Acta Cybernetica **18** (2007) 61–75.

Functional Equations, Constraints, Definability of Function Classes, and Functions of Boolean Variables*

Miguel Couceiro† and Stephan Foldes‡

Abstract

The paper deals with classes of functions of several variables defined on an arbitrary set A and taking values in a possibly different set B. Definability of function classes by functional equations is shown to be equivalent to definability by relational constraints, generalizing a fact established by Pippenger in the case $A = B = \{0, 1\}$.

Conditions for a class of functions to be definable by constraints of a particular type are given in terms of stability under certain functional compositions. This leads to a correspondence between functional equations with particular algebraic syntax and relational constraints with certain invariance properties with respect to clones of operations on a given set.

When $A = \{0, 1\}$ and B is a commutative ring, such B-valued functions of n variables are represented by multilinear polynomials in n indeterminates in $B[X_1, \ldots, X_n]$. Functional equations are given to describe classes of field-valued functions of a specified bounded degree. Classes of Boolean and pseudo-Boolean functions are covered as particular cases.

Keywords: Function classes, class composition, stability, functional equations, relational constraints, function class definability, ring-valued functions, multilinear polynomial representations, linear equations, field-valued functions of Boolean variables, Boolean functions, pseudo-Boolean functions.

1 Introduction and Basic Definitions

For arbitrary sets B and C, by a *C-valued function on B* we mean a map

$$f : B^n \to C$$

*The work of the first named author was partially supported by the Graduate School in Mathematical Logic MALJA. Supported in part by grant #28139 from the Academy of Finland

†Department of Mathematics, Statistics and Philosophy University of Tampere Kansleririnne 1, 33014 Tampere, Finland E-mail: Miguel.Couceiro@uta.fi

‡Institute of Mathematics, Tampere University of Technology PL553, 33101 Tampere, Finland, E-mail: stephan.foldes@tut.fi