Structural Properties of High-Multiplicity Bin Packing

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The Bin Packing Problem

Given a set of item sizes $s_1, \ldots, s_d \in (0, 1]$ and multiplicities $b_1, \ldots, b_d$ of the corresponding item sizes.
Objective: Find a packing into as few unit sized bins as possible.

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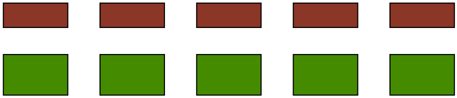
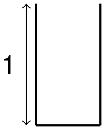
Example:

Item sizes: $s_1 = \frac{1}{5}, s_2 = \frac{1}{3}$

Multiplicities: $b_1 = 5, b_2 = 5$





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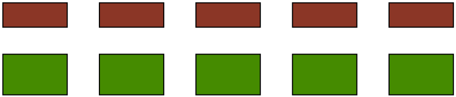

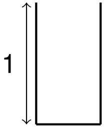

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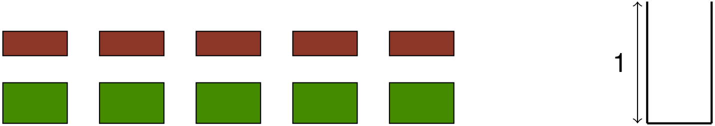
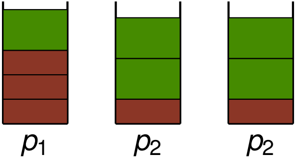


Solution



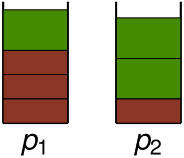
The Knapsack Polytopes

The Knapsack polytope

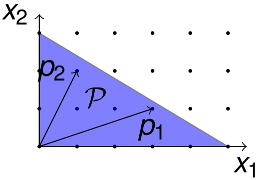
$$\mathcal{P} = \{x \in \mathbb{R}^d \mid s_1 x_1 + \ldots + s_d x_d \leq 1, x \geq 0\}$$

for given sizes $s_1, \ldots, s_d \in (0, 1]$.

Example



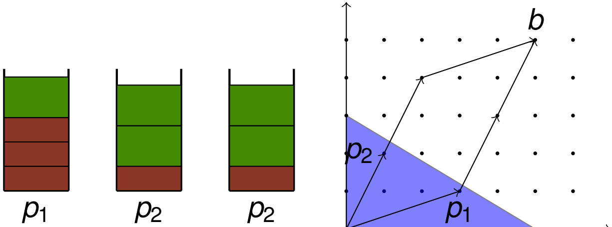
$$\mathcal{P} = \{x \in \mathbb{R}^2 \mid \frac{1}{5}x_1 + \frac{1}{3}x_2 \leq 1, x \geq 0\}$$



Geometric Interpretation

Consider a multiplicity λ_p for each $p \in \mathcal{P} \cap \mathbb{Z}^d$.
Every solution of the bin packing problem can be written as a sum

$$\sum_{p \in \mathcal{P} \cap \mathbb{Z}^d} \lambda_p p = b.$$



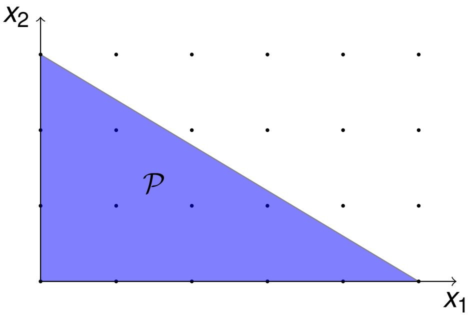
Integer Programming Formulation

Observation

The vector $(x_p)_{p \in \mathcal{P} \cap \mathbb{Z}^d}$ belongs to the system

$$
\begin{aligned}
\min \sum_{p \in \mathcal{P} \cap \mathbb{Z}^d} x_p & \\
\sum_{p \in \mathcal{P} \cap \mathbb{Z}^d} x_p p_i &= b_i \quad \text{for all } 1 \leq i \leq d \\
x &\in \mathbb{Z}_{\geq 0}^{\mathcal{P} \cap \mathbb{Z}^d}
\end{aligned}
$$

Integer Points in the Knapsack Polytope



The number of integer points $p \in \mathcal{P} \cap \mathbb{Z}^d$ is bounded by $O((\frac{1}{s})^d)$, where s is the smallest item size.

Integer Programming Formulation

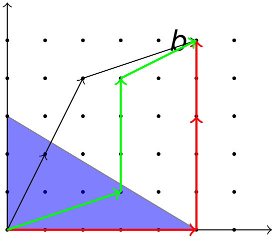
Observation

$$\begin{aligned} \sum_{p \in \mathcal{P} \cap \mathbb{Z}^d} x_p &= m \\ \sum_{p \in \mathcal{P} \cap \mathbb{Z}^d} x_p p_i &= b_i \quad \text{for all } 1 \leq i \leq d \\ x &\in \mathbb{Z}_{\geq 0}^{\mathcal{P} \cap \mathbb{Z}^d} \end{aligned}$$

Using Lenstra: Running time of roughly $O((\frac{1}{s})^{d \cdot (\frac{1}{s})^d})$.

Structural Properties

Arguing about the set of possible solutions $\lambda \in \mathbb{Z}_{\geq 0}^{\mathcal{P} \cap \mathbb{Z}^d}$.



The Structure of Solutions

Theorem (Eisenbrand, Shmonin)

There exists an integral vector $\lambda \in \mathbb{Z}_{\geq 0}^{\mathcal{P} \cap \mathbb{Z}^d}$ *with* $\sum_{p \in \mathcal{P} \cap \mathbb{Z}^d} \lambda_p p = b$ *and*

$$|supp(\lambda)| \leq 2^d.$$

The Structure of Solutions

Theorem (Goemans, Rothvoß)

There exists a set $X \subseteq \mathcal{P} \cap \mathbb{Z}^d$ with $|X| \leq d^{O(d)}(\log \Delta)^d$ such that for any point $b \in \mathbb{Z}^d$, there exists an integral vector $\lambda \in \mathbb{Z}_{\geq 0}^{\mathcal{P} \cap \mathbb{Z}^d}$ such that $b = \sum_{p \in \mathcal{P} \cap \mathbb{Z}^d} \lambda_p p$ and

1. $\lambda_p \leq 1 \qquad \forall p \in (\mathcal{P} \cap \mathbb{Z}^d) \setminus X$
2. $|supp(\lambda) \cap X| \leq 2^{2d}$
3. $|supp(\lambda) \setminus X| \leq 2^{2d}$

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Theorem (Goemans, Rothvoß)

Bin packing/makespan scheduling with d different item sizes can be solved in time $(\log \Delta)^{2^{O(d)}}$, where Δ is the maximum over all multiplicities b and denominators in s.

Integer Programming Formulation

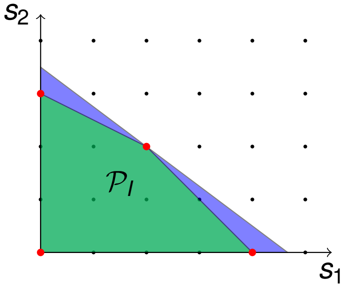
Algorithm by Goemans and Rothvoß

- Guess the support $\hat{X}$ in X with $|\hat{X}| \leq 2^{2d}$.
- Solve the following IP:

$$\begin{aligned} \sum_{p \in \hat{X}} \lambda_p p + \sum_{i=1}^{2^{2^{O(d)}}} q^{(i)} &= b \\ \sum_{j=1}^{d} s_j q_j^{(i)} &\leq 1 \quad \text{for each } q^{(i)} \\ \lambda \in \mathbb{Z}_{\geq 0}^{\hat{X}}, \quad q^{(i)} &\in \mathbb{Z}_{\geq 0}^{d} \end{aligned}$$

Vertices of the Integer Polytope

Integer Polytope $\mathcal{P}_I = Conv(\mathcal{P} \cap \mathbb{Z}^d)$



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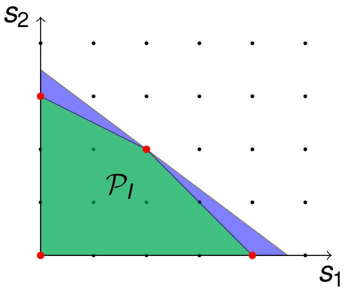

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Theorem (Cook, Hartmann, Kannan, McDiarmid)

For polytope $\mathcal{P} = \{x \in \mathbb{R}^d \mid Ax \leq c\}$ the integer polytope $\mathcal{P}_I$ has at most $m^d \cdot O((\log \Delta)^d)$ vertices.

Structure Theorem

Theorem (Jansen, K.)

Let $V_I \subseteq \mathcal{P} \cap \mathbb{Z}^d$ be the set of vertices of the integer polytope $\mathcal{P}_I$. Then for any vector $b \in \mathbb{Z}^d$, there exists an integral vector $\lambda \in \mathbb{Z}_{\geq 0}^{\mathcal{P} \cap \mathbb{Z}^d}$ such that $b = \sum_{p \in \mathcal{P} \cap \mathbb{Z}^d} \lambda_p p$ and

1. $\lambda_p \leq 2^{2^{O(d)}} \quad \forall p \in (\mathcal{P} \cap \mathbb{Z}^d) \setminus V_I$
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Theorem (Jansen, K.)

The bin packing problem can be solved in time $|V_I|^{2^{O(d)}} \cdot (\log \Delta)^{O(1)}$ and hence in fpt-time, parameterized by the number of vertices V_I.

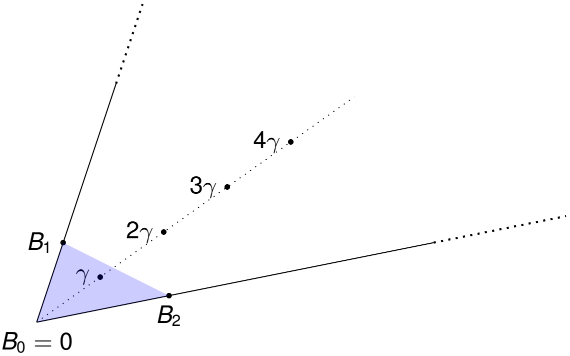
Consider the case that $\mathcal{P}_I$ is a simplex with vertices $\mathcal{B} = \{B_0, B_1, \ldots, B_d\} \subset \mathbb{Z}^d$.

Given solution $\lambda \in \mathbb{Z}_{\geq 0}^{\mathcal{P} \cap \mathbb{Z}^d}$ with $b = \sum_{p \in \mathcal{B}} \lambda_p p + \sum_{p \in (\mathcal{P} \cap \mathbb{Z}^d) \setminus \mathcal{B}} \lambda_p p$.

Main Issue

How do we handle large multiplicities λ_γ with $\gamma \notin \mathcal{B}$?

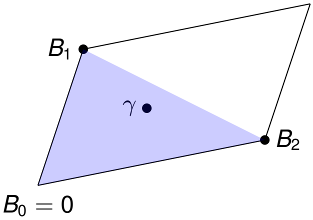
Moving Weight into the Basis



Can we move weight from a multiplicity $\lambda_\gamma \in 2^{2^{\Omega(d)}}$ to the vertices $B_1, \ldots, B_d$?

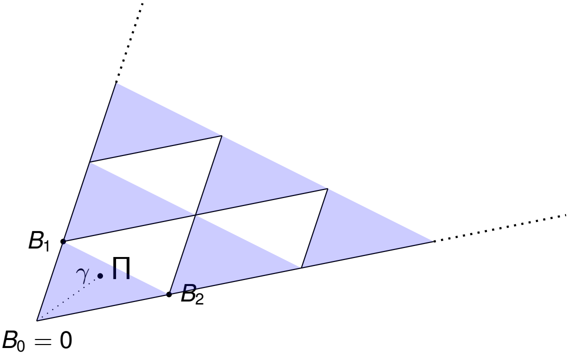
The Fundamental Parallelepiped

$$\Pi = \{x_0 B_0 + x_1 B_1 + \ldots + x_d B_d \mid x_i \in [0, 1]\}$$



Partitioning the Cone

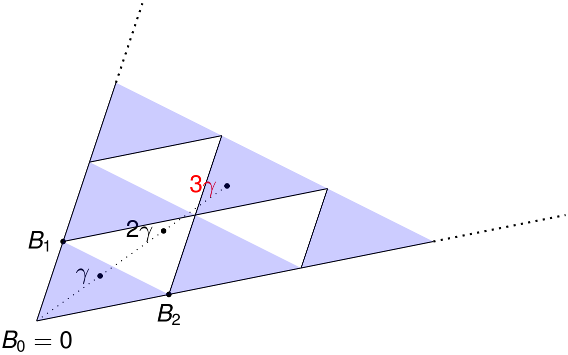
$$Cone(B) = \{\sum_{p \in \mathcal{B}} \lambda_p p + \Pi \mid \lambda \in \mathbb{Z}_{\geq 0}^{\mathcal{B}}\}$$



Partitioning the Cone

Suppose for a multiplicity $K \in \mathbb{Z}_{>1}$ and some $\lambda \in \mathbb{Z}^{\mathcal{B}}_{\geq 0}$ that

$$K\gamma \in \mathcal{P} + \sum_{p \in \mathcal{B}} \lambda_p p$$



Partitioning the Cone

Then $K\gamma$ can be written as

$$K\gamma = \gamma' + \sum_{p \in \mathcal{B}} \lambda_p p$$

for some $\gamma' \in \mathcal{P} \cap \mathbb{Z}^d$

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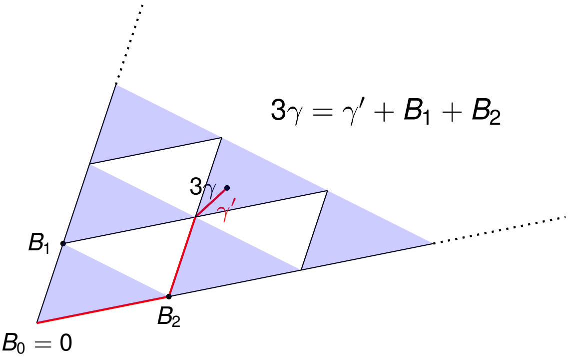
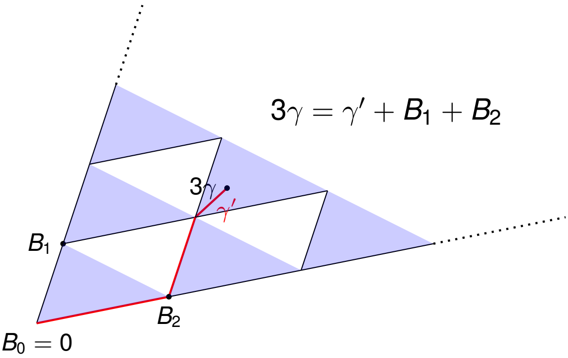


Integrality of γ'

$$\gamma' = \underbrace{K\gamma}_{\in \mathbb{Z}^d} - \underbrace{\sum_{p \in \mathcal{B}} \lambda_p p}_{\in \mathbb{Z}^d}$$



$$3\gamma = \gamma' + B_1 + B_2$$

Conclusion:

If there exists a multiplicity $K > 1$ such that

$$K\gamma \in \mathcal{P} + \sum_{p \in \mathcal{B}} \lambda_p p$$

then more weight can be shifted into the basis $\mathcal{B}$.

$\gamma \in \mathcal{P} \Leftrightarrow$ there exists $x \in [0,1]^{d+1}$ with $\mathcal{B}(x) = \gamma$ and $\sum x_i = 1$

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Splitting Kx into the integral part $\lfloor Kx \rfloor$ and the fractional part $\{Kx\}$.

$$(\lfloor Kx \rfloor)_i = \lfloor Kx_i \rfloor$$

$$\{Kx\} = Kx - \lfloor Kx \rfloor$$

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$$\{Kx\} = Kx - \lfloor Kx \rfloor$$

$$\begin{aligned} K\gamma &= K\mathcal{B}x = \mathcal{B}(Kx) = \mathcal{B}(\lfloor Kx \rfloor + \{Kx\}) \\ &= \underbrace{\mathcal{B}(\lfloor Kx \rfloor)}_{=\sum_{p \in \mathcal{B}} \lambda_p p} + \underbrace{\mathcal{B}(\{Kx\})}_{\in \Pi} \end{aligned}$$

Condition

$$K\gamma \in \mathcal{P} + \sum_{p\in\mathcal{B}} \lambda_p p \quad \Leftrightarrow \quad \sum\{Kx\}_i = 1$$

Example

$$x = \underbrace{\begin{pmatrix} 0.3 \\ 0.4 \\ 0.1 \\ 0.2 \end{pmatrix}}_{\sum=1},$$

Example

$$x = \underbrace{\begin{pmatrix} 0.3 \\ 0.4 \\ 0.1 \\ 0.2 \end{pmatrix}}_{\sum = 1}, \{2x\} = \underbrace{\begin{pmatrix} 0.6 \\ 0.8 \\ 0.2 \\ 0.4 \end{pmatrix}}_{\sum > 1},$$

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$$\{5x\} = \underbrace{\begin{pmatrix} 0.5 \\ 0.0 \\ 0.5 \\ 0.0 \end{pmatrix}}_{\sum=1}$$

Directed Diophantine Approximation:

Lemma

For given vector $x \in \mathbb{R}^d_{\geq 0}$ there exists a multiplicity $K \leq 2^{2^{O(d)}}$ such that

$$\sum \{Kx\}_i = 1.$$

A Sketch of the Proof:

Suppose the components of x are sorted by their sizes i.e.

$$x_1 \geq x_2 \geq \ldots \geq x_d.$$

- Is there a component d' with a big jump in size i.e. $x_{d'} > x_{d'+1} \prod_{i=1}^{d'} \frac{1}{x_i}$?
- Partition $[0, 1]^{d'}$ into boxes B_x by partitioning each component $1 \leq i \leq d'$ into intervals $[kx_i, (k+1)x_i)$ for $1 \leq k \leq \lfloor \frac{1}{x_i} \rfloor$.
- There are at most $2^{2^{O(d)}}$ many boxes B_x.

A Sketch of the Proof:

There exist two multiplicities $K, K' \in 2^{2^{O(d)}}$ with $K' > K > 1$ and a box $B_x \subset [0,1]^{d'}$ such that

$$\{Kx\}, \{K'x\} \in B_x.$$

It holds that

- $(K' - K + 1)x_i \geq 0$,
- $\sum \{Kx\}_i = \sum \{K'x\}_i$,

Hence,

$$\sum \{(K' - K + 1)x\}_i = 1$$

Integer Programming Formulation

Algorithm by Goemans and Rothvoß

- Guess the support $\hat{X}$ in V_I with $|\hat{X}| \leq 2^{2d}$.
- Solve the following IP:

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\sum_{p \in \hat{X}} \lambda_p p + \sum_{i=1}^{2^{2^{O(d)}}} q^{(i)} &= b \\
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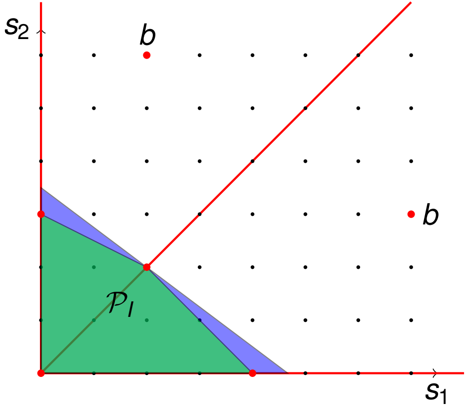
\lambda \in \mathbb{Z}_{\geq 0}^{\hat{X}}, \quad q^{(i)} &\in \mathbb{Z}_{\geq 0}^{d}
\end{aligned}$$

Guessing the support of $\hat{X}$

Using information from the solution of the relaxed linear program.



Conjecture:

Proximity

Given basic feasible solution x of the relaxed linear program. Then there exists an integral solution y such that

$$\|x - y\|_1 \leq f(d)$$

for some function f.

Main Open Question:

- Is there an fpt-algorithm for the bin packing/makespan scheduling problem parameterized by d?
- What about other objectives?
- Allowing an ϵ error in makespan scheduling.