Almost Optimal Inapproximability of Multidimensional Packing Problems

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Plan

1. Introduction
2. Results
3. Vector Bin Packing
4. Summary
5. Vector Scheduling
6. Vector Bin Covering

Packing Problems

- Input: n items with sizes $a_1, a_2, \ldots, a_n \in [0, 1]$.
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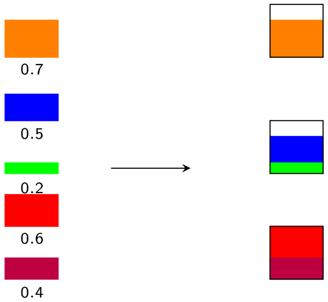

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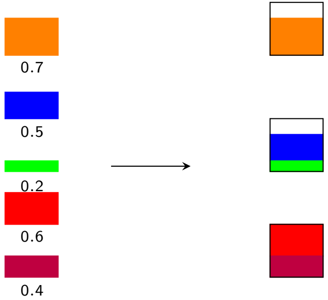

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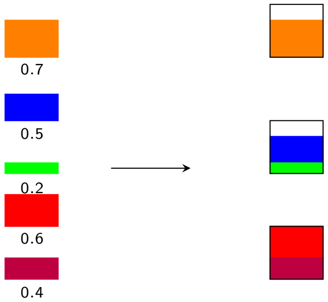

- Multiprocessor Scheduling: Fixed number of bins, Minimize the maximum load.
- Bin Covering: Maximize the number of bins, each bin should get at least 1 load.

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- All of them have a Polynomial Time Approximation Scheme (PTAS): $(1+\epsilon)$-factor approximation algorithm running in time poly$(n, \frac{1}{\epsilon})$.
- Asymptotic approximation for Bin Packing: We study the setting when the optimal value is large enough.

Multidimensional Packing Problems

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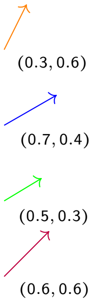
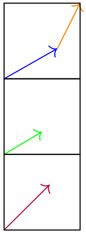
- Motivation:
  1. Fundamental problems in theory, well studied in the approximation algorithms community.
  2. Most of the scheduling problems in practice are multidimensional.

Vector Bin Packing

Vector Bin Packing

- Input: n d-dimensional vectors $v_1, v_2, \ldots, v_n \in [0,1]^d$.
- Assign the jobs to minimum number of machines such that in each machine, the sum is at most 1 in every coordinate.





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- When d is part of the input, essentially tight : $\Omega(d^{1-\epsilon})$ NP-hardness (Chekuri, Khanna, SICOMP 2004)

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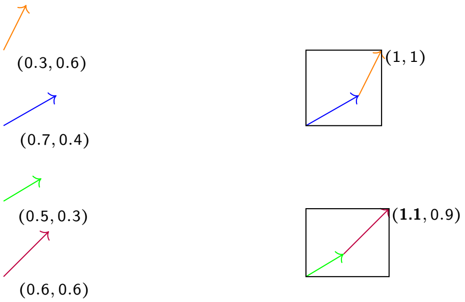
- Question: Is there a constant factor approximation for Vector Bin Packing when the dimension is fixed?

Vector Scheduling

Vector Scheduling

- Input is $v_1, v_2, \ldots, v_n \in [0,1]^d$ and number of machines m.
- Find $f : [n] \to [m]$. Load vector on a machine $j \in [m] : \sum_{i \in [n]: f(i)=j} v_i$.
- Objective: Minimize the maximum ℓ_∞ norm of the load vectors.



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Our results

Problem	Subcase	Best Algorithm	Best Hardness
VBP	$d = 1$	PTAS	NP-Hard
	Fixed d	$\ln d + O(1)$	$\Omega(\log d)$
	Arbitrary d	$1 + \epsilon d + O\left(\ln \frac{1}{\epsilon}\right)$	$d^{1-\epsilon}$
VS	$d = 1$	PTAS	NP-Hard
	Fixed d	PTAS	NP-Hard
	Arbitrary d	$O\left(\frac{\log d}{\log \log d}\right)$	$\Omega\left((\log d)^{1-\epsilon}\right)$ $\left(\mathrm{NP} \not\subseteq \mathrm{ZPTIME}\left(n^{(\log n)^{O(1)}}\right)\right)$
VBC	$d = 1$	FPTAS	NP-Hard
	Arbitrary d	$O(\log d)$	$\Omega\left(\frac{\log d}{\log \log d}\right)$

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Vector Bin Packing

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- Configuration: A subset $\{i_1, i_2, \ldots, i_k\} \subseteq [n]$ is called a configuration if

$$\left\| v_{i_1} + v_{i_2} + \ldots + v_{i_k} \right\|_\infty \leq 1$$

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- Objective is to use the minimum number of configurations to cover $[n]$.
- Using the minimum number of sets from a given family to cover all the elements: Set Cover Problem.

$$\boxed{\text{Vector Bin Packing}} \longrightarrow \boxed{\text{Set Cover}}$$

Reversing the Reduction

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Which Set Cover instances can be formulated as d-dimensional Vector Bin Packing instances?

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Which Set Cover instances can be formulated as d-dimensional Vector Bin Packing instances?

- Given a set family $\mathcal{F} \subseteq 2^{[n]}$, goal is to find vectors $v_1, v_2, \ldots, v_n \in [0,1]^d$ such that for every set $S \subseteq [n]$, $S \in \mathcal{F}$ if and only if

$$\left\| \sum_{i \in S} v_i \right\|_\infty \leq 1$$

Packing Dimension

- Packing Dimension of $\mathcal{F} \subseteq 2^{[n]}$ pdim($\mathcal{F}$) is the smallest positive integer d such that there exist vectors $v_1, v_2, \ldots, v_n \in [0,1]^d$ with

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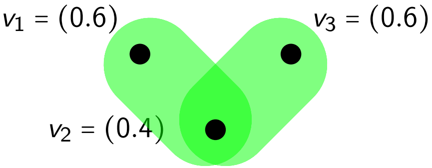
  - No isolated elements: For every $i \in [n]$, there exists $S \in \mathcal{F} : i \in S$.
- These two conditions are sufficient.



$\mathcal{F} = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}\}$

Hardness of VBP

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Packing Dimension of $\mathcal{F}$ is at most d

Set Cover is Hard to approximate within $f(d)$ on $\mathcal{F}$

$\downarrow$

VBP is Hard to approximate within $f(d)$

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- Answer: Simple Bounded Set Families. ☺

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- (k, Δ)-Bounded: Each set has cardinality at most k, and each element appears in at most Δ sets.

Set Cover on Simple Bounded Set Families

Set Cover is hard to approximate within $\Omega(\ln k)$ factor on simple (k, Δ)-bounded set families with $\Delta = k^{O(1)}$, when k is a large constant.

- Proof: Essentially (Anil Kumar, Arya, Ramesh, ICALP 2000), start with a modified Label Cover instance.

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Packing Dimension of Simple Bounded Set Families

Suppose that $\mathcal{F}$ is a simple (k, Δ)-bounded set family with no isolated elements. Then,

$$\mathrm{pdim}(\mathcal{F}^{\downarrow}) \leq (k\Delta)^{O(1)}$$

A property of Packing Dimension

Sub-additivity of Packing Dimension

$\mathrm{pdim}(\mathcal{F}_1 \cap \mathcal{F}_2)$ is at most $\mathrm{pdim}(\mathcal{F}_1) + \mathrm{pdim}(\mathcal{F}_2)$.

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- Proof: $f_1 : [n] \to [0,1]^{d_1}$ such that for every $S \subseteq [n]$, $S \in \mathcal{F}_1$ if and only if

$$\left\| \sum_{i \in S} f_1(i) \right\|_\infty \leq 1$$

Similarly, $f_2 : [n] \to [0,1]^{d_2}$.

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  Similarly, $f_2 : [n] \to [0,1]^{d_2}$.
- Define $f : [n] \to [0,1]^{d_1+d_2}$ as $(f_1(i), f_2(i))$. For every $S \subseteq [n]$,

$$\left\| \sum_{i \in S} f(i) \right\|_\infty \leq 1$$

  if and only if $S \in \mathcal{F}_1$ and $S \in \mathcal{F}_2$.

Sunflower Bouquets

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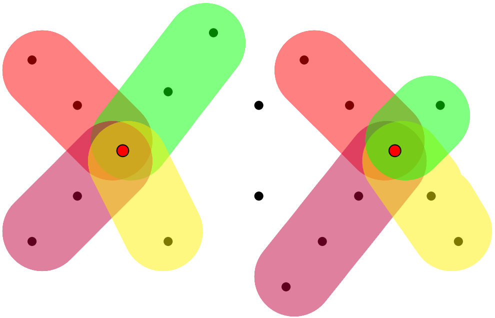
- Yes, using Sunflower-Bouquets.

(k, Δ)-Sunflower Bouquet with core U

1. Every set $S \in \mathcal{F}$ intersects with U exactly once.
2. Intersection of any two sets is either empty or is in U.
3. Each set has cardinality at most k, each element appears in at most Δ sets.



Embedding of Sunflower Bouquets

Embedding of Sunflower Bouquets (Main Technical Lemma)

Suppose that $\mathcal{F} \subseteq 2^{[n]}$ is a (k, Δ)-Sunflower with core U. Then, there exists an embedding $f : [n] \to [0,1]^d$ with $d = \text{poly}(k, \Delta)$ such that

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1. For every set $S \in \mathcal{F}$, $\|\sum_{i \in S} f(i)\|_\infty \leq 1$.
2. For every set $S \notin \mathcal{F}^{\downarrow}$ with $S \cap U \neq \phi$, $\|\sum_{i \in S} f(i)\|_\infty > 1$.

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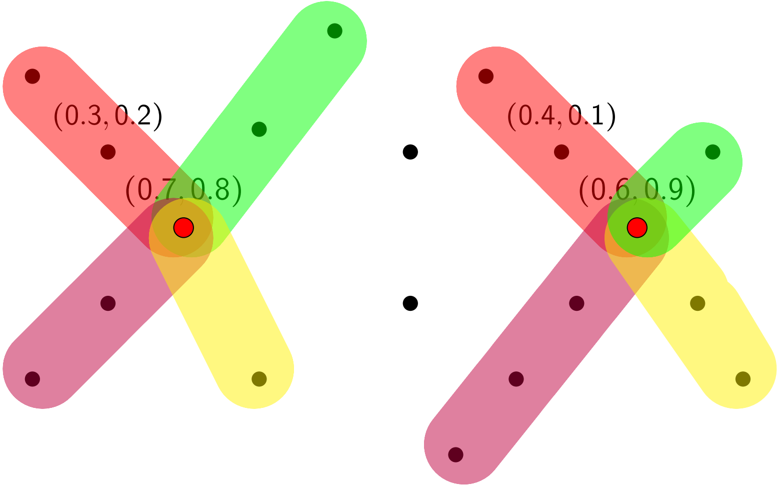
3. For every set S with $S \cap U \neq \phi$ and $|S| \leq k$, $\|\sum_{i \in S} f(i)\|_\infty \leq 1$.

- High level idea: many embeddings, each satisfying 1. and 3.
  1. Eliminating "inter-sunflower" sets.
  2. Pinning the "intra-sunflower" sets.

Inter-Sunflower sets



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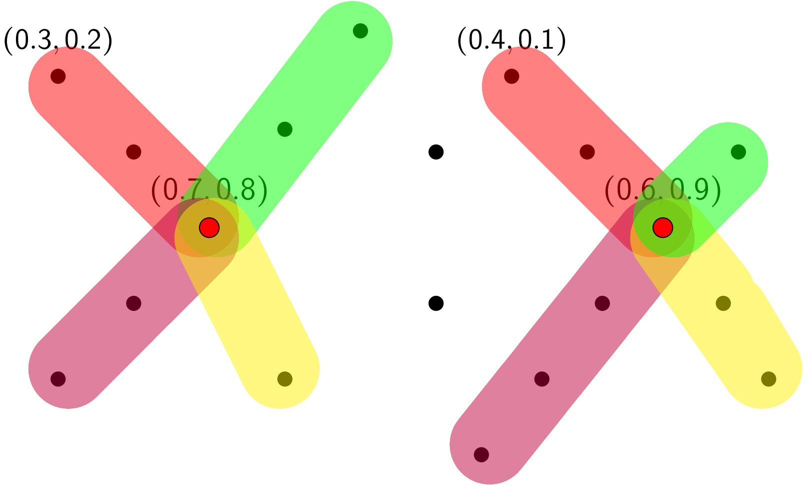
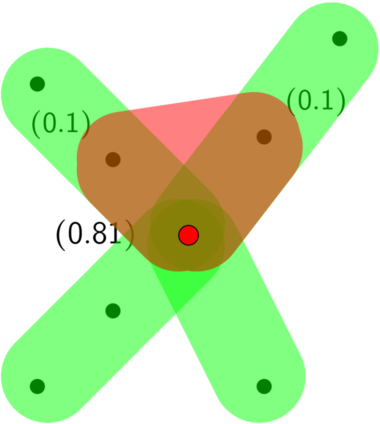


Intra-Sunflower sets



- For every minimal set $S \notin \mathcal{F}^{\downarrow}$ with $S \cap U \neq \phi$, we create a coordinate such that the sum is greater than 1.
- There are only $O((k\Delta)^2)$ such minimal sets in a sunflower.

Packing Dimension of Simple Bounded Set Families

- $\mathcal{F} \subseteq 2^{[n]}$: Simple (k, Δ)-Bounded set family.

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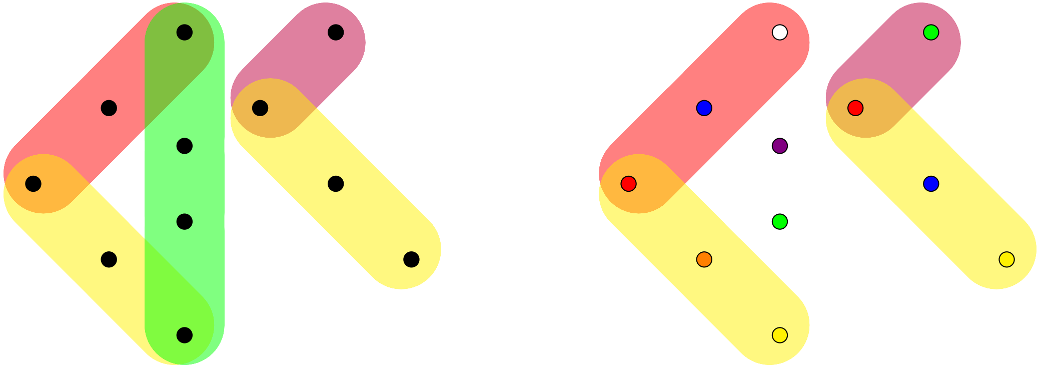
- Note: $L = O\left((k\Delta)^2\right)$ suffices.
- $\mathcal{F}_i \subseteq \mathcal{F}$: all the sets that hit the ith color. $\mathcal{F}_i$ is a Sunflower-Bouquet!



Packing Dimension of Simple Bounded Set Families

- Use the Embedding of Sunflower-Bouquets for every $\mathcal{F}_i$, $f_i : [n] \to [0,1]^d$ with $d = \text{poly}(k, \Delta)$ such that
  1. For every set $S \in \mathcal{F}_i$, $\|\sum_{i \in S} f(i)\|_\infty \leq 1$.
  2. For every set $S \notin \mathcal{F}_i^\downarrow$ with $S \cap U_i \neq \phi$, $\|\sum_{i \in S} f(i)\|_\infty > 1$.
  3. For every set S with $S \cap U_i \neq \phi$ and $|S| \leq k$, $\|\sum_{i \in S} f(i)\|_\infty \leq 1$.

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  3. For every set S with $S \cap U_i \neq \phi$ and $|S| \leq k$, $\|\sum_{i \in S} f(i)\|_\infty \leq 1$.
- Final embedding $f = (f_1, f_2, \dots, f_L)$
  1. For every set $S \in \mathcal{F}$, $\|\sum_{i \in S} f(i)\|_\infty \leq 1$.
  2. For every set $S \notin \mathcal{F}^{\downarrow}$, $\|\sum_{i \in S} f(i)\|_\infty > 1$.
- Completes the proof that $\text{pdim}(\mathcal{F})$ is at most $\text{poly}(k, \Delta)$.

Hardness of Vector Bin Packing

- The Packing Dimension of Simple (k, Δ)-Bounded set families is at most $\text{poly}(k, \Delta)$.
- Set Cover on Simple (k, Δ)-Bounded set families is hard to approximate within $\Omega(\log k)$ when $\Delta = k^{O(1)}$ and k is a large constant.

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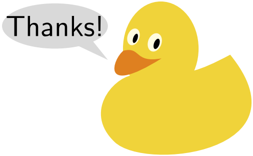
- Set Cover on Simple (k, Δ)-Bounded set families is hard to approximate within $\Omega(\log k)$ when $\Delta = k^{O(1)}$ and k is a large constant.
- Both together prove that d-dimensional Vector Bin Packing is hard to approximate within $\Omega(\log d)$ factor when d is a large constant.

Plan

Summary

- Hardness of Multidimensional Packing Problems: Vector Bin Packing, Vector Scheduling, Vector Bin Covering.
- Vector Bin Packing: Packing Dimension.
- Open Problems: Other applications of Packing Dimension? Hardness of Geometric Bin Packing? Better hardness for 2-dimensional Bin Packing?



Plan

Our results

Vector Bin Packing

Assuming $P \neq NP$, when d is fixed, Vector Bin Packing is hard to approximate within $\Omega(\log d)$ factor.

Vector Scheduling

Assuming NP has no quasipolynomial time algorithms, Vector Scheduling has no $\Omega((\log d)^{1-\epsilon})$ factor polynomial time algorithms for all $\epsilon > 0$.

Vector Bin Covering

Assuming $P \neq NP$, Vector Bin Packing has no polynomial time algorithm with approximation factor $\Omega\left(\frac{\log d}{\log \log d}\right)$.

Vector Scheduling

- Input: $v_1, v_2, \ldots, v_n \in [0,1]^d$, and m, the number of machines.
- Find $f : [n] \to [m]$. Load vector on a machine $j \in [m] : \sum_{i \in [n] : f(i) = j} v_i$.
- Objective: Minimize the maximum ℓ_∞ norm of the load vectors.

Monochromatic Clique

- Monochromatic Clique(k, B): Given a graph $G = ([n], E)$, and parameters $k := k(n), B := B(n)$, the goal is to distinguish between
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- $B = 2$: Standard k-coloring of graphs.
- Problem gets easier as B increases.
- When $B = \sqrt{n}$, the problem can be solved in polynomial time.

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- Define $\mathcal{V} = \{v_1, v_2, \ldots, v_n\} \subseteq \{0, 1\}^d$ as

$$(v_i)_j = \begin{cases} 1 & \text{if } i \in T_j \\ 0 & \text{otherwise.} \end{cases}$$

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  The number of machines is equal to k.
- Completeness: If G is k-colorable, there is an assignment with maximum ℓ_∞ value 1.
- Soundness: If in any assignment of k-colors to the vertices of G there exists a clique of size B, the load is at least B in any scheduling.

Hardness of Monochromatic Clique

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- Question: Can we prove improved hardness of Monochromatic Clique (k, B) when B is larger?
- Answer: Yes, using Lexicographic graph product based amplification.

Hardness of Vector Scheduling

Graph Coloring Hardness of Approximation (Khot, FOCS 2001)

↓ Elementary Ramsey Theoretic Argument

Strong Monochromatic Clique Hardness

↓ Amplification using Lexicographic Product

Monochromatic Clique Hardness

↓ Above reduction

Vector Scheduling Hardness of Approximation

Strong Monochromatic Clique

- Strong Monochromatic Clique(k, B, C): A generalization of Monochromatic Clique: Given a graph $G = ([n], E)$, parameters $k := k(n), B := B(n), C$, the goal is to distinguish between
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Hardness of Strong Monochromatic Clique

Assuming NP has no quasipolynomial time algorithm, there exist constant $\gamma > 0$ and $k := k(n)$ such that Strong Monochromatic Clique $(k, (\log n)^{\gamma}, C)$ has no quasipolynomial time algorithm for all integers $C \geq 1$.

Lexicographic Product

- $G = G_1 \times G_2$. Vertex set of G is $V_1 \times V_2$.
- Two vertices (u_1, u_2) and (v_1, v_2) are adjacent in G if
  - (u_1, v_1) are adjacent in G_1, or
  - $u_1 = v_1$, and (u_2, v_2) are adjacent in G_2

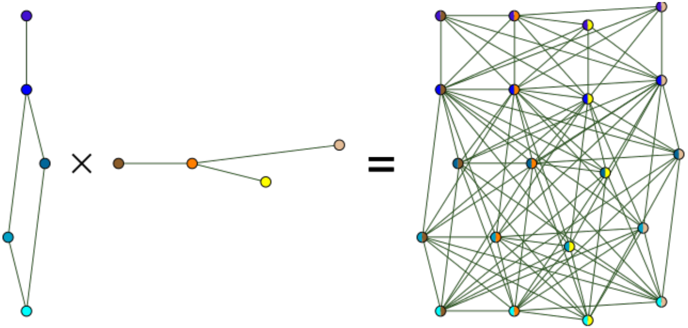

Image Credit: Wikipedia, Author=David Eppstein

Hardness Amplication using Lexicographic Product

- Reduction from Strong Monochromatic Clique (k, B, C) to Monochromatic Clique (k^C, B^C).

Hardness Amplication using Lexicographic Product

- Reduction from Strong Monochromatic Clique (k, B, C) to Monochromatic Clique (k^C, B^C).
- Let $G^2 = G \times G$.
- Completeness: If $\chi(G) \le k$, then $\chi(G^2) \le k^2$. If $c : V(G) \to [k]$ is a proper k-coloring of G, simply assign $(c(u_1), c(u_2))$ to (u_1, u_2).

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- Soundness: Suppose that in any assignment of k^2 colors to $V(G)$, there is a monochromatic clique of size B.
- Consider an assignment $c : V(G^2) \to [k^2]$.
    - For every $u \in V(G)$, there is a clique of size B $(u, v_1), (u, v_2), \dots, (u, v_B)$ in G^2 that are all assigned the same color $c'(u)$.

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  - c' itself is a coloring of $V(G)$, there is a clique of size B, $u_1, u_2, \ldots, u_B$ that have the same c' value.

Hardness Amplification Using Lexicographic Product

- Let $G' = G^C$.
    1. Completeness: If $\chi(G) \leq k$, then $\chi(G') \leq k^C$.
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- Polynomial time reduction from Strong Monochromatic Clique (k, B, C) to Monochromatic Clique (k^C, B^C).

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Plan

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- View each vector as a vertex of a hypergraph and each coordinate as an edge of the hypergraph.
- Given a hypergraph, objective is to partition the vertex set into maximum number of parts such that every part hits each edge.
- Such a coloring of hypergraphs: Rainbow Coloring.

Hypergraph Rainbow Coloring

- Hypergraph $H = (V, E)$, find $c : V \to [k]$ such that

$$\bigcup_{v \in e} c(v) = [k] \quad \forall e \in E$$

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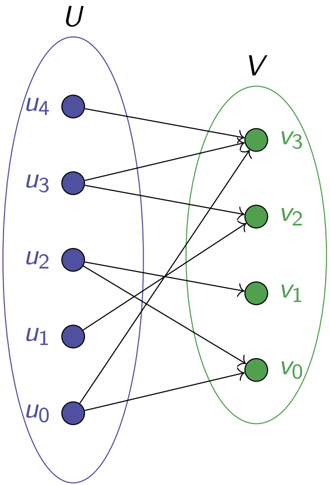
- When $k = 2$, same as 2-coloring of hypergraphs. NP-Hard.

Approximate Rainbow Coloring Hardness

Given a hypergraph H with m edges, it is NP-Hard to distinguish between

1. H is 2-colorable.
2. H cannot be rainbow colored with $\Omega\left(\frac{\log m}{\log \log m}\right)$ colors.

Hardness of approximate Rainbow Coloring



Label Cover

In the Label Cover problem, the input is a bipartite graph $U \cup V, E$ with projection constraints $\Pi_e : \Sigma \to \Sigma$ on each edge.

- The objective is to assign labels from Σ to the vertices to satisfy as many constraints as possible.
- Very hard to approximate: NP-Hard to find a labeling satisfying ϵ fraction of the constraints on fully satisfiable instances.
- "Mother of most optimal inapproximability results".

Approximate Rainbow Coloring Hardness

- Gadget reduction based on Label Cover-Long Code framework.

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- Gadget reduction based on Label Cover-Long Code framework.
- Difference from the previous results: we now use very weak Label Cover hardness.
- We just use NP-Hardness of Label Cover, so alphabet size is $O(1)$.