Lower Bound for 3-Batched Bin Packing

The talk is based on the paper

J. Balogh, J. Békési, G. Galambos, Gy. Dósa, and Z. Tan, Lower Bound for 3-Batched Bin Packing, *Discrete Optimization*, **21**(2016): 14–24.

Bin packing problem

Given a list L of n items, where the size of the i-th item is denoted by a_i, $0 < a_i \leq 1$.

Our task is to pack the items into the minimum number of unit-capacity bins in such a way that the sum of the elements in a box does not exceed its capacity.

In this talk we deal only with the one dimensional case.

Bin packing problem

Two main research directions:

- finding exact algorithms,
- design and analysis of approximation algorithms.

The efficiency of the algorithms can be measured by

- worst-case analysis,
- probabilistic analysis,
- empirical analysis.

Bin packing problem

For the analysis of approximation algorithms, we use the following notations:

- $OPT(L)$ denotes the minimal number of bins necessary to pack the elements of list L.
- $A(L)$ denotes the number of bins used by an approximation algorithm A.

Bin packing problem

Absolute approximation ratio (**AR**): infimum $R \geq 1$, such that for all input list L:

$$A(L) \leq R \, OPT(L).$$

Asymptotic approximation ratio (**AAR**): infimum $R \geq 1$, such that for all input list L:

$$A(L) \leq R \, OPT(L) + c,$$

where c is independent from the size of the input.

Online bin packing algorithms

A special class of approximation algorithms are the

Online algorithms

- We have to pack the items in that order as they arrive in the list and we have no any information about the forthcoming ones (neither their numbers, nor their sizes).
- The packed items must not be removed later.

These constraints have the consequence that there is no on-line algorithm whose asymptotic competitive ratio (ACR) would be close to 1.

Classification of the BP algorithms

- On-line,
- Off-line,
- Semi-on-line:
  - extra information about the input,
  - extra operations are allowed during the packing,
  - combination of the above cases.

Classification of the BP algorithms

Extra information about the input

- Preordered input in nonincreasing order,
- lookahead elements (before assigning the current one),
- known optimum.

Classification of the BP algorithms

Extra operations are allowed:

- repack elements,
- using buffers for temporary packing.

Extra information and operations:

- batched bin packing.

Batched bin packing problem

We consider a special relaxation of the well-known online bin packing problem called *batched bin packing problem* (BBPP, Gutin et al.)

The elements come in batches and one batch is available for packing in a given time. If we have *K* batches then we call the problem as *K*-BBPP.

K-BBPP:

- K=1 – Offline problem
- K unbounded – Online problem

Gutin et al. gave a 1.3871… lower bound for the asymptotic competitive ratio of any on-line 2-BBPP algorithm.

Gutin, G., Jensen, T., Yeo, A.: Batched bin packing. Discrete Optimization, 2(1), 71-82 (2005)

Batched bin packing problem

Algorithms:

- Disjunctive model: the algorithm must use separate bins for the different batches.
- Augmenting model: The algorithm may use existing bins while packing the next batches.

Leah Epstein analysed the disjunctive model completely and defined an algorithm in the augmenting model for K=2 with ACR 1.5.

Epstein, L., More on batched bin packing. Operations Research Letters 44(2) (2016)

An example for the *ACR* of arbitrary online algorithm (3/2):

L_1=6n pieces of 0.15 (1/7+ε),

L_2=6n pieces of 0.34 (1/3+ε),

L_3=6n pieces of 0.51 (1/2+ε).

For the above lists

$OPT(L_1)=n$ $OPT(L_1L_2)=3n$ $OPT(L_1L_2L_3)=6n$

Online bin packing algorithms

Using L_1, L_2, L_3 we can prove that no online algorithm can pack better all the 3 sublists, than 1.5-times the optimum. This means that there is no online algorithm with *ACR* better than 1.5.

Technique of the proof: we can use Linear Programming or combinatorial calculations.

3-batched bin packing problem

Using L_1, L_2, L_3 we can prove that no online algorithm can be better than 1.5-times the optimum for 3-BBPP.

Can we prove higher lower bound?

L_1=n pieces of 1/25,

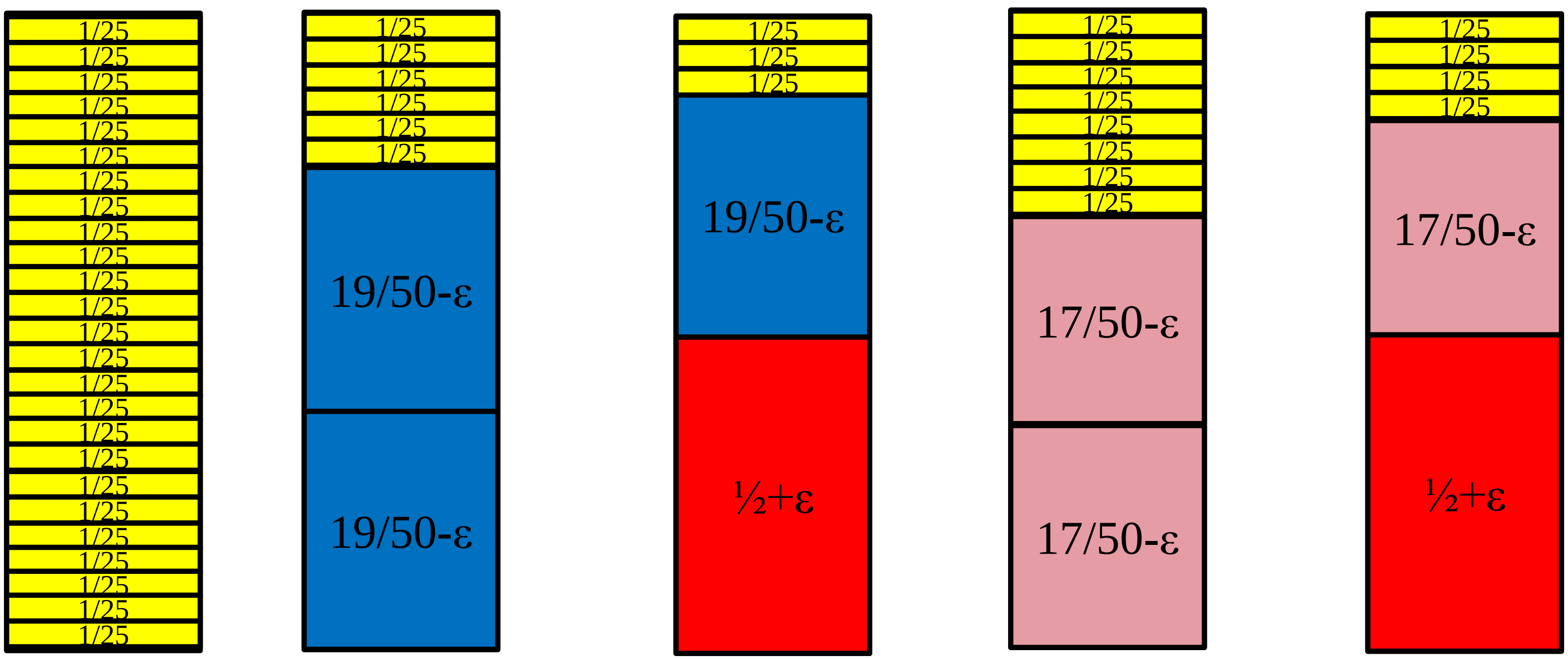
L_2=n/3 pieces of 1/2-3/25-ε = 19/50-ε ,

L_3=n/3 pieces of 1/2+ε.

L_4=n/4 pieces of 1/2-4/25-ε = 17/50-ε ,

L_5=n/4 pieces of 1/2+ε.

$OPT(L_1)$=n/25 $OPT(L_1L_2)$=n/6 $OPT(L_1L_2L_3)$=n/3 $OPT(L_1L_4)$=n/8 $OPT(L_1L_4L_5)$=n/4



3-batched bin packing problem

L_1=n pieces of 1/25,
L_2=n/3 pieces of 1/2-3/25-ε = 19/50-ε ,
L_3=n/3 pieces of 1/2+ε.
L_4=n/4 pieces of 1/2-4/25-ε = 17/50-ε ,
L_5=n/4 pieces of 1/2+ε.

$OPT(L_1)$=n/25 $OPT(L_1L_2)$=n/6 $OPT(L_1L_2L_3)$=n/3 $OPT(L_1L_4)$=n/8 $OPT(L_1L_4L_5)$=n/4

R=7300/4863≈1.5011…

3-batched bin packing problem

$L_1 = 6j \cdot n$ pieces of small items with sizes $\varepsilon = \frac{1}{6j}$, $j \geq 4$ fixed integer.

$L_{2,k} = \frac{6j}{j-k} n$ pieces of items with sizes $\frac{1}{3} + \frac{3k-1}{3}\varepsilon$, $k = 1, \ldots, j-1$.

$L_{3,k} = \frac{6j}{j-k} n$ pieces of items with sizes $\frac{1}{2} + \frac{\varepsilon}{3}$, $k = 1, \ldots, j-1$.

For the above lists

$OPT(L_1) = n$.

$OPT(L_1 L_{2,k}) = \frac{3j}{j-k} n$.

$OPT(L_1 L_{2,k} L_{3,k}) = \frac{6j}{j-k} n$.

Using an LP model we can prove a lower bound greater than 1.51 for a fixed j.

3-batched bin packing problem

min R

subject to

$$\sum_{i_1=1}^{6j} i_1\, x^k_{i_1,i_2,i_3} = 6jn$$

$$\sum_{i_1=2j-2k+1}^{4j-k} x^k_{i_1,1,0} + 2\sum_{i_1=1}^{2j-2k} x^k_{i_1,2,0} + x^k_{0,1,1} + 2x^k_{0,2,0} = \frac{6j}{j-k}n$$

$$x^k_{0,1,1} + x^k_{0,0,1} = \frac{6j}{j-k}n$$

$x^k_{i_1,i_2,i_3}$ denotes the number of (i_1, i_2, i_3) type bins while we pack the batches L_1, $L_{2,k}$, $L_{3,k}$.

3-batched bin packing problem

$$\sum_{i_1=1}^{6j} x^k_{i_1,i_2,i_3} \le R \cdot OPT(L_1)$$

$$\sum_{i_1=1}^{6j} x^k_{i_1,i_2,i_3} + x^k_{0,1,1} + x^k_{0,2,0} \le R \cdot OPT(L_1 L_{2,k})$$

$$\sum_{i_1=1}^{6j} x^k_{i_1,i_2,i_3} + x^k_{0,1,1} + x^k_{0,2,0} + x^k_{0,0,1} \le R \cdot OPT(L_1 L_{2,k} L_{3,k})$$

$k = 1, \ldots, j-1.$

3-batched bin packing problem

min R

subject to

$$\sum_{i_1=1}^{6j} i_1\, z^k_{i_1,i_2,i_3} = 6j$$

$$\sum_{i_1=1}^{6j} i_1\, z^k_{i_1,i_2,i_3} \leq R$$

$$\sum_{i_1=2j-2k+1}^{4j-k} i_1\, z^k_{i_1,1,0} + 2 \sum_{i_1=4j-k+1}^{6j} i_1\, z^k_{i_1,i_2,i_3} \leq 6j\, \frac{3R-4}{2(j-k)}$$

$k = 1, \ldots, j-1.$

3-batched bin packing problem

where

$$z^k_{i_1,i_2,i_3} = \frac{x^k_{i_1,i_2,i_3}}{n}$$

We do not need to consider all conditions from the $k = 1, \dots, j-1$ possible ones. We assume that d of them is enough, i.e. erasing the conditions for $k = d+1, \dots, j-1$, the lower bound remains the same.

3-batched bin packing problem

We multiply the first equation by -1, the second inequality by $2(j - d)$. Furthermore, we multiply the first condition of the last $j - 1$ inequalities $2j - d + 2$ and all the other ones by 2.

Making the linear combinations with the above coefficients and doing some calculations we get a formula for the lower bound. This formula has 2 parameters, j and d.

3-batched bin packing problem

$$f := \frac{30j^2 + 18j - 12j \cdot d + 24j(j-1)ln\frac{j-2}{j-d-1}}{20j^2 + 16j - 11j \cdot d + 2d + 18j(j-1)ln\frac{j-2}{j-d-1}} \leq R$$

$$d = 1, \dots, j-2$$

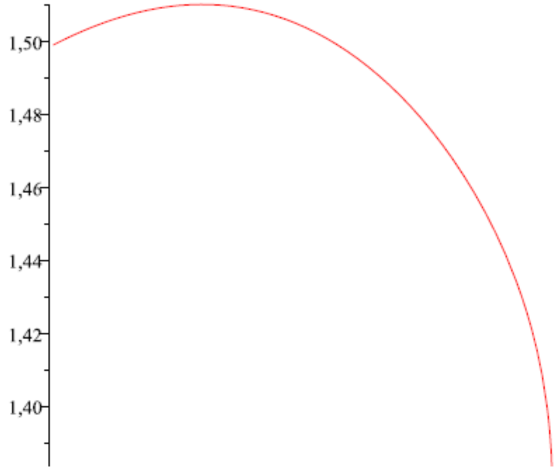


Figure 1.
Graph of the function $f(d)$ for $j = 100$.

3-batched bin packing problem

j	d	$f(j,d)$
5	1	1.480075901
10	3	1.494928787
20	6	1.503357743
50	15	1.508573181
100	30	1.510335641
200	60	1.511221192
500	152	1.511757013
1000	305	1.511935384

Table 1:
The values of $f(j,d)$

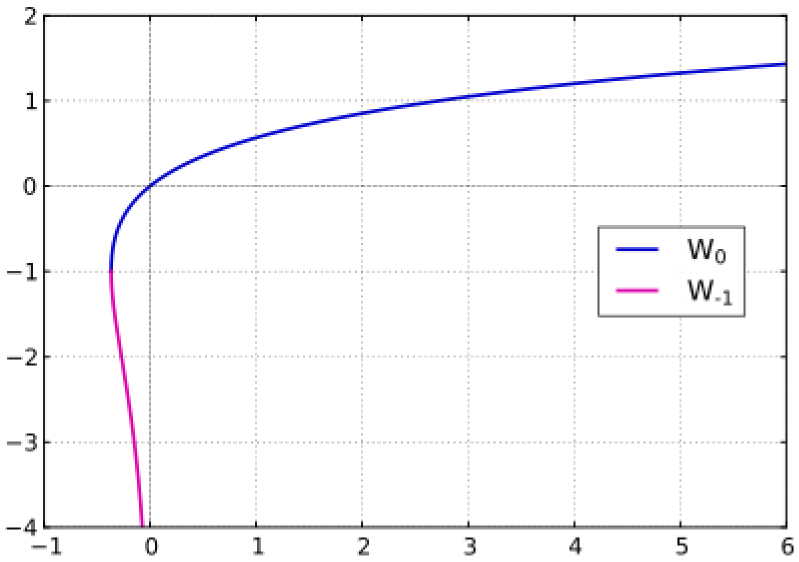
3-batched bin packing problem

$$d_0 = \frac{1}{4}\frac{9j-4}{W_{-1}\left(-\frac{1}{4}\frac{e^{\frac{-1}{8}\cdot\frac{23j-2}{j-1}}(9j-4)}{j-2}\right)} + j - 1$$

$$R \geq \frac{32W_{-1}\left(-\frac{9}{4}e^{-\frac{23}{8}}\right)+36}{24W_{-1}\left(-\frac{9}{4}e^{-\frac{23}{8}}\right)+33} \approx 1.51211383.$$

3-batched bin packing problem

The Lambert function: $W(x)e^{W(x)} = x$



Source: https://en.wikipedia.org/wiki/Lambert_W_function

Discussion and open problems

It is possible to extend the construction for K-BBPP and for parametric variants.

J. Balogh, J. Békési, G. Dósa, L. Epstein, A. Levin, Lower bounds for batched bin packing, submitted to Journal of Combinatorial Optimization, 2021.

Using the construction for larger K will not improve the lower bound for the classical online case.

There is no algorithm for $K \geq 3$.

Discussion and open problems

A new lower bound for the classical online version using branching and full adaptivity (1.54278).

J. Balogh, J. Békési, G. Dósa, L. Epstein, A. Levin, A New Lower Bound for Classic Online Bin Packing. Algorithmica (2021). https://doi.org/10.1007/s00453-021-00818-7

To be able to construct it, we needed to study the algorithms (Advanced Harmonic with ACR below 1.57829).

J. Balogh, J. Békési, G. Dósa, L. Epstein, A. Levin, A new and improved algorithm for online bin packing, In: Azar, Y., Bast, H., Herman, G. (eds.), 26th Annual European Symposium on Algorithms (ESA 2018), LIPIcs, Vol. 112, pp. 5:1–5:14, Dagstuhl, Germany, 2018. ISBN 978-3-95977-081-1

Thank you for your attention!