Limited Codes Associated with Petri Nets

Genjiro Tanaka

Abstract

The purpose of this paper is to investigate the relationship between limited codes and Petri nets. The set $M$ of all positive firing sequences which start from the positive initial marking $\mu$ of a Petri net and reach $\mu$ itself forms a pure monoid $M$ whose base is a bifix code. Especially, the set of all elements in $M$ which pass through only positive markings forms a submonoid $N$ of $M$. Also $N$ has a remarkable property that $N$ is pure. Our main interest is in the base $D$ of $N$. The family of pure monoids contains the family of very pure monoids, and the base of a very pure monoid is a circular code. Therefore, we can expect that $D$ may be a limited code. In this paper, we examine “small” Petri nets and discuss under what conditions $D$ is limited.

Keywords: free monoid, Petri net, code, prefix code, circular code, limited code

1 Introduction

Let $A$ be an alphabet, $A^*$ the free monoid over $A$, and $1$ the empty word. Let $A^+ = A^* - \{1\}$. A word $v \in A^*$ is a right factor of a word $u \in A^*$ if there is a word $w \in A^*$ such that $u = vw$. The right factor $v$ of $u$ is called proper if $v \neq u$. For a word $w \in A^*$ and a letter $x \in A$ we let $|w|_x$ denote the number of $x$ in $w$. The length $|w|$ of $w$ is the number of letters in $w$.

A non-empty subset $C$ of $A^+$ is said to be a code if for $x_1, \ldots, x_p, y_1, \ldots, y_q \in C$, $p, q \geq 1$,

$$x_1 \cdots x_p=y_1 \cdots y_q \implies p=q, x_1=y_1, \ldots, x_p=y_p.$$  

A subset $M$ of $A^*$ is a submonoid of $A^*$ if $M^2 \subseteq M$ and $1 \in M$. Every submonoid $M$ of a free monoid has a unique minimal set of generators $C = (M - \{1\}) - (M - \{1\})^2$. $C$ is called the base of $M$. A submonoid $M$ is right unitary in $A^*$ if for all $u, v \in A^*$,

$$u, uv \in M \implies v \in M.$$  

*Dept. of Computer Science, Shizuoka Institute of Science and Technology, Fukuroi-shi, 437-8555 Japan. E-mail: tanaka@cs.sist.ac.jp
\[ M \text{ is called left unitary in } A^* \text{ if it satisfies the dual condition. A submonoid } M \text{ is biunitary if it is both left and right unitary.} \]

**Definition 1.1.** Let \( M \) be a submonoid of a free monoid \( A^* \), and \( C \) its base.

If \( CA^* \cap C = \emptyset \) (resp. \( A^* C \cap C = \emptyset \)), then \( C \) is called a prefix (resp. suffix) code over \( A \). \( C \) is called a biunitary code if it is a prefix and suffix code.

A submonoid \( M \) of \( A^* \) is right unitary (resp. biunitary) if and only if its minimal set of generator is a prefix code (resp. biunitary code) ([1, p.46],[3, p.108]).

**Definition 1.2.** A Petri net is a 4-tuple, \( PN = (P, A, W, \mu_0) \) where \( P = \{p_1, p_2, \ldots, p_m\} \) is a finite set of places, \( A = \{t_1, t_2, \ldots, t_n\} \) is a finite set of transitions such that \( P \cap A = \emptyset \) and \( P \cup A \neq \emptyset \), \( W : (P \times A) \cup (A \times P) \rightarrow \{1, 2, \ldots\} \) is a weight function, \( \mu_0 : P \rightarrow \{0, 1, 2, \ldots\} \) is the initial marking.

Let \( t \in A \), and let \( t = \{p \in P | (p, t) \in P \times A \} \) and \( t' = \{p \in P | (t, p) \in A \times P \} \). In this paper we shall assume that a Petri net has no isolated transitions, i.e., no \( t \) such that \( t \cap t' = \emptyset \). A transition \( t \) is said to be enabled in a marking \( \mu_0 \), if \( W(p, t) \leq \mu_0(p) \) for all \( p \in t \). A firing of an enabled transition \( t \) removes \( W(p, t) \) tokens from each input place \( p \in t \), and adds \( W(t, p) \) tokens to each output place \( p \in t' \). A firing of an enabled transition \( t \) in \( \mu_0 \) produces a new marking \( \mu_1 \)

\[ \mu_1(p) = \mu_0(p) - W(p, t) + W(t, p) \]

for any \( p \in P \), denoted by \( \mu_1 = \delta(\mu, t) \). A string \( w = t_1 t_2 \ldots t_r, t_i \in A \), of transitions is said to be a firing sequence from \( \mu_0 \) if there exist markings \( \mu_i, 1 \leq i \leq r \), such that \( \delta(\mu_{i-1}, t_i) = \mu_r \) for all \( i, 1 \leq i \leq r \). In this case, \( \mu_r \) is reachable from \( \mu_0 \) by \( w \) and we write \( \delta(\mu_0, w) = \mu_r \). The set of all possible markings reachable from \( \mu_0 \) is denoted by \( \text{Re}(\mu_0) \), and the set of all possible sequences from \( \mu_0 \) is denoted by \( \text{Seq}(\mu_0) \). The function \( \delta : \text{Re}(\mu_0) \times A \rightarrow \text{Re}(\mu_0) \) is called a next-state function of a Petri net \( PN \) [5,p.23]. We note that the above condition for \( r = 0 \) is understood to be \( \mu_0 \in \text{Re}(\mu_0) \). A marking \( \mu \) is said to be positive if \( \mu(p) > 0 \) for all \( p \in P \).

A sequence \( t_1 t_2 \ldots t_n \in \text{Seq}(\mu_0) \), \( t_i \in A \), is called a positive sequence from \( \mu_0 \) if \( \delta(\mu_0, t_1 t_2 \ldots t_i) \) is positive for all \( i, 1 \leq i \leq n \). The set of all positive sequences from \( \mu_0 \) is denoted by \( \text{PSeq}(\mu_0) \). Let \( P = \{p_1, p_2, \ldots, p_n\} \). A marking \( \mu \) can be represented by a vector \( \mu = (\mu(p_1), \mu(p_2), \ldots, \mu(p_n)) \).

For every \( t \in A \) the vector \( \Delta t \) is defined by

\[ \Delta t = (\Delta t(p_1), \Delta t(p_2), \ldots, \Delta t(p_n)), \quad n = |P|, \]

where \( \Delta t(p) = -W(p, t) + W(t, p) \). For a sequence \( w = t_1 t_2 \ldots t_n \in \text{Seq}(\mu_0) \) and \( p \in P \), \( \Delta w = \sum_{i=1}^{n} \Delta t_i \) and \( \Delta w(p) \) is a \( p \)-th component of a vector \( \Delta w \), i.e., \( \Delta w(p) = \sum_{i=1}^{n} \Delta t_i(p) \). Note that if \( \delta(\mu_0, w) = \mu_1, w \in \text{Seq}(\mu_0) \), then \( \mu_1 = \mu_0 + \Delta w \) as a vector.
2 Some codes related to Petri nets

For a Petri net $PN = (P, A, W, \mu)$ and a subset $X \subseteq Re(\mu)$ we can define a deterministic automaton $A(PN)$ as follows: $Re(\mu), A, \delta: Re(\mu) \times A \rightarrow Re(\mu)$, $\mu$, and $X$, are regarded as the state set, the input set, the next-state function, the initial state, and the final set of $A(PN)$, respectively (For basic concepts of automata, refer to [1,p.10]). By using such automata, in [9] we defined four kinds of prefix codes and examined fundamental properties of these codes.

The set $Stab(PN) = \{w \mid w \in Seq(\mu) \text{ and } \delta(\mu, w) = \mu\}$ forms a submonoid of $A^*$. If $Stab(PN) \neq \{1\}$, then we denote the base of $Stab(PN)$ by $S(PN)$. Since $S(PN)A^+ \cap S(PN) = \emptyset, S(PN)$ is a prefix code over $A$.

A submonoid $M$ of $A^*$ is called pure [6] if for all $x \in A^*$ and $n \geq 1$,

$$x^n \in M \implies x \in M.$$ 

A subsemigroup $H$ of a semigroup $S$ is extractable in $S$ [8, p.191] if

$$x, y \in S, z \in H, xzy \in H \implies xy \in H.$$ 

**Proposition 2.1.** $Stab(PN)$ is an extractable pure monoid.

**Proof.** It is clear that $Stab(PN)$ is right unitary. Let $y, xy \in Stab(PN)$. Then $x$ is a sequence from the initial marking $\mu$. Since $\Delta y = 0$ (the zero vector) and $\Delta(xy) = \Delta x + \Delta y = 0$, we have $x \in Stab(PN)$. Thus $Stab(PN)$ is left unitary. Therefore $Stab(PN)$ is biunitary.

Assume that $x^n \in Stab(PN), n \geq 1$. Then it is obvious that $x$ is a sequence from $\mu$. Since $\Delta(x^n)\Delta x = 0$, we have $\Delta x = 0$. Thus $x \in Stab(PN)$, and $Stab(PN)$ is pure.

Let $x, y \in A^*$ and $z, xzy \in Stab(PN)$. If $x = 1$, then $z, xzy \in Stab(PN)$. Since $Stab(PN)$ is biunitary, we have $y \in Stab(PN)$ and $xy \in Stab(PN)$. Similarly $y = 1$ implies $xy \in Stab(PN)$. Suppose that $x, y \in A^+$. $y$ is a sequence from $\mu + \Delta x z = \mu + \Delta x$. Thus $xy$ is a sequence from $\mu$. From $\mu + \Delta(xzy) = \mu + \Delta(xy) = \mu$ we have $xy \in Stab(PN)$. \qed

**Definition 2.1.** Let $PN = (P, A, W, \mu)$ be a Petri net with a positive marking $\mu$. Define the subset $D(PN)$ as the set of all positive sequence $w$ of $S(PN)$.

Since $D(PN)$ is a subset of a bifix code $S(PN)$, also $D(PN)$ is a bifix code over $A$ if $D(PN) \neq \emptyset$. By the same argument mentioned above, we have

**Proposition 2.2.** If $D(PN) \neq \emptyset$, then $D(PN)^*$ is an extractable pure monoid.

**Example 2.1.** Let $PN = ([p, q], \{a, b\}, W, \mu)$ be a Petri net defined by $W(a, p) = W(p, b) = W(q, a) = W(b, q) = 1, \mu(p) = \mu(q) = 2$. Then $D(PN) =$
\{ab, ba\}, therefore \{ab, ba\} is pure [1, p.324].

**Proposition 2.3.** If \(x, y \in A^+\), \(z, xzy \in D(PN)\), then \(xz^2y \subseteq D(PN)\).

**Proof** Let \(x, y \in A^+\), \(z, xzy \in D(PN)\) and \(\mu\) an initial marking of \(PN\).

First we show that \(xy \in D(PN)\). \(x\) is a positive sequence from \(\mu\), and \(y\) is a positive sequence from \(\mu + \Delta(xz) = \mu + \Delta x\). Therefore \(xy \in P\text{Seq}(\mu)\). Since \(\Delta(xy) = \Delta(xy) = 0\), we have that \(xy \in D(PN)\), so that \(xy = d_1 \cdots d_m\) for some \(d_i \in D(PN)\), \(m \geq 1\). We have the following three cases.

Case (a). \(m = 1\), \(xy = d_1 \in D(PN)\)

Case (b). \(m \geq 2\), \(x = d_1 u\) for some \(u \in A^*\).

Case (c). \(m \geq 2\), \(d_1 = xu, y = uv, v = d_2 \cdots d_m\) for some \(u, v \in A^*\).

In Case (b) we have \(d_1, xzy = d_1 uzy \in D(PN)\), but it does not occur since \(D(PN)\) is a prefix code. In Case (c), if \(u = 1\), then \(d_1, xzy = d_1 y \in D(PN)\).

This contradicts the fact that \(D(PN)\) is a prefix code. Therefore \(u \neq 1\) and it follows that both \(d_m\) and \(xzy = xzu \cdots d_m\) are elements of \(D(PN)\). This contradicts the fact that \(D(PN)\) is a suffix code. Therefore only Case (a) is possible to occur. Thus \(xy \in D(PN)\).

Next we show that \(x, y \in A^+\), \(z, xzy \in D(PN)\) implies \(xz^2y \in D(PN)\). Since \(z\) is a positive sequence from \(\mu + \Delta x = \mu + \Delta (xz)\), we have \(xz^2 \in P\text{Seq}(\mu)\). Since \(y\) is a positive sequence from \(\mu + \Delta (xz) = \mu + \Delta (xz^2)\), we have \(xz^2y \in P\text{Seq}(\mu)\).

Therefore, from \(\Delta(xz^2y) = \Delta(xy) = 0\) we have \(xz^2y \in D(PN)^*\).

Thus

\[
xz^2y = d_1 \cdots d_m, \; d_i \in D(PN), \; m \geq 1.
\]

We have the following four cases.

Case 1. \(m = 1\), \(xz^2y = d_1 \in D(PN)\),

Case 2. \(m \geq 2\), \(d_1 = xz^2u, y = ud_2 \cdots d_m\) for some \(u \in A^*\),

Case 3. \(m \geq 2\), \(d_1 = xzu, z = uv, v = d_2 \cdots d_m\) for some \(u, v \in A^*\),

Case 4. \(m \geq 2\), \(d_1 = xu, z = uv, vzy = d_2 \cdots d_m, \; u, v \in A^*\),

Case 5. \(m \geq 2\), \(x = d_1 u\), for some \(u \in A^*\).

In Case 2, \(d_m, xzy = xzu \cdots d_m \in D(PN)\). Thus Case 2 cannot occur since \(D(PN)\) is a suffix code. In Case 3, \(d_m \in D(PN)\), and \(xzy = xuvy = xud_2 \cdots d_m \in D(PN)\). However Case 3 cannot occur since \(D(PN)\) is a suffix code. Since \(D(PN)\) is a prefix code, Case 4 and Case 5 cannot occur. Therefore only Case 1 is possible to occur. Thus \(xz^2y \in D(PN)\).

Now suppose that \(x, y, z, xz^ny \in D(PN), n \geq 2\). Then, \(xz^{n-1}, y \in A^+\), \(z, (xz^{n-1})yz \in D(PN)\). Therefore we have \((xz^{n-1})z^2y = xz^{n+1}y \in D(PN)\).

Let \(C\) be a code over \(A\). \(C\) is an *infix code* ([7,p.129]), if for all \(x, y, z \in A^*\),

\[
z, xzy \in C \implies x = y = 1.
\]

**Proposition 2.4.** If \(D(PN)\) is a non-empty finite set, then \(D(PN)\) is an infix code.
Proof Let $x, y \in A^*$, $z, xyz \in D(PN)$. $x = 1, y \neq 1$ or $x \neq 1, y = 1$ cannot occur because $D(PN)$ is a bifix code. Therefore either $x = y = 1$ or $x, y \in A^+$. By Proposition 2.3, $x, y \in A^+$ and $z, xyz \in D(PN)$ follow that $xz^*y \in D(PN)$. This contradicts the fact that $D(PN)$ is a finite set. Thus we have $x = y = 1$.

Example 2.2. (1). Let $PN = (\{p, q, r\}, \{a, b, c, d\}, W, \mu_0)$ be a Petri net such that $W(p, a) = W(a, q) = W(q, b) = W(b, r) = W(r, c) = W(c, q) = W(q, d) = W(d, p) = 1$. $\mu_0 = (2, 1, 1)$. Then $D(PN) = a(bc)^*d$. Therefore the infinite code $D(PN)$ is infix. Thus the converse of Proposition 2.4 is false.

(2). Let $PN = (\{p\}, \{a, b\}, W, \mu_0)$ be a Petri net such that $W(a, p) = 1, W(p, b) = 1, \mu_0 = (1)$. Then $ab, a^2b^2 \in D(PN)$. Therefore the infinite code $D(PN)$ is not infix.

3 Limited code

A submonoid $M$ of $A^*$ is very pure if for all $u, v \in A^*$,

$$uv, vu \in M \Rightarrow u, v \in M.$$ 

The base of a very pure monoid is called a circular code.

Let $p, q \geq 0$ be two integers. A code $C$ is called $(p, q)$-limited if for any sequence $u_0, u_1, \ldots, u_{p+q}$ of words in $A^*$, the assumptions $u_{i-1}u_i \in C^*$ ($1 \leq i \leq p + q$) imply $u_p \in C^*$.

Any limited code is circular ([1, p.329, Proposition 2.1]). If a subset $C$ of $A^*$ is a bifix $(1,1)$-limited code, then for any $u_0, u_1, u_2 \in A^*$ such that $u_0u_1, u_1u_2 \in C^*$ we have $u_1 \in C^*$. Thus $u_0u_1, u_1u_2 \in C^*$. This implies that $u_0, u_1, u_2 \in C^*$ because $C^*$ is biunitary. Therefore $C$ is $(p, q)$-limited for all $p, q$ with $p + q = 2$.

Let $PN_0 = (\{p\}, \{a, b\}, W, \mu_0)$ be a Petri net such that $W(a, p) = \alpha, W(p, b) = \beta, \mu_0 = (\lambda_p)$, $\lambda_p > 0$.

Consider the set $\Omega$ of all positive markings in $PN_0$;

$$\Omega = \{\mu \mid \mu = \mu_0 + \Delta w, w \in PSeq(\mu_0)\}.$$ 

Let $g = gcd(\alpha, \beta)$ be the greatest common divisor of $\alpha$ and $\beta$, and let $N = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. Then we have

(0) $D(PN_0)$ is dense, that is, $D(PN_0) \cap A^*wA^* \neq \emptyset$ for every $w \in A^*$.

(1) If $\lambda_p < g$, then $\Omega = \{\lambda_p + ng \mid n \in \mathbb{N}\}$.

(2) If $\lambda_p = sg$, $s \geq 1$, $s \in \mathbb{N}$, then $\Omega = \{ng \mid n \geq 1, n \in \mathbb{N}\}$.

(3) If $\lambda_p = sg + t_p$, $s \geq 0$, $0 < t_p < g$, then $\Omega = \{t_p + ng \mid n \geq 0, n \in \mathbb{N}\}$. 

Limited Codes Associated with Petri Nets 221
Proposition 3.1. If \( \lambda_p > \text{gcd}(\alpha, \beta) \), then \( D(PN_0) \) is not circular.

Proof. Let \( D = D(PN_0) \), and let \( g = \text{gcd}(\alpha, \beta) \). Note that \( \mu_0 = \lambda_p \). We have the following two cases:

Case 1. \( g = \alpha \) or \( g = \beta \).

Case 2. \( \alpha \neq \beta \), \( \alpha \geq 2 \), \( \beta \geq 2 \), \( \text{gcd}(\alpha', \beta') = 1 \).

Case 1-(i). If \( \alpha = \text{gcd}(\alpha, \beta) \), \( \beta = k \alpha, k > 1 \), then \( ab^{k-1}b, ab^{-1}ba \in D \) and \( a \notin D^* \). Therefore \( D \) is not circular.

Case 1-(ii). If \( \beta = \text{gcd}(\alpha, \beta) \), \( \alpha = k \beta, k > 1 \), then \( ab^{k-1}b, bba^{-1}b \in D \) and \( b \notin D^* \). Thus \( D \) is not circular.

Case 1-(iii). If \( \alpha = \text{gcd}(\alpha, \beta) \), \( \alpha = \beta, \) then \( ab, ba \in D \). Consequently \( D \) is not circular.

Case 2. Since \( g = \text{gcd}(\alpha, \beta) \), there exist some integers \( x' \) and \( y' \) such that \( \alpha x' + \beta y' = g \).

Case 2-(i). We consider the case \( \alpha x' + \beta y' = g, x' > 0, y' < 0 \). We set \( x = x', y = -y' \), then \( \alpha x - \beta y = g, x > 0, y > 0 \). Since \( \alpha x = \beta y + g > \beta i \), for \( i = 1, \ldots, y \), \( b^\alpha \) is a sequence from \( \lambda_p + \Delta(a^\alpha) \), and \( \lambda_p + \Delta(a^\alpha b^\beta) = \lambda_p + g \). Consequently \( a^\alpha b^\beta \) is also a sequence from \( \lambda_p + \Delta(a^\alpha b^\beta) \), therefore \( (a^\alpha b^\beta)^j \in P\text{Seq}(\mu_0) \). Similarly we have \( (a^\alpha b^\beta)^j \in P\text{Seq}(\lambda_\beta) \) and \( \lambda_p + \Delta((a^\alpha b^\beta)^j) = \lambda_p + \beta g \). Thus \( (a^\alpha b^\beta)^j b \in D(PN_0) \).

Case 2-(ii). We consider the case \( -\alpha x + \beta y = g \) for some positive integers \( x \) and \( y \). Thus \( a(a^\alpha b^\beta)^j \in P\text{Seq}(\lambda_p) \), \( 1 \leq j \leq \alpha' \). Thus \( a(a^\alpha b^\beta)^j \in D \). On the other hand, from \( \lambda_p > g \), and \( \alpha' \geq 2 \) we have \( \lambda_p + \Delta(a^\alpha b^\beta) \geq 0 \). Thus \( a^\alpha b^\beta a \in P\text{Seq}(\lambda_p) \). It follows that \( a^\alpha ((a^\alpha b^\beta)^j)^{-1} \in D \). However \( a^\alpha b^\beta \notin D^* \). Therefore \( D \) is not circular.

Proposition 3.2. If \( \lambda_p \leq \text{gcd}(\alpha, \beta) \), then any nonempty subset of \( D(PN_0) \) is \((p,q)\)-limited for all \( p, q \) with \( p + q = 2 \).

Proof. Let \( D = D(PN_0) \) and \( g = \text{gcd}(\alpha, \beta) \). Let \( d \) be an arbitrary element in \( D \). Then \( d \) has a proper right factor \( v \neq 1, d \), because \( a, b \notin D \). Let \( d = uv, u, v \in A^+D \).

First, we shall show that \( \Delta v \leq -g \). Assume the contrary. Then \( \Delta v > -g \), and we have \( \Delta v \geq 0 \) since \( \Delta v \) is a multiple of \( g \). If \( \Delta v = 0 \), then \( \Delta u = 0 \) since \( \Delta d = \Delta(uv) = \Delta u + \Delta v = 0 \). Therefore we get \( u \in D^* \). This contradicts the fact that \( D \) is a prefix code. Thus we have \( \Delta v > 0 \), it follows that \( \Delta v > 0 \) and \( \Delta u = -\Delta(v) \leq -g \). Then we have \( \mu_0 + \Delta u = \lambda_p + \Delta u \leq \lambda_p - g \leq 0 \), showing that \( u \notin P\text{Seq}(\mu_0) \) and contradicting \( d \in D \). Therefore we have prove \( \Delta v \leq -g \).

Next we shall show that any nonempty subset \( C \) of \( D \) is \((1,1)\)-limited. Note that \( C \) is a biunitary code. Suppose that \( u_0, u_1, u_2 \in A^* \) and \( u_0 u_1, u_1 u_2 \in C^* \). If \( u_i = 1 \) for some \( i, 0 \leq i \leq 2 \), then \( u_0, u_1, u_2 \in D^* \) since \( C^* \) is biunitary. We assume that \( u_i \neq 1 \) for all \( i, 0 \leq i \leq 2 \). We may write
\[ u_0 = v_0 x_1, u_1 = y_1 w_1, x_1 y_1 \in C, v_0, w_1 \in C^* \]
If \( y_1 \neq 1 \) and \( y_1 \notin C \), then \( y_1 \) is a proper right factor of \( d_i \in D \). Therefore \( \Delta(y_1) \leq -g \) as mentioned above. It follows \( \lambda_p + \Delta y_1 \leq 0 \), and \( y_1 \notin PSeq(\mu_0) \). However, \( u_1u_2 = y_1w_1w_2 \in C^* \subseteq D^* \). Thus \( y_1 \in PSeq(\mu_0) \). This is a contradiction. Therefore \( y_1 = 1 \) or \( y_1 \in C \). Thus \( u_1 \in C^* \).

Let \( PN_1 = \{\{p, q\}, \{a, b\}, W, \mu_0\} \) be a Petri net such that \( W(a, p) = \alpha > 0 \), \( W(p, b) = \alpha' > 0 \), \( W(q, a) = \beta > 0 \), \( W(b, q) = \beta' > 0 \), \( \mu_0(p) = \lambda_p, \mu_0(q) = \lambda_q \).

We examine the code \( D(PN_1) \) associated with Petri net \( PN_1 \).

Suppose that \( D(PN_1) \neq \emptyset \) and \( w \in D(PN_1) \). Let \( n = |w|_a \) and \( m = |w|_b \), then \( \Delta(w)\Delta(a) + m\Delta(b) = 0 \). Consequently the linear equation

\[
\begin{pmatrix}
\alpha & -\alpha' \\
-\beta & \beta'
\end{pmatrix}
\begin{pmatrix}
 n \\
m
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

has a non-trivial solution. Thus \( \alpha'\beta = \alpha'\beta \). Therefore, if \( D(PN_1) \neq \emptyset \), then \( PN_1 = \{\{p, q\}, \{a, b\}, W, \mu_0\} \) has the following form:

\( W(a, p) = \alpha, W(p, b) = k\alpha, W(q, a) = \beta, W(b, q) = k\beta \), for some \( k > 0 \).

Here we assume that \( k \) is a positive integer. That is, we define a Petri net \( PN_1 = \{\{p, q\}, \{a, b\}, W, \mu_0\} \) as follows:

\[
\Delta(a) = \begin{pmatrix}
\alpha \\
-\beta
\end{pmatrix}, \quad \Delta(b) = \begin{pmatrix}
-k\alpha \\
k\beta
\end{pmatrix},
\]

where \( k \) is a positive integer.

We define an integer \( M_p \) as follows

\[
M_p = \begin{cases}
\frac{\lambda_p}{\alpha} - 1, & \text{if } \frac{\lambda_p}{\alpha} \text{ is an integer}, \\
\left\lfloor \frac{\lambda_p}{\alpha} \right\rfloor, & \text{if } \frac{\lambda_p}{\alpha} \text{ is not an integer},
\end{cases}
\]

where \( \lfloor \cdot \rfloor \) is the symbol of Gauss. Similarly we define an integer \( M_q \) as follows, \( M_q = \frac{\lambda_q}{\sigma} - 1 \) if \( \frac{\lambda_q}{\sigma} \) is an integer, and \( M_q = \left\lfloor \frac{\lambda_q}{\sigma} \right\rfloor \) if \( \frac{\lambda_q}{\sigma} \) is not an integer. Note that \( a, a^2, \ldots, a^{M_q+1} \in PSeq(\mu_0) \) and \( a^{M_q+1} \notin PSeq(\mu_0) \). If \( M_p \geq k \), then \( b \in PSeq(\mu_0) \).

Now, we observe that \( M_p + M_q \leq k - 1 \) implies \( D(PN_1) = \emptyset \). If \( M_p + M_q \leq k - 1 \), then \( M_p \leq k - 1 \). It follows that \( b \notin PSeq(\mu_0) \). Furthermore, if \( M_q = 0 \), then \( a \notin PSeq(\mu_0) \) and \( PSeq(\mu_0) = \{1\} \). If \( M_q > 0 \), we have

\( a^i \in PSeq(\mu_0), 1 \leq i \leq M_q, \) and \( \Delta a^i(p) = \lambda_p + \alpha i \leq \lambda_p + \alpha M_q \).

Assume that \( \lambda_p + \alpha M_q - \alpha k > 0 \). If \( \frac{\lambda_p}{\alpha} \) is an integer, then \( M_p + 1 + M_q - k > 0 \). This contradicts the hypothesis. If \( \frac{\lambda_p}{\alpha} = \alpha M_p + s \) for some \( s, 1 \leq s < \alpha \), then

\[
0 < M_p + M_q + \frac{s}{\alpha} - k < M_p + M_q + 1 - k.
\]
This also contradicts our hypothesis. Therefore we get $\lambda_p + \alpha M_q - \alpha k \leq 0$, showing that $b$ is not enabled in $\mu_0 + \Delta(a')$, $1 \leq i \leq M_q$. Therefore $PSeq(\mu_0) = \{1, a, a^2, \ldots, a^{M_q}\}$. Thus the condition $M_p + M_q \leq k - 1$ implies $D(PN_1) = 0$.

**Lemma 3.3.** Let $d = uv \in D(PN_1)$, $u, v \in A^+$. 
(1) If $M_p = 0$ and $M_q \geq k$, then $\Delta v(p) \leq -\alpha$. 
(2) If $M_p \geq k$ and $M_q = 0$, then $\Delta v(q) \leq -\beta$.

*Proof.* (1). Suppose that $\Delta v(p) > -\alpha$. Then, since $\Delta v(p)$ is a multiple of $\alpha$, we have $\Delta v(p) \geq 0$. Note that

$$\Delta v = |v|_a \Delta a + |v|_a \Delta b = \left(\frac{|v|_a - k|v|_b}{|v|_a - k|v|_b}\right) \cdot \alpha_b$$

If $\Delta v(p) = 0$, then $|v|_a - k|v|_b = 0$. Thus $\Delta v = 0$, it follows that $\Delta u = 0$ and $u \in D(PN_1)^*$. This contradicts the fact that $D(PN_1)$ is a prefix code. Thus $\Delta v(p) \geq \alpha$. It implies that $\Delta u(p) = -\Delta v(p) \leq -\alpha$. Since $M_p = 0$, $\mu_0(p) + \Delta u(p) < \lambda_p - \alpha \leq 0$. This yields $u \notin PSeq(\mu_0)$. This is a contradiction. Therefore we have $\Delta v(p) \leq -\alpha$. 

(2). Proof is omitted.

□

**Proposition 3.4.** We have 
(1) If $M_p + M_q > k, M_p \geq k$ and $M_q \geq 1$, then $D(PN_1)$ is not circular. 
(2) If $M_p + M_q > k, k > M_p \geq 1, M_q > 1$, then $D(PN_1)$ is not circular. 
(3) If $M_p + M_q = k$, then $D(PN_1)$ is a singleton. 
(4) If $M_p = 0, M_q > k$, then any nonempty subset of $D(PN_1)$ is $(p, q)$-limited for all $p, q$ with $p + q = 2$. 
(5) If $M_p > k, M_q = 0$, then any nonempty subset of $D(PN_1)$ is $(p, q)$-limited for all $p, q$ with $p + q = 2$.

*Proof.* Let $D = D(PN_1)$.

(1) From $M_p \geq k$ it follows that $b \in PSeq(\mu_0)$ and $ba^{k-1} \in D$. On the other hand, from $\lambda_p > k\alpha$ we have $\lambda_p + \alpha - k\alpha > \alpha$. Thus $ab \in PSeq(\mu_0)$. Since $\lambda_p + (k-1)\beta > (k-1)\beta$, we have $abak^{k-1} \in D$. Therefore $D$ is not circular.

(2) Let $k = M_p + r$Since $M_p + M_q > k$, we have $M_q > r$. It follows that $a^i \in PSeq(\mu_0)$. Since $\lambda_p + ra - k\alpha = \lambda_p - \alpha M_p > 0$, we have $a^ib \in PSeq(\mu_0)$. Consequently $a^{i+1}b^\alpha \in D$. On the other hand we have $a^{i+1}b^\alpha \in D$ since $M_q > r$. Therefore $D$ is not circular.

(3) First we consider the case that $M_p \geq 1, M_q \geq 1$. Since $M_p = k - M_q < k$, we have $b \notin PSeq(\mu_0)$. It is obvious that $a^i \in PSeq(\mu_0)$ for $i = 0, 1, \cdots, M_q$. If $i \leq M_q - 1$, then from $\lambda_p + ia - k\alpha = \lambda_p - M_p\alpha - (M_q - i)\alpha$ and $\lambda_p - M_p\alpha \leq \alpha$, we get $\lambda_p + ia - k\alpha \leq 0$, showing that $a^i b \notin PSeq(\mu_0), 0 \leq i \leq M_q - 1. It is obvious that $a^{M_q}b \in PSeq(\mu_0)$ and

$$\mu_0 + \Delta(a^{M_q}b) = \left(\frac{\lambda_p - \alpha M_p}{\lambda_q + \beta M_p}\right).$$
Lemma 3.3, we have
\[ \text{(5) The proof of (5) is similar to the proof of (4), therefore it is omitted.} \]

Let \( C \) be a nonempty subset of \( D \). Suppose that \( w_0, w_1, w_2 \in A^* \) and \( w_0w_1, w_1w_2 \in C^* \). If \( w_i = 1 \) for some \( i, 0 \leq i \leq 2 \), then \( w_0, w_1, w_2 \in C^* \) since \( C^* \) is biunitary. We assume that \( w_i \neq 1 \) for all \( i, 0 \leq i \leq 2 \). We may write
\[
w_0 = w_0x_1, \quad w_1 = y_1v_1, \quad x_1y_1 \in C \text{ for some } x_1, y_1 \in A^*, \quad u_0, v_1 \in C^*.
\]
If either \( y_1 = 1 \) or \( y_1 \in C \), then \( w_1 \in C^* \). Assume that \( y_1 \neq 1 \) and \( y_1 \notin C \). Then \( y_1 \) is a proper right factor of an element in \( D \). Since \( M_p \leq \alpha, \lambda_p + 2 \Delta y_1 \leq 0 \) by Lemma 3.3, we have \( y_1 \notin PSeq(\mu_0) \). However \( w_1w_2 = y_1v_1w_2 \in C^* \subset D^* \). Thus \( y_1 \in PSeq(\mu_0) \). This is a contradiction. Therefore \( y_1 \in C \cup \{1\} \). This yields \( w_1 \in C^* \).

(5) The proof of (5) is similar to the proof of (4), therefore it is omitted.

\[ \square \]

**Remark 3.1.** In the above proof for (3) we have \( D = \{w\}, \quad w = a^{M_0}b_{a^{k-M_0}}, \quad M_0 \neq 0, \quad k - M_0 \neq 0 \). \( D \) is \( (s, t) \)-limited for all \( s, t \geq 0 \) with \( s + t = 3 \). For any \( n, m \geq 0 \) the code \( D = \{a^nba^{k-n}\} = \{a^nba^m\}, k + m \), is realizable as a Petri net code which is produced by the Petri net \( PN_1 \) such that \( W(a, p) = W(q, a) = 1, \quad W(p, b) = W(b, q) = k, \quad \mu_0(p) = m + 1, \quad \mu_0(q) + 1. \)

Let \( PN = (P, A, W, \mu_0) \) be a Petri net. By \( PRec(\mu_0) \) we denote the set of all possible positive markings reachable from \( \mu_0 \). For a Petri net \( PN \) we define a deterministic automaton \( A(PN) \) as follows:
\[ PRec(\mu_0), \quad A, \quad \delta : PRec(\mu_0) \times A \rightarrow PRec(\mu_0), \quad \mu_0, \text{ and } \{\mu_0\}, \]
are regarded as the state set, the input set, the next-state function, the initial state, and the final set of \( A(PN) \), respectively.

**Corollary 3.5.** Let \( n \) and \( k \) be arbitrary integers such that \( n > k > 1 \). Define the automaton
\[ A_{(n,k)} = (\{1, 2, \ldots, n\}, \{a, b\}, f, 1, \{1\}) \]
by \( f(i, a) = i + 1, 1 \leq i \leq n - 1, \quad f(j, b) = j - k, k + 1 \leq j \leq n. \) Then any nonempty subset of the base of language \( L(A_{(n,k)}) \) recognized by \( A_{(n,k)} \) is a \( (p,q) \)-limited code for all \( p, q \) with \( p + q = 2 \).

**Proof.** We define the \( PN_1 = (\{p, q\}, \{a, b\}, W, \mu_0) \) as follows:
\[ W(a, p) = 1, W(p, b) = k, W(b, q) = k, W(q, a) = 1, \quad \mu_0(p) = 1, \mu_0(q). \]
Then \( M_p = 0, M_q - 1 \geq k. \) Therefore, by Proposition 3.4, \( D(PN) \) is \( (1,1) \)-limited. Since \( A(PN_1) \) is isomorphic to \( A_{(n,k)} \) as an automaton, we have Corollary 3.5. \( \square \)
Proposition 3.6. Let $PN = \{(p_1, \cdots, p_n), \{a_1, \cdots, a_n\}, W, \mu_0\}, n \geq 2$, be a Petri net such that $W(p_i, a_i) = \alpha_i, W(a_{i+1}, p_{i+1}) = \beta_i, 1 \leq i \leq n - 1$, and $W(p_n, a_n) = \alpha_n, W(a_1, p_1) = \beta_n, \mu_0 = (\lambda_1, \cdots, \lambda_n), \mu_0(p_i) = \lambda_i, 1 \leq i \leq n$. Furthermore let $g_j = \gcd(\beta_{j-1}, \alpha_j), 2 \leq j \leq n$. If $\lambda_1/\alpha_1 > 1$ and $\lambda_i \leq g_i$ for all $i = 2, \cdots, n$, and if $D(PN) \neq \emptyset$, then any nonempty subset of $D(PN)$ is $(p, q)$-limited for all $p, q$ with $p + q = 2$.

Proof. We set $D = D(PN)$. Since $\lambda_i \leq g_i$ for all $i = 2, \cdots, n$, we have $D \subseteq a_1A^+$. Let $d \in D, d = auv, u \in A^+, v \in A^+$. Note that $v$ is a proper right factor of an element in $D$.

First we show that $\Delta v(p_i) \leq 0$ for all $i = 2, \cdots, n$. Suppose that $\Delta v(p_j) > 0$ for some $j \geq 2$. Since $\Delta v(p_j) > 0$ is a linear combination of $\beta_{j-1}$ and $\alpha_j$, $\Delta v(p_j)$ is a multiple of $g_j$. Therefore $\Delta v(p_j) > 0$ implies $\Delta(v)(p_j) \geq g_j$. Thus $-\Delta v(p_j) \leq -g_j$. On the other hand, $\Delta d = \Delta(a_1u) + \Delta v = 0$, and we have $\Delta(a_1u) = -\Delta v$. Therefore $\Delta(a_1u)(p_j) = -\Delta v(p_j) \leq -g_j$. However, $\mu_0(p_j) + \Delta(a_1u)(p_j) \leq \lambda_j - g_j$. This contradicts the fact that $a_1u \in PSeq(\mu_0)$. Consequently we have that $\Delta v(p_i) \leq 0$ for all $i, (i \geq 2)$.

Next we show that $v \notin PSeq(\mu_0)$. To prove this we show that there exists some $p_t, t \geq 2$, such that $\Delta v(p_t) \leq -g_t$. Suppose the contrary. Then $\Delta v(p_i) = 0$ for all $i \geq 2$. Let $x_j$ be the number of the letter $a_j$ in $v$, then

$$
\Delta(v) = \begin{pmatrix}
-\alpha_1 & 0 & \cdots & 0 & \beta_n \\
\beta_1 & -\alpha_2 & \cdots & 0 & 0 \\
0 & \beta_2 & \cdots & 0 & 0 \\
0 & 0 & \cdots & \beta_{n-1} & -\alpha_n
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_n
\end{pmatrix}
= \begin{pmatrix}
\Delta(v)(p_1) \\
0 \\
0 \\
0
\end{pmatrix}.
$$

We regard the equation above as a system of linear equations. Since $D \neq \emptyset$, the determinant of a matrix $(\Delta a_1, \cdots, \Delta a_n)$ is zero. That is, $\alpha_1\alpha_2 \cdots \alpha_n = \beta_1\beta_2 \cdots \beta_n$. Since there exists a solution, we must have $\Delta v(p_1) = 0$. Consequently $\Delta(a_1u) = -\Delta v = 0$ and $a_1u \in PSeq(\mu_0)$. Therefore $a_1u \in D^*$. This contradicts the fact that $D$ is a prefix code. Thus we have proved that $\Delta v(p_t) \leq -g_t$ for some $p_t, t \geq 2$. This means that $v \notin PSeq(\mu_0)$ since $\mu_0(p_t) + \Delta(v)(p_t) \leq \lambda_j - g_t \leq 0$.

Finally we prove that any nonempty subset $C$ of $D$ is $(1, 1)$-limited. Suppose that $w_0, w_1, w_2 \in A^+$, and $w_0w_1, w_1w_2 \in C^*$. We may write

$$w_0 = u_0x_1, \quad w_1 = y_1v_1, \quad x_1y_1 \in C \quad \text{for some} \quad x_1, y_1 \in A^+, \quad u_0, v_1 \in C^*.$$

Note that $y_1$ is a right factor of an element of $D$. If $y_1 \neq 1$ and $y_1 \notin C$, then $y_1 \notin PSeq(\mu_0)$ as we mentioned above. Therefore $w_1w_2 = y_1v_1w_2 \notin PSeq(\mu_0)$. This contradicts our hypothesis. Thus $y_1 = 1$ or $y_1 \in C$. This shows that $w_1 \in C^*$. \qed
Let \( PN_2 = \{ \{p_1, p_2\}, \{a, b, c\}, W, \mu_0\} \) be a Petri net such that
\[
W(a, p_1) = \alpha_1, W(p_1, b) = \alpha_2, W(b, p_2) = \beta_1, W(p_1, c) = \alpha_3, W(p_2, c) = \beta_2,
\]
\( \mu_0(p_1) = \lambda_1, \mu_0(p_2) = \lambda_2. \)

**Lemma 3.7.** Let \( PN_2 \) be a Petri net mentioned above, and let 
\( \alpha = \gcd(\alpha_1, \alpha_2, \alpha_3), \beta = \gcd(\beta_1, \beta_2). \) Suppose that \( D(PN_2) \neq \emptyset \) and \( \lambda_1 \leq \alpha, \lambda_2 \leq \beta. \) If \( d \in D(PN_2) \) and \( v, v \neq 1, \) is a proper right factor of \( d, \) then we have one of the following:
\[
\begin{align*}
(1) & \quad \Delta v(p_1) \leq -\alpha, \quad \Delta v(p_2) \leq -\beta, \\
(2) & \quad \Delta v(p_1) = 0, \quad \Delta v(p_2) \leq -\beta, \\
(3) & \quad \Delta v(p_1) \leq -\alpha, \quad \Delta v(p_2) = 0.
\end{align*}
\]

**Proof.** Let \( D = D(PN_2). \) It is obvious that \( b \) and \( c \) are not enabled in \( \mu_0, \) so that \( D \subset aA^+. \) Let \( d \in D, d = au, u \in A^+, v \in A^+. \) If \( u = 1, \) then 
\( \Delta v = -\Delta u \) and (3) holds. Assume \( u \neq 1. \) If \( \Delta v(p_1) > 0, \) then \( \Delta v(p_1) \geq \alpha \) because \( \Delta v(p_1) \) is a multiple of \( \alpha. \) Thus \( \Delta (au)(p_1) = -\Delta v(p_1) \leq -\alpha. \) Hence 
\( \mu_0(p_1) + \Delta(au)(p_1) \leq \mu_0(p_1) - \alpha \leq 0. \) This contradicts the fact that \( au \in P Seq(\mu_0) \).

Therefore \( \Delta v(p_1) \leq 0. \) Similarly we have \( \Delta v(p_2) \leq 0. \) If \( \Delta v(p_1) = 0 \) and 
\( \Delta v(p_2) = 0, \) i.e., \( \Delta v = 0, \) then \( \Delta (au) = 0. \) Since \( au \in P Seq(\mu_0) \), we have \( au \in D^*, \)
contradicting the fact that \( D \) is a prefix code. Therefore at least one of (1),(2) or (3) occurs. \( \square \)

**Proposition 3.8.** If \( D(PN_2) \neq \emptyset \) and \( \lambda_1 \leq \alpha, \lambda_2 \leq \beta, \) then any nonempty subset of \( D(PN_2) \) is \((p, q)\)-limited for all \( p, q \) with \( p + q = 2. \)

**Proof.** Let \( D = D(PN_2). \) We note that for a right factor \( v, v \neq 1, \) of an element 
\( d \in D \) the vector \( \mu_0 + \Delta v \) is not positive by Lemma 3.7. That is, \( v \notin P Seq(\mu_0). \)
Let \( C \) be a nonempty subset of \( D. \) Assume \( w_0, w_1, w_2 \in A^+ \) and \( w_0w_1w_2 \in C^+. \)
If either \( w_0 = 1 \) or \( w_1 = 1, \) then \( w_1 \in C^+. \) Therefore we consider the case where
\( w_0 \neq 1 \) and \( w_1 \neq 1. \) Let \( w_0w_1 = d_1 \cdots d_m, d_i \in \mathbb{C}, 1 \leq i \leq m. \) There exist
an integer \( i, 1 \leq i \leq m, \) and \( u, v \in A^+ \) such that \( w_0 = d_1 \cdots d_{i-1}u, d_i = uv, w_1 = vd_{i+1} \cdots d_m. \) If \( v \neq 1 \) and \( v \notin D, \) then \( v \) is a proper right factor \( d_i. \) Thus 
\( v \notin P Seq(\mu_0). \) However, from \( w_1w_2 = ved_{i+1} \cdots d_mw_2 \in C^+, \) we have \( v \in P Seq(\mu_0). \)
This is a contradiction. Hence \( v = 1 \) or \( d \in C. \) It follows that \( w_1 \in C^+. \) Thus \( C \) is
(1,1)-limited. \( \square \)

Let \( PN_3 = \{ \{p, q\}, \{a, b, c\}, W, \mu_0\} \) be a Petri net such that \( W(a, p) = \alpha, \)
\( W(q, a) = \beta, W(p, b) = \alpha + \beta, W(b, q) = \alpha + \beta, W(c, p) = \beta, W(q, c) = \alpha, \mu_0(p) = \lambda_p, \mu_0(q) = \lambda_q. \)

**Lemma 3.9.** Let \( PN_3 \) be a Petri net mentioned above. If \( \beta < \lambda_p \leq \alpha + \beta \) and 
\( \beta < \lambda_q \leq \alpha, \) then for any \( u \in P Seq(\mu_0) \) we have one of the following:
\[
\begin{align*}
(1) \Delta u &= \left( \frac{k(\alpha - \beta)}{k(\alpha - \beta)} \right), \quad k \geq 0, \\
(2) \Delta u &= \left( \frac{k(\alpha - \beta) + l\alpha}{k(\alpha - \beta) - l\beta} \right), \quad k \geq 0, \quad l \geq 1,
\end{align*}
\]
(3) \( \Delta u = \begin{pmatrix} k(\alpha - \beta) - l\beta \\ k(\alpha - \beta) + l\alpha \end{pmatrix}, \ k \geq 0, \ l \geq 1. \)

**Proof.** We shall prove the lemma by induction on the length of \( u \) in \( PSeq(\mu_0) \).

Since \( \beta < \lambda_p \leq \alpha + \beta \), and \( \beta < \lambda_q \leq \alpha \), only \( a \) is enabled in \( \mu_0 \). That is, a positive sequence of length 1 is only \( a \), and \( \Delta a \) is of the form (2).

Since \( \lambda_q - \beta < \alpha - \beta < \alpha \), \( c \) is not enabled in \( \mu_0 + \Delta a \). \( \Delta(a^2) \) is of the form (2), and \( \Delta(ab) = (-\beta, \alpha) \) is of the form (3). \( c \) is not enabled in \( \mu_0 + \Delta(ab) \). \( \Delta(a^2) \) is of form (2). \( \Delta(a^2b), \Delta(aba) \) and \( \Delta(abc) \) are of the form (1).

Now we suppose that for \( u \in PSeq(\mu_0), \ |u| > 3 \), the vector \( \Delta u \) has a form (1), (2) or (3). Let \( x \) be an element in \( \{a, b, c\} \) such that \( ux \in PSeq(\mu_0) \). We shall show that \( \Delta(ux) \) is of the form (1), (2), or (3).

Case 1. \( \Delta u = (k(\alpha - \beta), k(\alpha - \beta)) \). \( \Delta(ua) \) is of the form (2). If \( k = 0 \), then both \( b \) and \( C \) are not enabled in \( \mu_0 + \Delta u \). For \( k \geq 1 \), \( \Delta(ab) = ((k - 1)(\alpha - \beta) - 2\beta, (k - 1)(\alpha - \beta) + 2\alpha) \) is of the form (3). \( \Delta(uc) = ((k - 1)(\alpha - \beta) + \alpha, (k - 1)(\alpha - \beta) - \beta) \) is of the form (2).

Case 2. \( \Delta u = (k(\alpha - \beta) + l\alpha, k(\alpha - \beta) - l\beta) \). \( \Delta(uu) \) is of the form (2). If \( l = 1 \), then \( \Delta(ub) \) is of the form (3). If \( l \geq 2 \), then \( \Delta(ab) = ((k + 1)(\alpha - \beta) + (l - 2)\alpha, (k + 1)(\alpha - \beta) - (l - 2)\beta) \) is the form (1) or (2). If \( k = 0 \), then \( c \) is not enabled in \( \mu_0 + \Delta u \). For \( k \geq 1 \), \( \Delta(uc) = ((k - 1)(\alpha - \beta) + 2\alpha, (k - 1)(\alpha - \beta) - 2\beta) \).

Case 3. \( \Delta u = (k(\alpha - \beta) - l\beta, k(\alpha - \beta) + l\alpha) \). \( \Delta(ua) = ((k + 1)(\alpha - \beta) - (l - 1)\beta, (k + 1)(\alpha - \beta) + (l - 1)\alpha) \) is of the form (1) or (3). \( \Delta(uc) = (k(\alpha - \beta) - (l - 1)\beta, k(\alpha - \beta) + (l - 1)\alpha) \). If \( k = 0 \), then \( b \) is not enabled in \( \mu_0 + \Delta u \). For \( k \geq 1 \), \( \Delta(ab) = ((k - 1)(\alpha - \beta) - (l + 2)\beta, (k - 1)(\alpha - \beta) + (l + 2)\alpha) \). Thus Lemma 3.9 is proved. \( \square \)

**Proposition 3.10.** If \( D(PN_3) \neq \emptyset \), and if \( \beta < \lambda_p \leq \alpha + \beta \) and \( \beta < \lambda_q \leq \alpha \), then any nonempty subset of \( D(PN_3) \) is \( (p, q) \)-limited for all \( p, q \) with \( p + q = 2 \).

**Proof.** Let \( D = D(PN_3) \), and let \( u, v \neq 1 \), be a proper right factor of \( d \in D \).

First we shall show that \( v \notin PSeq(\mu_0) \). Let \( d = uv, u, v \in A^+ \), then \( \Delta v = -\Delta u \).

Since \( u \in PSeq(\mu_0) \), by Lemma 3.9 we have one of the following:

(i) \( \Delta v = \begin{pmatrix} -k(\alpha - \beta) \\ -k(\alpha - \beta) \end{pmatrix}, \ k \geq 0, \ l \geq 0. \)

(ii) \( \Delta v = \begin{pmatrix} -k(\alpha - \beta) - l\alpha \\ -k(\alpha - \beta) + l\beta \end{pmatrix}, \ k \geq 0, \ l \geq 1. \)

(iii) \( \Delta v = \begin{pmatrix} -k(\alpha - \beta) + l\beta \\ -k(\alpha - \beta) - l\alpha \end{pmatrix}, \ k \geq 0, \ l \geq 1. \)

We consider Case (iii). If \( v \in PSeq(\mu_0) \), then, by Lemma 3.9 we have the following three cases. Case (iii)-(1)

\( \Delta v = \begin{pmatrix} x(\alpha - \beta) \\ x(\alpha - \beta) \end{pmatrix}, \ k \geq 0, \ l \geq 1, \ x \geq 0. \)
In this case, there is not a solution for $x$

Case (iii)-(2)

\[ \Delta v = \begin{pmatrix} -k(\alpha - \beta) + 1\beta \\
-k(\alpha - \beta) - la_x \end{pmatrix} = \begin{pmatrix} x(\alpha - \beta) + ya \\
x(\alpha - \beta) - y\beta \end{pmatrix}, \quad k \geq 0, \quad l \geq 1,\quad x \geq 0, \quad y \geq 1. \]

In this case, only one solution of linear system is a non-positive $(x,y) = (-k+l,l)$.

Case (iii)-(3)

\[ \Delta v = \begin{pmatrix} -k(\alpha - \beta) + 1\beta \\
-k(\alpha - \beta) - la_x \end{pmatrix} = \begin{pmatrix} x(\alpha - \beta) - y\beta \\
x(\alpha - \beta) + ya \end{pmatrix}, \quad k \geq 0, \quad l \geq 1,\quad x \geq 0, \quad y \geq 1. \]

In this case, only one solution of linear system is a non-positive $(x,y) = (-k,-l)$. Therefore in any cases we have $v \not\in \text{P Seq}(\mu_0)$. Similarly, in Case (i) or Case (ii) we cannot write $\Delta v$ in the form (1), (2) or (3) of Lemma 3.9. Therefore $v \not\in \text{P Seq}(\mu_0)$.

Let $C$ be a subset of $D$. Assume $w_0, w_1, w_2 \in A^*$ and $w_0w_1, w_1w_2 \in C^*$. If either $w_0 = 1$ or $w_1 = 1$, then $w_1 \in C^*$. Therefore we consider the case where $w_0 \neq 1$ and $w_1 \neq 1$. Let $w_0w_1 = d_1 \cdots d_m$, $d_j \in C, 1 \leq j \leq m$. There exist an integer $i, 1 \leq i \leq m$, and $u, v \in A^*$ such that $w_0 = d_1 \cdots d_{i-1}u, \quad d_i = uv$, $w_1 = vd_{i+1} \cdots d_m$. If $v \neq 1$ and $v \neq d_i$, then $v$ is a proper right factor $d_i$. By using the above fact that $v \not\in \text{P Seq}(\mu_0)$, we obtain $w_1 \not\in \text{P Seq}(\mu_0)$. This is a contradiction. Thus we have either $v = 1$ or $v = d_i$ which implies $w_1 \in C^*$. Thus $C$ is $(1,1)$-limited.

When a submonoid of a free monoid is given, it seems complicated to judge whether the submonoid is pure or not. This is because we have to show it by the treatment of many different cases of words which belong to the submonoid. Also it doesn’t seem easy to decide whether the base of a pure monoid is limited or not. Proposition 2.1 and 2.2 ensure that any submonoid generated by a code $D(PN)$ or $S(PN)$ is always pure. The proof techniques of Proposition 3.2-3.10 which use the properties of right factors of the elements in $D(PN)$ may be usable to decide whether $D(PN)$ is limited or not in other Petri nets.

References


1981.


Received 8th May 2008