertions hold:*
*(i)* $C^R(P) \geq width(P) \cdot trace(P)$.
*(ii)* $C^R(P) \geq \frac{1}{2}(n+1- width(P)) \cdot (n+1-(width\ (P)+ trace(P)))$.
*(iii) For any* $0 < \varepsilon < 1$, *if* $width(P) \leq (1-\varepsilon) \cdot n$ *then* $C^R(P) \geq \frac{\varepsilon^3}{2-\varepsilon} n \cdot width(P)$.

**Proof.** Assertion (i) is a straightforward consequence of Theorem 2.3. To prove (ii) choose in Lemma 3.1. $r$ =width $(P) - 1$ and apply Lemma 3.2. to the reduced property re $\alpha(P|n - (\text{width } (P) - 1))$. Finally Theorem 2.3. yields the result. If trace$(P) \geq \frac{\varepsilon^2}{2-\varepsilon} \cdot n$, then assertion (iii) follows from (i), else it can be proved, using (ii) and assumption width$(P) \leq (1-\varepsilon) \cdot n$.

Before we state our main results we apply this method to some special graph properties. For this reason let us denote the property that an $n$-vertex graph contains a Hamiltonian cycle by $PH_n$ and the property that an $n$-vertex graph has a vertex colouring with $k$ colours by $PC_{k,n}$.

**Theorem 3.4**

$$C^R(PH_n) \geq \frac{1}{8} \cdot (n^2 - 1)$$

$$C^R(PC_{n,k}) \geq \binom{n+1-k}{2}.$$

**Proof.** We have only to determine the width and trace of the given properties. The required values are:

$$width(PH_n) = \lceil \frac{n}{2} \rceil,$$

$$trace(PH_n) = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 2, & \text{if } n \text{ is even} \end{cases}$$

Since $PC_{k,n}$ itself is not monotone, we consider instead of $PC_{k,n}$ the property $\neg PC_{k,n}$ which is non-trivial, monotone and obviously, $C^R(\neg PC_{k,n}) = C^R(PC_{k,n})$. It can be seen that the corresponding values are:

$$width(\neg PC_{k,n}) = k$$

$$trace(\neg PC_{k,n}) = 1.$$

The following theorem improves the known reductions of non-trivial, monotone graph properties to bipartite graph properties. Although King [3] has already stated a similar result, the new approach can help to prove better uniform lower bounds, since the reduction is to colour classes both of size $\Omega(n)$

**Theorem 3.5** *The randomized decision tree complexity of any non-trivial, monotone graph property* $P \in \mathcal{P}_n$ *is*

$$C^R(P) \geq \min\{\frac{1}{40} \cdot n^{3/2}, C^R(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)\}.$$

**Proof.** We have only to consider the case that the property $P$ can't be reduced to a non-trivial bipartite graph property bipart$P|\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil)$. This implies $K^*_{\lfloor\frac{n}{2}\rfloor} \in P$ or $K^*_{\lceil\frac{n}{2}\rceil} \in P^D$. Therefor, it remains only to prove, that for any $P \in \mathcal{P}_n$, if $K^*_{\lceil\frac{n}{2}\rceil} \in P$ then $C^R(P) \geq \frac{1}{40} \cdot n^{3/2}$ holds.

Let us suppose that we found a property $P \in \mathcal{P}_n$ with $K^*_{\lceil\frac{n}{2}\rceil} \in P$ and $C^R(P) < \frac{1}{40} \cdot n^{3/2}$. Let us construct the following sequence of induced graph properties

$$\{P_i | 0 \leq i \leq \lfloor\frac{1}{2}n^{1/2}\rfloor\}, P_i = ind(P|\lceil\frac{3}{4}n + \frac{1}{2}i \cdot n^{1/2}\rceil).$$

Since $K^*_{\lceil\frac{n}{2}\rceil} \in P$ and for any $i$ $P_i$ is an induced property of $P$ on at least $\lceil\frac{3}{4}n\rceil$ vertices, $P_i$ is non-trivial and

$$C^R(P_i) \leq C^R(P) < \frac{1}{40}n^{3/2} \tag{1}$$

$K^*_{\lceil\frac{n}{2}\rceil} \in P_0$ implies width$(P_0) \leq \lceil\frac{n}{2}\rceil - 1 \leq \frac{1}{2}n$. Assertion (iii) of Lemma 3.3. yields $C^R(P_0) \geq \frac{1}{20}n\cdot$ width$(P_0)$. Hence

$$width(P_0) \leq \lfloor\frac{1}{2}n^{1/2}\rfloor. \tag{2}$$

Obviously $G \in P_i$ implies $G \in P_{i+1}$. Therefore

$$width(P_{i+1}) \leq width(P_i), i \geq 0. \tag{3}$$

Let us suppose, that for some $i \geq 0$ width$(P_{i+1})$ = width$(P_i)$ holds. Then trace$(P_{i+1}) \leq$ trace$(P_i)$. Now Lemma 3.3. yields

$$\begin{aligned}
C^R(P_{i+1}) \geq\; & \frac{1}{2}(\lceil\frac{3}{4}n + \frac{1}{2}(i+1) \cdot n^{1/2}\rceil + 1 - width(P_{i+1})) \cdot \\
& (\lceil\frac{3}{4}n + \frac{1}{2}(i+1) \cdot n^{1/2}\rceil + 1 - (width(P_{i+1}) + trace(P_{i+1}))) \\
\geq\; & \frac{1}{2} \cdot (\frac{3}{4}n + \frac{1}{2}n^{1/2} - width(P_0)) \cdot \\
& (\frac{3}{4}n + \frac{1}{2}i \cdot n^{1/2} + 1 - (width(P_i) + trace(P_i)) + \frac{1}{2}n^{1/2}) \\
\geq\; & \frac{3}{16}n^{3/2} > \frac{1}{40}n^{3/2},
\end{aligned}$$

which contradicts (1).

The sequence of positive integers $\{$width $(P_i) | 0 \leq i \leq \lfloor\frac{1}{2}n^{1/2}\rfloor\}$ therefore decreases strictly monotone, and so

$$width(P_0) \geq \lfloor\frac{1}{2}n^{1/2}\rfloor + 1,$$

which contradicts (2).

Since our assumption $C^R(P) < \frac{1}{40}n^{3/2}$ led to a contradiction we have completed the proof.

A straightforward consequence of the improved reduction is the following result.

**Theorem 3.6** *For the randomized decision tree complexity of any subgraph isomorphism property $P_G \in \mathcal{P}_n$*

$$C^R(P_G) = \Omega(n^{3/2}).$$

**Proof.** According to Theorem 3.5., we have only to settle the case that $\text{bipart}(P_G | \lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$ is nontrivial. Depending on $\text{width}(P_G)$ and $\text{trace}(P_G)$ we shall distinguish three cases.

*Case 1.* Assume that $\text{width}(P_G) \geq \frac{1}{4}n$. Since $\text{bipart}(P_G | \lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$ is non-trivial, we get $\text{width}(P_G) \leq \frac{1}{2}n$ and assertion (iii) of Lemma 3.3. implies a lower bound of $\Omega(n^2)$.

*Case 2.* If $\text{width}(P_G) < \frac{1}{4}n$ and $\text{trace}(P_G) < \frac{2}{3}n$, then we can apply assertion (ii) of Lemma 2.5. and get also a lower bound of $\Omega(n^2)$.

*Case 3.* Suppose that $\text{width}(P_G) < \frac{1}{4}n$ and $\text{trace}(P_G) \geq \frac{2}{3}n$. Since $\text{trace}(P_G) \geq \frac{2}{3}n$ the corresponding bipartite graph property $\text{bipart}(P_G | \lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$ has only such minimal graphs $H$ for that $D_V(H) \geq \frac{1}{6}n$ holds. If $\text{bipart}(P_G | \lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$ has a minimal graph with at least $n^{3/2}$ edges we can apply Theorem 2.3. Otherwise we can apply Theorem 2.5. In both cases we get a lower bound of $\Omega(n^{3//2})$ which completes the proof.

Before we prove the sharper version of Theorem 2.6. we consider some special bipartite graph properties. Let us denote by $S_{n,m}$ the graph which has one vertex of positive degree in $V$ and $m$ edges.

**Lemma 3.7** *Let $P_S \in \mathcal{P}_{n,m}$ denote the property of containing a subgraph isomorph to $S_{n,m}$. Then*

$$C^R(P_S) \geq \frac{1}{2} m \cdot n.$$

**Proof.** (analogue to Yao [7]). We consider the dual property $P_S^D$, which is easy to see to contain exactly those graphs, that have no isolated nodes in colour class $W$. According to Lemma 2.2. wee choose as a "hard" input distribution the uniform distribution over all minimal graphs. Let be $A$ an optimal deterministic decision tree, that computes our $P_S^D$. We denote by $X_i(G)$ the number of edges incident to $w_i$ asked by $A$. Then

$$\begin{aligned} C^R(P_S^D) &\geq E(\sum_{i=1}^{m} X_i(G)) \\ &= \sum_{i=1}^{m} E(X_i(G)) \end{aligned}$$

Since for any value of $i$ we have to find one edge out of $n$ edges, we get

$$E(X_i(G)) \geq \frac{1}{2} n$$

and finally

$$C^R(P_S) \geq \frac{1}{2} m \cdot n.$$

**Lemma 3.8** *Let $P \in \mathcal{P}_{n,m}$ such a property, that every $G \in \min(P)$ has exactly $k \leq \frac{1}{2}n$ vertices of positive degree in colour class $V$. Then*

$$C^R(P) \geq \frac{1}{6} m \cdot n.$$

**Proof.** We consider the reduced graph property $P' = red_V(P|n+1-k)$. Obviously, $P'$ is non-trivial and $\min(P')$ contains up to isomorphy exactly one graph. This graph has exactly one vertex with positive degree $(d)$ in the colour class $V'$. We distinguish two cases.

*Case 1.* Assume that $d \leq \frac{2}{3}m$. Since the minimal graphs of $P'$ and $P'^D$ can't be packed as bipartite graphs, any $G \in \min(P'^D)$ has at least $(n+1-k)\cdot(m+1-d) \geq \frac{1}{6}m \cdot n$ edges. Hence Theorem 2.1. implies the required lower bound.

*Case 2.* If $d > \frac{2}{3}m$ then let us consider the induced property $ind_W(P'|d)$ on colour classes of size $n+1-k$ and $d$, respectively. Since $ind_W(P'|d) = S_{n+1-k,d}$, Lemma 3.7. yields the statement.

**Lemma 3.9** *The randomized decision tree complexity of any subgraph isomorphism property $P_G \in \mathcal{P}_n$ with width at most $\frac{2}{3}n$ fulfilles*

$$C^R(P_G) \geq \frac{1}{24}(n^2 - 1).$$

**Proof.** Depending on width$(P_G)$ and trace$(P_G)$ we distinguish six cases.

*Case 1.* If $\frac{1}{2}n \leq width(P_G) \leq \frac{2}{3}n$ and trace$(P_G) \geq \frac{1}{12}n$, then assertion (i) of Lemma 3.3. implies the lower bound.

*Case 2.* If $\frac{1}{2}n \leq$ width$(P_G) \leq \frac{2}{3}n$ and trace$(P_G) < \frac{1}{12}n$, then assertion (ii) of Lemma 3.3. implies the lower bound.

*Case 3.* If $K^*_{\lfloor n/2 \rfloor} \in P_G$, then width$(P_G)$+ trace$(P_G) \leq \frac{1}{n}$. Therefore, by assertion (ii) of Lemma 3.3., we obtain $C^R(P_G) \geq \frac{1}{8}n^2 \geq \frac{1}{24}n^2$.

So far we have considered all the cases, when $P_G$ can't be reduced to a non-trivial bipartite graph property bipart$(P_G)|\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$.

*Case 4.* If $\frac{n}{2} \leq$ width$(P_G) < \frac{n}{2}$, then we can apply assertion (iii) of Lemma 3.3. for $\varepsilon = \frac{1}{2}$ and get the required lower bound.

*Case 5.* If width$(P_G) < \frac{n}{4}$ and trace$(P_G) \geq \frac{5}{6}n$, then $G$ contains width$(P_G)$ vertices with degree at least $\frac{5}{9}n$. In our reduction to the bipartite graph property bipart$P_G|\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$ we have to put them all into $V$. On the other hand, these vertices build a covering of the graph $G$. Hence $G$ contains no edge independent of this vertex set. Therefore any minimal graph of the property bipart$(P_G|\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$ has exactly width$(P_G)$ vertices of positive degree in $V$ and Lemma 3.8. implies

$$C^R(P_G) \geq (bipart(P_G|\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)) \geq \frac{1}{24} \cdot (n^2 - 1)$$

*Case 6.* If width$(P_G) < \frac{n}{4}$ and trace$(P_G) < \frac{5}{9}n$, then by assertion (ii) of Lemma 3.3., we get that

$$C^R(P_G) \geq \frac{1}{2} \cdot \frac{3}{4}n \cdot (\frac{3}{4}n - \frac{5}{9}n) \geq \frac{1}{24}n^2,$$

which completes the proof.

The following statement is an immediate consequence of this theorem and generalizes the results of Yao [6].

**Assertion 3.10** *For every $\varepsilon > 0$ we can find a $\lambda > 0$ which depends only on $\varepsilon$, such that the randomized decision tree complexity of any subgraph isomorphism property $P_G \in \mathcal{P}_n$ with $\overline{d}(G) \leq \varepsilon$ fulfilles:*

$$C^R(P_G) \geq \lambda(\varepsilon) \cdot n^2.$$

After finishing this manuscript the author has learnt that M. Karpinski et al [2] independently proved Theorem 3.5.

## References

[1] P. Hajnal, Complexity of graph properties, An $\Omega n^{\frac{4}{3}})$ lower bound on the randomized complexity of graph properties. Combinatorica, 11(2) (1991), 131-143.

[2] M. Karpinski, V. King, M. Szegedy, personal communication, 1990.

[3] V. King, Lower bounds on the complexity of graph properties, Proc. 20th ACM STOC (1988), 468-476.

[4] R. Rivest and Vuillemin, On recognizing graph properties from adjacency matrices, Theor. Comp. Sci. 3 (1976), 371-384.

[5] M. Saks and A. Widgerson, Probabilistic Boolean decision trees and the complexity of evaluating game trees, Proc. 27th IEEE FOCS (1986), 29-38.

[6] A. Yao, Probabilistic computation: towards a unified measure of complexity, Proc. 18th IEEE FOCS (1977), 222-227.

[7] A. Yao, Lower bounds to randomized algorithms for graph properties, Proc. 28th IEEE FOCS (1987), 393-400.

*Received April 25, 1989.*