Acta Cybernetica 14 (2000) 541-552.

Equivalence of Mealy and Moore Automata

István Babcsányi *

Abstract

It is proved here that every Mealy automaton is a homomorphic image of a Moore automaton, and among these Moore automata (up to isomorphism) there exists a unique one which is a homomorphic image of the others. A unique simple Moore automaton **M** is constructed (up to isomorphism) in the set $MO(\mathbf{A})$ of all Moore automata equivalent to a Mealy automaton **A** such that **M** is a homomorphic image of every Moore automaton belonging to $MO(\mathbf{A})$. By the help of this construction, it can be decided in steps $|X|^k$ that automaton mappings inducing by states of a k-uniform finite Mealy [Moore] automaton are equal or not. The structures of simple k-uniform Mealy [Moore] automata are described by the results of [1]. It gives a possibility for us to get the k-uniform Mealy [Moore] automata from the simple k-uniform Mealy [Moore] automata. Based on these results, we give a construction for finite Mealy [Moore] automata.

1 Preliminaries

Let X be a nonempty set. A *Mealy automaton* (*over* X) is a system $\mathbf{A} = (A, X, Y, \delta, \lambda)$ consisting of a (nonempty) state set A, the input set X, a (nonempty) output set Y, a transition function $\delta : A \times X \to A$ and a surjective output function $\lambda : A \times X \to Y$.

A *Moore automaton* (*over* X) is a system $\mathbf{A} = (A, X, Y, \delta, \mu)$ consisting of a (nonempty) state set A, the input set X, a (nonempty) output set Y, a transition function $\delta : A \times X \to A$ and a surjective sign function $\mu : A \to Y$.

If A, X and Y are finite, the Mealy [Moore] automaton **A** is called *finite*.

For arbitrary Moore automaton $\mathbf{A} = (A, X, Y, \delta, \mu)$, the system $\mathbf{A}_\lambda = (A, X, Y, \delta, \lambda)$ with $\lambda = \mu\delta$ is a Mealy automaton over X. The Mealy automaton $\mathbf{A}_\lambda$ is called *the Mealy automaton associated with the Moore automaton* **A**. It is said that λ is *the output function of the Moore automaton* **A**. The Mealy automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$ fulfils the *Moore criterion* if

$$\delta(a_1, x_1) = \delta(a_2, x_2) \implies \lambda(a_1, x_1) = \lambda(a_2, x_2)$$

for every $a_1, a_2 \in A$ and $x_1, x_2 \in X$. If $\mu : A \to Y$ is a surjective mapping such that $\lambda = \mu\delta$, the Moore automaton $\mathbf{A}_\mu = (A, X, Y, \delta, \mu)$ is called *a Moore*

*Department of Algebra, Mathematical Institute, Technical University of Budapest, 1521 Budapest, Műegyetem rkp. 9., Hungary, E-mail: babcs@math.bme.hu

automaton associated with the Mealy automaton **A**. Furthermore, we say that μ is *a sign function of the Mealy automaton* **A**. We note that the output function λ is determined by restriction of μ to the subset $\delta(A,X) = \{\delta(a,x); a \in A, x \in X\}$ of A. Thus, the restrictions of all sign functions of the Mealy automaton **A** to $\delta(A,X)$ are equal. The Mealy automaton $\mathbf{A} = (A,X,Y,\delta,\lambda)$ is called *real* if there exist $a_1, a_2 \in A$ and $x_1, x_2 \in X$ such that

$$\delta(a_1,x_1) = \delta(a_2,x_2) \quad \text{and} \quad \lambda(a_1,x_1) \neq \lambda(a_2,x_2).$$

Let Z^* and Z^+ denote the free monoid and the free semigroup over a nonempty set Z, respectively. If $\mathbf{A} = (A,X,Y,\delta,\lambda)$ is a Mealy automaton, the functions δ and λ can be extended to $A \times X^*$ in the usual forms as follows:

$$\delta(a,e) = a, \quad \delta(a,px) = \delta(a,p)\delta(ap,x),$$

$$\lambda(a,e) = e, \quad \lambda(a,px) = \lambda(a,p)\lambda(ap,x),$$

where $a \in A$, $p \in X^+$, ap denotes the last letter of $\delta(a,p)$ and e denotes the empty word. ([5], [2]). If $\mathbf{A} = (A,X,Y,\delta,\mu)$ is a Moore automaton, the extension of δ is similar to the case when **A** is a Mealy automaton. The extension of μ to A^+ is given by

$$\mu(a_1a_2\ldots a_k) = \mu(a_1)\mu(a_2)\ldots\mu(a_k) \quad (a_1,a_2,\ldots,a_k \in A).$$

It means that if $\lambda = \mu\delta$, then

$$\lambda(a,p) = \mu(\delta(a,p)),$$

for all $a \in A$, $p \in X^+$. But $\lambda(a,e) = e$ and $\mu(\delta(a,e)) = \mu(a)$ for all $a \in A$.

The Mealy [Moore] automaton $\mathbf{A}' = (A',X,Y',\delta',\lambda'[\mu'])$ is a *subautomaton* of the Mealy [Moore] automaton **A** if $A' \subseteq A$, $Y' \subseteq Y$, δ' and λ' $[\mu']$ are restrictions of δ and λ $[\mu]$ to $A' \times X$ $[A']$.

Let $\mathbf{A}_i = (A_i,X,Y,\delta_i,\lambda_i[\mu_i])$ $(i = 1,2)$ be arbitrary Mealy [Moore] automata over X. We say that a mapping $\varphi : A_1 \to A_2$ is a *homomorphism* of $\mathbf{A}_1$ into $\mathbf{A}_2$ if

$$\varphi(\delta_1(a,x)) = \delta_2(\varphi(a),x), \quad \lambda_1(a,x) = \lambda_2(\varphi(a),x) \quad [\mu_1(a) = \mu_2(\varphi(a))]$$

for all $a \in A$ and $x \in X$. It is easy to see that

$$\lambda_1(a,p) = \lambda_2(\varphi(a),p)$$

for all $p \in X^*$. The mapping $\varphi : A_1 \to A_2$ is called a *homomorphism* of a Moore automaton $\mathbf{A}_1$ into a Mealy automaton $\mathbf{A}_2$ if φ is a homomorphism of $(\mathbf{A}_1)_\lambda$ into $\mathbf{A}_2$. We note that every homomorphic image of a real Mealy automata is real, too.

Every state $a \in A$ of a Mealy automaton **A** induces a mapping $\alpha_a : X^* \to Y^*$ given by $\alpha_a(p) = \lambda(a,p)$ $(p \in X^*)$. The mapping $\alpha : X^* \to Y^*$ is called *automaton mapping* if there exist a Mealy automaton **A** and a state $a \in A$ such that $\alpha = \alpha_a$. The mapping $\alpha : X^* \to Y^*$ is an automaton mapping if and only if it preserves the length of words and the map of every prefix of a word is a prefix of the image word. The Mealy automata **A** and **B** are called *equivalent* if $\{\alpha_a; a \in A\} = \{\alpha_b; b \in B\}$. The Mealy automaton **A** and the Moore automaton **B** are *equivalent* if **A** and $\mathbf{B}_\lambda$ are equivalent. Similarly, the Moore automata **A** and **B** are *equivalent* if $\mathbf{A}_\lambda$ and $\mathbf{B}_\lambda$ are equivalent.

An equivalence relation ρ of state set A of a Mealy [Moore] automaton **A** is called a *congruence* on **A** if

$$(a,b) \in \rho \Longrightarrow (\delta(a,x), \delta(b,x)) \in \rho, \quad \lambda(a,x) = \lambda(b,x) \quad [\mu(a) = \mu(b)]$$

for all $a, b \in A$ and $x \in X$. The ρ-class of **A** containing the state a is denoted by $\rho[a]$. The greatest congruence on **A** is the relation $\rho_{\mathbf{A}}$ $[\pi_{\mathbf{A}}]$ defined by

$$(a,b) \in \rho_{\mathbf{A}}[\pi_{\mathbf{A}}] \quad \Longleftrightarrow \quad \lambda(a,p) = \lambda(b,p) \quad [\mu(\delta(a,p)) = \mu(\delta(b,p))]$$

for all $p \in X^*$. Denoting the identity relation on the state set A by ι_A, we say that **A** is *simple* if $\rho_{\mathbf{A}} = \iota_A$ $[\pi_{\mathbf{A}} = \iota_A]$, that is, **A** and $\mathbf{A}/\rho_{\mathbf{A}}$ $[\mathbf{A}/\pi_{\mathbf{A}}]$ are isomorphic.

Since every homomorphic image of a Mealy automaton **A** is equivalent to **A** ([5], [7]), therefore we can give the automaton mappings with simple Mealy automata. The Mealy automata **A** and **B** are equivalent if and only if $\mathbf{A}/\rho_{\mathbf{A}}$ and $\mathbf{B}/\rho_{\mathbf{B}}$ are isomorphic ([5]). Thus, simple Mealy automata are equivalent if and only if they are isomorphic. For every Mealy automaton **A**, there exists a Moore automaton **B** such that **A** and **B** are equivalent ([4], [5], [6]). From this it follows that we can give the automaton mappings by simple Moore automata.

2 Moore automata equivalent to a Mealy automaton

For a Mealy automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$ over X, let us denote by $\mathbf{A}_Y = (A \times Y, X, Y, \delta_Y, \mu_Y)$ the Moore automaton over X for which

$$\delta_Y((a,y),x) = (\delta(a,x), \lambda(a,x)) \quad \text{and} \quad \mu_Y(a,y) = y \quad (a \in A, y \in Y, x \in X).$$

If $\lambda_Y = \mu_Y \delta_Y$, then

$$\lambda_Y((a,y),x) = \mu_Y(\delta_Y((a,y),x)) = \mu_Y(\delta(a,x), \lambda(a,x)) = \lambda(a,x)$$

for every $a \in A, y \in Y, x \in X$, and hence, $\mathbf{A}_Y$ is equivalent to **A**.

Lemma 1 *If the Mealy automaton $\mathbf{A}'$ is a homomorphic [isomorphic] image of the Mealy automaton $\mathbf{A}$, then $\mathbf{A}'_Y$ is a homomorphic [isomorphic] image of $\mathbf{A}_Y$.*

Proof. If φ is a homomorphism [isomorphism] of **A** onto $\mathbf{A}'$, the mapping $\psi : A \times Y \to A' \times Y$, such that

$$\psi(a,y) = (\varphi(a), y) \quad (a \in A, \ y \in Y),$$

is a homomorphism [isomorphism] of $\mathbf{A}_Y$ onto $\mathbf{A}'_Y$.

Consider the subautomata $\mathbf{M} = (M, X, Y, \delta'_Y, \mu'_Y)$ of $\mathbf{A}_Y$ where for every $a \in A$ there exists $y \in Y$ such that $(a, y) \in M$. Let $M(\mathbf{A})$ be the set of all such subautomata $\mathbf{M}$.

Lemma 2 *The Mealy automaton A is a homomorphic image of every automaton M in $M(A)$.*

Proof. It is easy to see that the mapping $\varphi : M \to A$, defined by $\varphi(a, y) = a$ $(a \in A)$, is a homomorphism of $\mathbf{M}_\lambda$ onto $\mathbf{A}$.

Theorem 1 *The Mealy automaton $A_1 = (A_1, X, Y, \delta_1, \lambda_1)$ is a homomorphic image of a Moore automaton $A_2 = (A_2, X, Y, \delta_2, \mu_2)$ if and only if there exists a homomorphic image of A_2 in $M(A_1)$.*

Proof. First, we note that every automaton $\mathbf{M} \in M(\mathbf{A}_1)$ is a Moore automaton. By Lemma 2, if there exists a homomorphic image of $\mathbf{A}_2$ in $M(\mathbf{A}_1)$, then $\mathbf{A}_1$ is a homomorphic image of $\mathbf{A}_2$.

Conversely, assume that φ is a homomorphism of the Moore automaton $\mathbf{A}_2$ onto the Mealy automaton $\mathbf{A}_1$. It is evident that by the state set $M = \{(\varphi(b), \mu_2(b)); b \in A_2\}$,

$$\mathbf{M} = (M, X, Y, \delta'_Y, \mu'_Y) \in M(\mathbf{A}_1).$$

We show that the mapping $\psi : A_2 \to M$, defined by

$$\psi(b) = (\varphi(b), \mu_2(b)) \quad (b \in A_2),$$

is a homomorphism of $\mathbf{A}_2$ onto $\mathbf{M}$. It is obvious that the mapping ψ is surjective. For every $b \in A_2$ and $x \in X$

$$\psi(\delta_2(b, x)) = (\varphi(\delta_2(b, x)), \mu_2(\delta_2(b, x))) = (\delta_1(\varphi(b), x), \lambda_2(b, x)) =$$

$$= (\delta_1(\varphi(b), x), \lambda_1(\varphi(b), x)) = \delta'_Y((\varphi(b), \mu_2(b)), x) = \delta'_Y(\psi(b), x),$$

$$\mu_2(b) = \mu'_Y(\varphi(b), \mu_2(b)) = \mu'_Y(\psi(b)).$$

Therefore, ψ is a homomorphism.

Theorem 2 *For every Mealy automaton A (up to isomorphism) there exists a unique automaton $M \in M(A)$ which is a homomorphic image of any automaton in $M(A)$.*

Proof. First, we give the automaton $\mathbf{M}$. If $A \neq \delta(A, X)$, let κ be a mapping of $A \setminus \delta(A, X)$ into Y. For all $a \in A$, consider the sets $Y_a \subseteq Y$ such that

$$\lambda(b, x) \in Y_a \iff \delta(b, x) = a \quad (b \in A, x \in X).$$

We define the sets M_a $(a \in A)$ as follows. If $a \in \delta(A, X)$, let $M_a = \{(a, y); y \in Y_a\}$, and if $a \notin \delta(A, X)$, let $M_a = \{(a, \kappa(a))\}$. Let $M = \cup_{a \in A} M_a$. Then $\mathbf{M} =$

$(M, X, Y, \delta_Y, \mu_Y) \in M(\mathbf{A})$. Let $M'(\mathbf{A})$ be the set of all such automata $\mathbf{M}$. If $A = \delta(A, X)$, then $|M'(\mathbf{A})| = 1$. We show that if $A \neq \delta(A, X)$, then all automata in $M'(\mathbf{A})$ are isomorphic. Assume that κ_i $(i = 1, 2)$ are arbitrary mappings of $A \setminus \delta(A, X)$ into Y and the automaton $\mathbf{M}_i \in M'(\mathbf{A})$ is defined by the mapping κ_i. It can be easily verified that the mapping $\varphi : M_1 \to M_2$, defined by

$$\varphi(a, y) = \begin{cases} (a, y) & \text{if } y \neq \kappa_1(a); \\ (a, \kappa_2(a)) & \text{if } y = \kappa_1(a), \end{cases}$$

is an isomorphism of $\mathbf{M}_1$ onto $\mathbf{M}_2$.

Now we show that for every $\mathbf{B} \in M(\mathbf{A})$, there is an $\mathbf{M} \in M'(\mathbf{A})$ such that $\mathbf{M}$ is a homomorphic image of $\mathbf{B}$. We define the following partition of the state set B:

$$B_a = \{(a, y); (a, y) \in B\} \quad (a \in A).$$

Take an automaton $\mathbf{M} \in M'(\mathbf{A})$ such that $M_a \subseteq B_a$ $(a \in A)$. By the definition of $M'(\mathbf{A})$, one can see that there exists such an automaton $\mathbf{M}$. Let ψ be an arbitrary mapping of B onto M for which

$$\{\psi(b); b \in B_a\} = M_a \quad \text{and} \quad \forall b \in M_a : \psi(b) = b.$$

It is clear that ψ is a homomorphism of $\mathbf{B}$ onto $\mathbf{M}$.

Lemma 3 ([7]) Let $\mathbf{A}$ be a Mealy automaton and $ME(\mathbf{A})$ be the set of all Mealy automata equivalent to $\mathbf{A}$. Then (up to isomorphism) there exists a unique simple Mealy automaton in $ME(\mathbf{A})$ which is a homomorphic image of every automaton in $ME(\mathbf{A})$.

We have a similar statement for Moore automata which are equivalent to a Mealy automaton.

Theorem 3 *Let $\mathbf{A}$ be a Mealy automaton and $MO(\mathbf{A})$ be the set of all Moore automata which are equivalent to $\mathbf{A}$. Then (up to isomorphism) there exists a unique simple Moore automaton in $MO(\mathbf{A})$ which is a homomorphic image of each automaton in $MO(\mathbf{A})$.*

Proof. Let $\mathbf{A}_0$ denote a simple Mealy automaton in $ME(\mathbf{A})$ which is homomorphic image of any automaton in $ME(\mathbf{A})$. By Lemma 3, such an automaton exists. Moreover, by Theorem 2, (up to isomorphism) there is a unique Moore automaton $\mathbf{M}_0 \in M(\mathbf{A}_0)$ which is homomorphic image of any automaton in $M(\mathbf{A}_0)$. Using the last fact, it can be seen that $\mathbf{M}_0$ is a simple Moore automaton.

Now, let $\mathbf{B}$ be an arbitrary Moore automaton equivalent to $\mathbf{A}$. We prove that $\mathbf{M}_0$ is a homomorphic image of $\mathbf{B}$. Since $\mathbf{B}$ is equivalent $\mathbf{A}$, $\mathbf{B}_\lambda \in ME(\mathbf{A})$, and hence, $\mathbf{A}_0$ is a homomorphic image of $\mathbf{B}$. This implies, by Theorem 1, that there is an $\mathbf{M} \in M(\mathbf{A}_0)$ such that $\mathbf{M}$ is a homomorphic image of $\mathbf{B}$, and therefore, $\mathbf{M}_0$ is a homomorphic image of $\mathbf{B}$ as well.

3 Uniform automata

Let $\mathbf{A} = (A, X, Y, \delta, \lambda[\mu])$ be a Mealy [Moore] automaton over X. Denote by $|p|$ the length of the word $p \in X^*$. Let $X^k = \{p \in X^*; |p| = k\}$ and $X(k) = \{p \in X^*; |p| \leq k\}$. For every nonnegative integer k, we define the equivalence relation η_k on A as follows:

$$(a,b) \in \eta_k \iff \lambda(a,p) = \lambda(b,p) \ [\mu(\delta(a,p)) = \mu(\delta(b,p))]$$

for all $p \in X(k)$. We note that if $\mathbf{A}$ is a Mealy automaton, the relation η_0 is the universal relation on A and η_1 is the output-equivalence of $\mathbf{A}$ ([2]). If $\mathbf{A}$ is a Moore automaton, η_0 is the sign-equivalence of $\mathbf{A}$ ([3]).

Lemma 4 *If a and b are arbitrary states of a Mealy [Moore] automaton $\mathbf{A} = (A, X, Y, \delta, \lambda[\mu])$, then*

$$(a,b) \in \eta_k \iff \lambda(a,p) = \lambda(b,p) \quad [\mu(a) = \mu(b),\ \lambda(a,p) = \lambda(b,p)]$$

for all $p \in X^k$.

Proof. If $(a,b) \in \eta_k$, the statement follows from the definition of η_k.

Conversely, assume that if $\mathbf{A}$ is a Mealy automaton, $\lambda(a,p) = \lambda(b,p)$, and if $\mathbf{A}$ is a Moore automaton, then $\mu(a) = \mu(b)$, $\lambda(a,p) = \lambda(b,p)$ holds for every $p \in X^k$. Take arbitrary words $q, r \in X^*$ such that $|q| \leq k$ and $|r| = k - |q|$. Then

$$\lambda(a,q)\lambda(aq,r) = \lambda(a,qr) = \lambda(b,qr) = \lambda(b,q)\lambda(bq,r).$$

Thus, $\lambda(a,q) = \lambda(b,q)$, which implies our statement.

The Mealy [Moore] automaton $\mathbf{A}$ is called *k-uniform* if $\eta_k = \rho_{\mathbf{A}}[\pi_{\mathbf{A}}]$. The k-uniform Mealy [Moore] automata are $(k+1)$-uniform. Every subautomaton of a k-uniform Mealy [Moore] automaton is k-uniform, too. An arbitrary homomorphic image of a Mealy [Moore] automaton is k-uniform if and only if it is k-uniform. The Mealy [Moore] automaton is said to be *uniform* if there exists a positive integer k such that it is k-uniform. Every finite Mealy [Moore] automaton is k-uniform for some positive integer k. Let α_a and α_b be automaton mappings induced by states a and b of a k-uniform finite Mealy [Moore] automaton $\mathbf{A} = (A, X, Y, \delta, \lambda[\mu])$, respectively. If $\alpha_a(p) = \alpha_b(p)$ for every $p \in X^k$, then $\alpha_a = \alpha_b$. Thus, it can be decided in $|X|^k$ steps whether two automaton mappings of this kind are equal or not.

Theorem 4 *If the Moore automaton $\mathbf{A} = (A, X, Y, \delta, \mu)$ is k-uniform, the Mealy automaton $\mathbf{A}_\lambda$ is (k+1)-uniform.*

Proof. We note that $\mathbf{A}_\lambda$ is (k+1)-uniform if and only if $\rho_{\mathbf{A}_\lambda} = \zeta_{k+1}$, where $(a,b) \in \zeta_{k+1}$ $(a, b \in A)$ if and only if $\lambda(a,p)) = \lambda(b,p))$ for all $p \in X(k+1)$.

Let the Moore automaton $\mathbf{A} = (A, X, Y, \delta, \mu)$ be k-uniform, that is, $\eta_k = \pi_{\mathbf{A}}$. Assume that $(a, b) \in \zeta_{k+1}$. Then,

$$\mu(\delta(a, x)) = \lambda(a, x) = \lambda(b, x) = \mu(\delta(b, x)),$$

$$\mu(\delta(\delta(a, x), q)) = \lambda(\delta(a, x), q) = \lambda(\delta(b, x), q) = \mu(\delta(\delta(b, x), q))$$

for every $x \in X$, $q \in X^k$. Thus, by Lemma 4, $(\delta(a, x), \delta(b, x)) \in \eta_k = \pi_{\mathbf{A}}$. This yields that

$$\lambda(\delta(a, x), r) = \mu(\delta(\delta(a, x), r)) = \mu(\delta(\delta(b, x), r)) = \lambda(\delta(b, x), r)$$

for all $r \in X^+$. Therefore, $(\delta(a, x), \delta(b, x)) \in \zeta_{k+1}$, that is, ζ_{k+1} is a congruence on $\mathbf{A}_\lambda$. Thus, $\zeta_{k+1} = \rho_{\mathbf{A}_\lambda}$. From this we get that $\mathbf{A}_\lambda$ is (k+1)-uniform.

Theorem 5 *The Mealy [Moore] automaton $\mathbf{A} = (A, X, Y, \delta, \lambda[\mu])$ is k-uniform if and only if $\eta_k = \eta_{k+1}$.*

Proof. Assume that the Mealy [Moore] automaton $\mathbf{A}$ is k-uniform, that is, $\eta_k = \rho_{\mathbf{A}}$. Since $\eta_{k+1} \subseteq \eta_k$ and $\cap_{k=0}^{\infty} \eta_k = \rho_{\mathbf{A}}[\pi_{\mathbf{A}}]$, therefore $\eta_k = \eta_{k+1}$.

Conversely, assume that $\eta_k = \eta_{k+1}$. If $\mathbf{A}$ is a Mealy automaton, η_0 is the universal relation on A. If $\eta_0 = \eta_1$, the relation η_1 is a congruence on $\mathbf{A}$. It yields that $\eta_0 = \eta_1 = \rho_{\mathbf{A}}$. Furthermore let us assume that $\mathbf{A}$ is a Mealy automaton and $1 \leq k$. Let $(a, b) \in \eta_k$. Since $\eta_k = \eta_{k+1}$, then $(a, b) \in \eta_{k+1}$. By Lemma 4, $\lambda(a, xp) = \lambda(b, xp)$ for every $x \in X$ and $p \in X^k$. From this it follows that

$$\lambda(\delta(a, x), p) = \lambda(\delta(b, x), p).$$

Moreover, if $\mathbf{A}$ is a Moore automaton,

$$\mu(\delta(a, x)) = \lambda(a, x) = \lambda(b, x) = \mu(\delta(b, x)),$$

that is, $(\delta(a, x), \delta(b, x)) \in \eta_k$. This results in that η_k is a congruence on $\mathbf{A}$, and so $\eta_k = \rho_{\mathbf{A}}[\pi_{\mathbf{A}}]$. Hence, $\mathbf{A}$ is k-uniform.

Lemma 5 *If a and b are arbitrary states of a Mealy [Moore] automaton $\mathbf{A} = (A, X, Y, \delta, \lambda[\mu])$, then*

$$(a, b) \in \eta_{k+1} \iff (a, b) \in \eta_k \quad \textit{and} \quad (\delta(a, x), \delta(b, x)) \in \eta_k, \quad \textit{for all} \quad x \in X.$$

Proof. Assume that $(a, b) \in \eta_{k+1}$. Since $\eta_{k+1} \subseteq \eta_k$, then $(a, b) \in \eta_k$. By Lemma 4, $\lambda(a, xp) = \lambda(b, xp)$ for every $x \in X$ and $p \in X^k$. But

$$\lambda(a, x)\lambda(\delta(a, x), p) = \lambda(a, xp) = \lambda(b, xp) = \lambda(b, x)\lambda(\delta(b, x), p),$$

and so

$$\lambda(\delta(a, x), p) = \lambda(\delta(b, x), p).$$

Moreover, if $\mathbf{A}$ is a Moore automaton,

$$\mu(\delta(a,x)) = \lambda(a,x) = \lambda(b,x) = \mu(\delta(b,x)).$$

By Lemma 4, this yields that $(\delta(a,x),\delta(b,x)) \in \eta_k$.

Conversely, assume that $(a,b) \in \eta_k$ and $(\delta(a,x),\delta(b,x)) \in \eta_k$ for every $x \in X$. If $x \in X$ and $q \in X^k$, then $\lambda(a,x) = \lambda(b,x)$ and $\lambda(\delta(a,x),q) = \lambda(\delta(b,x),q)$. From this it follows that

$$\lambda(a,xq) = \lambda(a,x)\lambda(\delta(a,x)q) = \lambda(b,x)\lambda(\delta(b,x),q) = \lambda(b,xq).$$

Moreover, if $\mathbf{A}$ is a Moore automaton, $\mu(a) = \mu(b)$. By Lemma 4, $(a,b) \in \eta_{k+1}$.

Theorem 6 *For every Mealy automaton* $\mathbf{A} = (A,X,Y,\delta,\lambda)$, $\mathbf{A}_Y$ *is k-uniform [simple] if and only if* $\mathbf{A}$ *is k-uniform [simple].*

Proof. If $a \in A$, $y \in Y$ and $p \in X^+$, then $\mu_Y(\delta_Y((a,y),p)) = \lambda(a,p)$.

We note that $\mathbf{A}_Y$ is k-uniform if $\pi_{\mathbf{A}_Y} = \zeta_k$, where ζ_k is an equivalence relation on $A \times Y$ for which

$$((a,y_1),(b,y_2)) \in \zeta_k \iff \mu_Y(\delta_Y((a,y_1),p)) = \mu_Y(\delta_Y((b,y_2),p))$$

for all $p \in X(k)$.

Assume that the Mealy automaton $\mathbf{A}$ is k-uniform. Consider two arbitrary elements (a,y_1) and (b,y_2) of $A \times Y$ with $((a,y_1),(b,y_2)) \in \zeta_k$. Then

$$y_1 = \mu_Y(a,y_1) = \mu_Y(b,y_2) = y_2,$$

$$\lambda(a,p) = \mu_Y(\delta_Y((a,y_1),p)) = \mu_Y(\delta_Y((b,y_2),p)) = \lambda(b,p)$$

for all $p \in X^k$. By Lemma 4, this implies $(a,b) \in \eta_k = \rho_{\mathbf{A}}$. By Theorem 5, $(a,b) \in \eta_{k+1}$, that is,

$$\mu_Y(\delta_Y((a,y_1),p)) = \lambda(a,p) = \lambda(b,p) = \mu_Y(\delta_Y((b,y_2),p))$$

for all $p \in X^{k+1}$ which results in $(a,b) \in \zeta_{k+1}$. Thus, $\zeta_k = \zeta_{k+1}$. By Theorem 5, $\mathbf{A}_Y$ is k-uniform.

Conversely, assume that $\mathbf{A}_Y$ is k-uniform. Let $(a,b) \in \eta_k$. If $y \in Y$, then $((a,y),(b,y)) \in \zeta_k = \pi_{\mathbf{A}_Y}$. By Theorem 5, $((a,y),(b,y) \in \zeta_{k+1}$, and thus $(a,b) \in \eta_{k+1}$. Therefore, $\eta_k = \eta_{k+1}$, that is, $\mathbf{A}$ is k-uniform.

We can prove, in a similar way, that $\mathbf{A}_Y$ is simple if and only if $\mathbf{A}$ is simple (see Lemma 2 in [1]).

By Theorem 6 and Lemma 2, every k-uniform Mealy automaton is equivalent to a k-uniform Moore automaton. By Theorem 3, among these Moore automata (up to isomorphism) there exists a unique simple k-uniform Moore automaton which is a homomorphic image of these Moore automata, that is, the cardinality of its state set is the least among these Moore automata.

4 Simple uniform automata

In this part of the paper, we describe the structure of the simple uniform Mealy [Moore] automata using the results of paper [1].

Lemma 6 (Lemma 3 in [1]) Every subautomaton of a simple Mealy [Moore] automaton **A** over X is simple and the subautomata of **A** are isomorphic if and only if they are equal.

Denote the set of mappings $\alpha^{(i)} : X^i \to Y$ by $\mathcal{A}^{(i)}$ for every integer $i > 0$. Consider the set $\mathcal{A} = \prod_{i=1}^{\infty} \mathcal{A}^{(i)}$. Let

$$\alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(i)}, \ldots) \quad (\alpha^{(i)} \in \mathcal{A}^{(i)}),$$

$$\alpha_x^{(i)}(x_1, x_2, \ldots, x_i) = \alpha^{(i+1)}(x, x_1, x_2, \ldots, x_i) \quad (x, x_1, x_2, \ldots, x_i \in X),$$

$$\alpha_x = (\alpha_x^{(1)}, \alpha_x^{(2)}, \ldots, \alpha_x^{(i)}, \ldots).$$

Assume that $\alpha_e = \alpha$ and let $\alpha_{px} = (\alpha_p)_x$ for every $p \in X^*$ and $x \in X$. Define the Mealy automaton $\underline{A} = (\mathcal{A}, X, Y, \delta, \lambda)$ with transition and output functions:

$$\delta(\alpha, x) = \alpha_x, \quad \lambda(\alpha, x) = \alpha^{(1)}(x) \quad (\alpha \in \mathcal{A},\ x \in X).$$

Theorem 7 (Theorem 4 in [1]) The Mealy automaton $\underline{A}$ is simple. A Mealy automaton $\mathbf{A} = (A, X, Y', \delta, \lambda)$ over X is simple if and only if it is isomorphic to a subautomaton of $\underline{A}$, where $Y' \subseteq Y$.

Theorem 8 (Theorem 5 in [1]) The Moore automaton $\underline{A}_Y$ is simple and $\underline{A}$ is a homomorpic image of $\underline{A}_Y$. A Moore automaton $\mathbf{A} = (A, X, Y', \delta, \mu)$ $(Y' \subseteq Y)$ over X is simple if and only if it is isomorphic to a subautomaton of $\underline{A}_Y$.

Consider the set $\mathcal{A}_k = \prod_{i=1}^{k} \mathcal{A}^{(i)}$ and a mapping $g : \mathcal{A}_k \to \mathcal{A}^{(k+1)}$. Let

$$\alpha_k = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}) \quad (\alpha^{(i)} \in \mathcal{A}^{(i)}),$$

$$\alpha_{k,g,x} = (\alpha_x^{(1)}, \alpha_x^{(2)}, \ldots, \alpha_x^{(k)}),$$

where $\alpha^{(k+1)} = g(\alpha_k)$.

We define the Mealy automaton $\underline{A}_{k,g} = (\mathcal{A}_k, X, Y, \delta, \lambda)$ with the following transition and output functions:

$$\delta(\alpha_k, x) = \alpha_{k,g,x}, \quad \lambda(\alpha_k, x) = \alpha^{(1)}(x) \quad (\alpha_k \in \mathcal{A}_k,\ x \in X).$$

Consider a nonempty set $H_0 \subseteq \mathcal{A}_k$. It is evident that

$$H_j = \{\alpha_{k,g,x}; \alpha_k \in H_{j-1}, x \in X\} \subseteq \mathcal{A}_k \ (j = 1, 2, \ldots).$$

If $H^{(j)} = H_0 \cup H_1 \cup \ldots \cup H_j$ for every nonnegative integer j, then $\mathbf{H}^{(j)}$ is a subautomaton of $\underline{A}_{k,g}$ if and only if $H^{(j+1)} \subseteq H^{(j)}$. We note that if X and Y are finite sets, then there exists a nonnegative integer j such that $H^{(j+1)} \subseteq H^{(j)}$.

Theorem 9 *A Mealy automaton* $\mathbf{A}$ *over* X *is simple k-uniform if and only if there exists a mapping* $g : \mathcal{A}_k \to \mathcal{A}^{(k+1)}$ *such that* $\mathbf{A}$ *is isomorphic to some subautomaton of* $\underline{\mathcal{A}}_{k,g}$.

Proof. As in the proof of Theorem 7, we can show that the Mealy automaton $\underline{\mathcal{A}}_{k,g}$ is simple. By Lemma 6, every subautomaton of $\underline{\mathcal{A}}_{k,g}$ is simple. On the other hand, it is easy to verify that the subautomata of $\underline{\mathcal{A}}_{k,g}$ are k-uniform.

Therefore, by Theorem 7, it is sufficient to show that every k-uniform subautomaton of $\underline{\mathcal{A}}$ is isomorphic to an automaton $\mathbf{H}^{(j)}$. Let $\underline{\mathcal{A}}' = (\mathcal{A}', X, Y', \delta_{\mathcal{A}'}, \lambda_{\mathcal{A}'})$ be a k-uniform subautomaton of $\underline{\mathcal{A}}$. Let

$$\alpha_k = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)})$$

for every $\alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}, \ldots) \in \mathcal{A}'$. Define a mapping $g : \mathcal{A}_k \to \mathcal{A}^{(k+1)}$ such that $g(\alpha_k) = \alpha^{(k+1)}$ for every $\alpha \in \mathcal{A}'$. Let $H_0 = \{\alpha_k; \alpha \in \mathcal{A}'\}$. Since $\underline{\mathcal{A}}'$ is a subautomaton of $\underline{\mathcal{A}}$, then $H_1 \subseteq H_0$. Thus, $\mathbf{H}^{(0)}$ is a subautomaton of $\underline{\mathcal{A}}_{k,g}$. The mapping $\varphi : \mathcal{A}' \to H_0$, for which $\varphi(\alpha) = \alpha_k$ $(\alpha \in \mathcal{A}')$, is an isomorphism of $\underline{\mathcal{A}}'$ onto $\mathbf{H}^{(0)}$.

Every finite Mealy [Moore] automaton is k-uniform for some nonnegative integer k. Thus, we get easily the following theorem from Theorem 9.

Theorem 10 *A finite Mealy automaton* $\mathbf{A}$ *over* X *is simple if and only if there exist a nonnegative integer* k *and a mapping* $g : \mathcal{A}_k \to \mathcal{A}^{(k+1)}$ *for which* $\mathbf{A}$ *is isomorphic to some subautomaton of* $\underline{\mathcal{A}}_{k,g}$.

By Theorems 6, 9 and 10, the following two theorems are true.

Theorem 11 *A Moore automaton* $\mathbf{A}$ *over* X *is simple k-uniform if and only if it is isomorphic to some subautomaton of* $\underline{(\mathcal{A}_{k,g})}_Y$.

Theorem 12 *A finite Moore automaton* $\mathbf{A}$ *over* X *is simple if and only if there exists a nonnegative integer* k *for which it is isomorphic to some subautomaton of* $\underline{(\mathcal{A}_{k,g})}_Y$.

Let $\underline{C} = (C, X, Y', \delta_C, \lambda_C)$ be a subautomaton of the automaton $\underline{\mathcal{A}}$. Consider a family of nonempty sets U_α $(\alpha \in C)$ such that $U_\alpha \cap U_\beta = \emptyset$ if $\alpha \neq \beta$. Let $U_C = \cup_{\alpha \in C} U_\alpha$. For all $x \in X$ and $\alpha \in C$, let $\varphi_{\alpha,x}$ be a mapping of U_α into U_{α_x}. Define the functions $\delta_{U_C}(a, x) = \varphi_{\alpha,x}(a)$ and $\lambda_{U_C}(a, x) = \alpha^{(1)}(x)$ for all $a \in U_\alpha$, $\alpha \in C$ and $x \in X$. It can be easily verified that $\mathbf{U}_C = (U_C, X, Y', \delta_{U_C}, \lambda_{U_C})$ is a Mealy automaton ([2]).

Lemma 7 *Every Mealy automaton* $\mathbf{A} = (A, X, Y', \delta, \lambda)$ $(Y' \subseteq Y)$ *equals an automaton* $\mathbf{U}_C$.

Proof. By Theorem 7, there exists an isomorphism φ of $\mathbf{A}/\rho_{\mathbf{A}}$ onto a subautomaton $\underline{C}$ of $\underline{A}$. Assume that $\varphi(\rho_{\mathbf{A}}[a]) = \alpha_a$, $U_{\alpha_a} = \rho_{\mathbf{A}}[a]$ $(a \in A)$, $\varphi_{\alpha_a,x} = \delta(a,x)$ and $U_C = \cup_{a\in A} U_{\alpha_a}$. Since

$$\lambda(a,x) = \lambda_{A/\rho_{\mathbf{A}}}(\rho_{\mathbf{A}}[a],x) = \lambda_{\underline{C}}(\alpha_a,x) = \alpha_a^{(1)}(x) = \lambda_{U_C}(a,x),$$

therefore $\mathbf{A} = \mathbf{U}_C$.

Theorem 13 *The automaton U_C is k-uniform if and only if $\underline{C}$ is simple k-uniform.*

Proof. It is evident that if the automaton $\mathbf{U}_C$ is k-uniform, then $\underline{C}$ is simple k-uniform.

Conversely, assume that the automaton $\underline{C}$ is simple k-uniform. Assume that $(a,b) \in \eta_k$ for some $a \in U_\alpha$ and $b \in U_\beta$. Then, by Lemma 4, for every $p \in X^k$

$$\lambda_C(\alpha,p) = \lambda_{U_C}(a,p) = \lambda_{U_C}(b,p) = \lambda_C(\beta,p).$$

But $\underline{C}$ is simple k-uniform, thus $\alpha = \beta$, that is, $a,b \in U_\alpha$. It means that $ap, bp \in U_{\alpha_p}$. Then, for all $x \in X$,

$$\lambda_{U_C}(a,px) = \lambda_{U_C}(a,p)\lambda_{U_C}(ap,x) = \lambda_{U_C}(b,p)\lambda_{U_C}(bp,x) = \lambda_{U_C}(b,px),$$

that is $(a,b) \in \eta_{k+1}$. By Theorem 5, $\mathbf{U}$ is a k-uniform automaton.

By Theorems 6 and 13, we get the following theorem:

Theorem 14 *The automaton $(U_C)_Y$ is k-uniform if and only if $\underline{C}_Y$ is simple k-uniform.*

By Theorems 10 and 12, we give a construction for finite simple Mealy and Moore automata. Thus, by using Theorems 13 and 14, we can give all finite Mealy and Moore automata.

Acknowledgement. The author express his thank to B. Imreh for his valuable comments on the original version of the manuscript.

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Received January, 2000