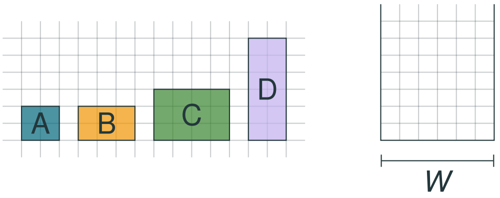
A Tight $(3/2 + \varepsilon)$ Approximation for Skewed Strip Packing

W. Gálvez, F. Grandoni, A. Jabal Ameli, K. Jansen, A. Khan and M. Rau

Introduction

Strip packing

- Input:
  - Strip with width $W \in \mathbb{N}$ and infinite height
  - Set of n items $\mathcal{I}$ with width $w_i \in \mathbb{N}_{\leq W}$ and height $h_i \in \mathbb{N}$



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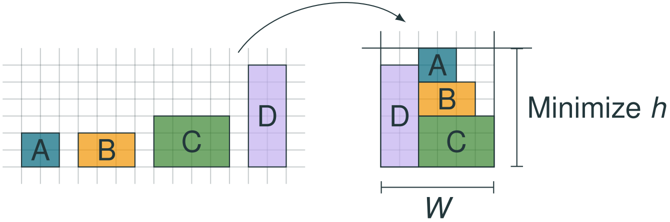
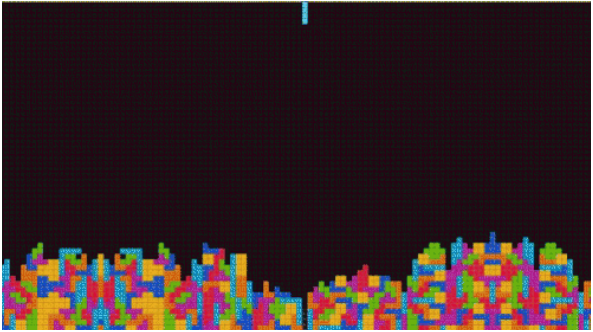
  - Strip with width $W \in \mathbb{N}$ and infinite height
  - Set of n items $\mathcal{I}$ with width $w_i \in \mathbb{N}_{\leq W}$ and height $h_i \in \mathbb{N}$
- Objective:
  - Find a *feasible* packing of the items in the strip with minimum height.



Strip Packing

Strip Packing bears a resemblance to Tetris.



Applications

- Manufacturing Processes

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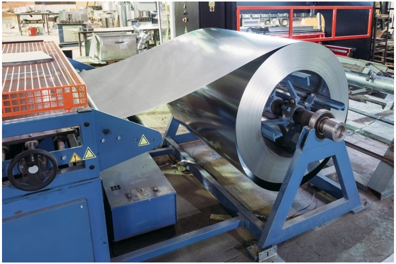
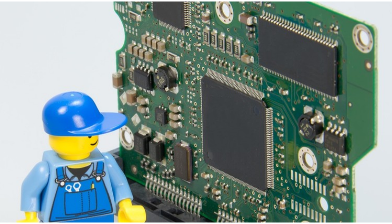


Applications

- VLSI Design



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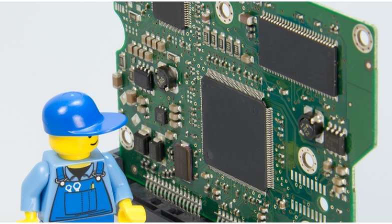
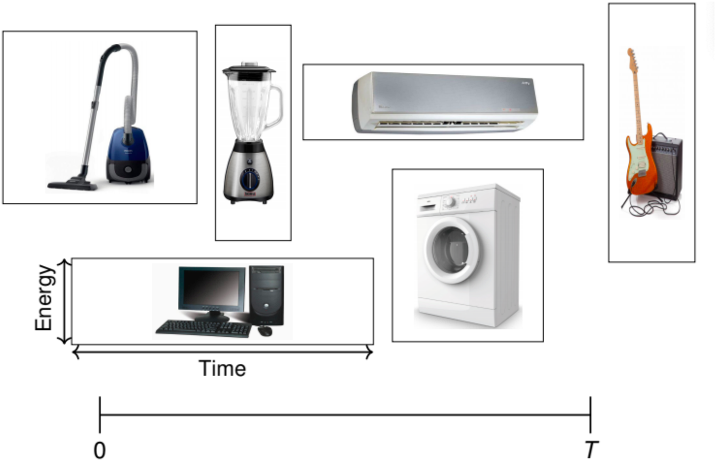


Applications

- Optimizing Energy Consumption



Applications

- Optimizing Energy Consumption

State Of the Art

$3/2 - \varepsilon$

There is no $(3/2 - \varepsilon)$-approximation unless $P = NP$

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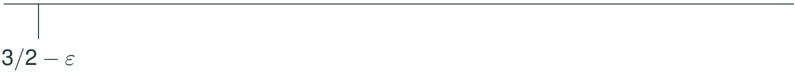

There is no $(3/2 - \varepsilon)$**-approximation unless** $P = NP$

By a simple reduction from the *Partition* problem.

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Pseudo-Polynomial Time $(n \cdot W \cdot h_{\max})^{\mathcal{O}(1)}$

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Algorithms

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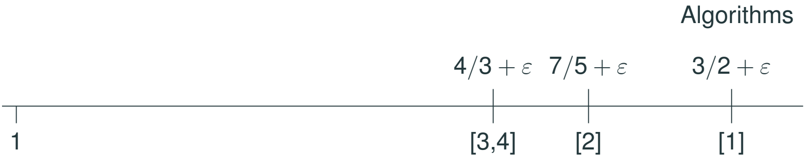

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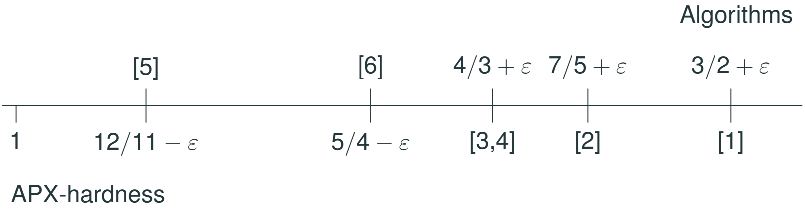

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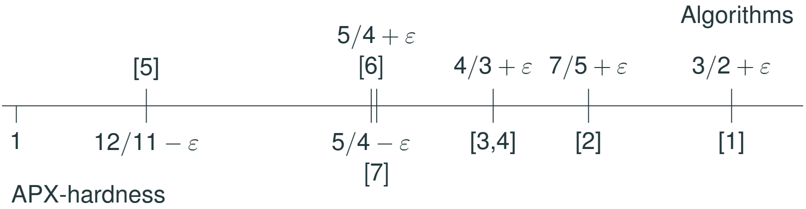

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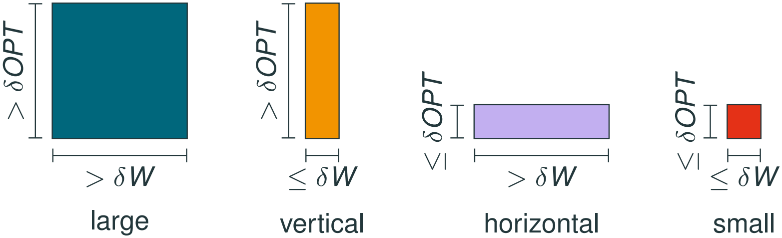
Standard Approach: Size classification

Classify items by size



Large items

Observation

- *There can be only* $\mathcal{O}_\delta(1)$ *large items.*
- *If we are given an instance containing only large items we are able to compute efficiently.*

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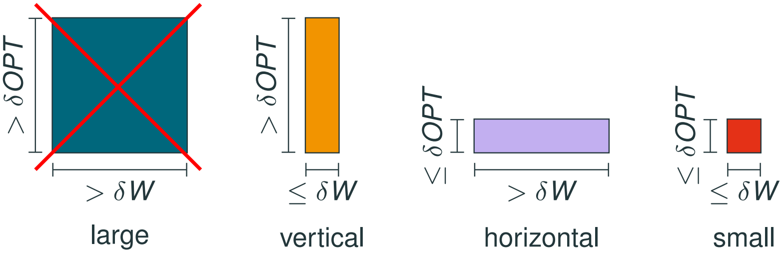
How about the complementary case?

Our problem

δ-skewed Strip Packing

Strip Packing restricted to instances where all items have either $w_i \leq \delta W$ or $h_i \leq \delta T$.



Our Results

Theorem

For any $\delta > 0$ and $\varepsilon > 0$, there is no polynomial time $(3/2 - \varepsilon)$-approximation for δ-skewed Strip Packing unless $P = NP$.

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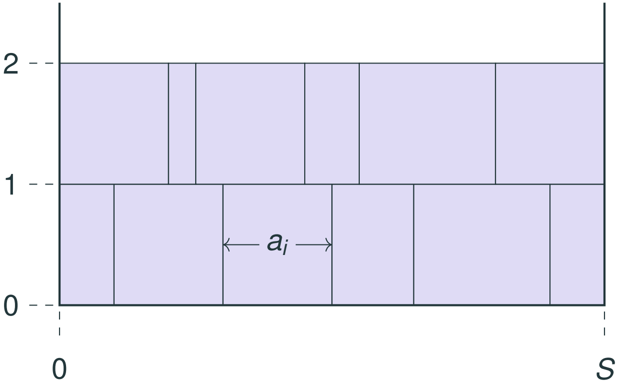
Theorem

For any $\varepsilon > 0$ there exists $\delta > 0$ such that there is a $(3/2 + \varepsilon)$-approximation for δ-skewed Strip Packing.

Hardness

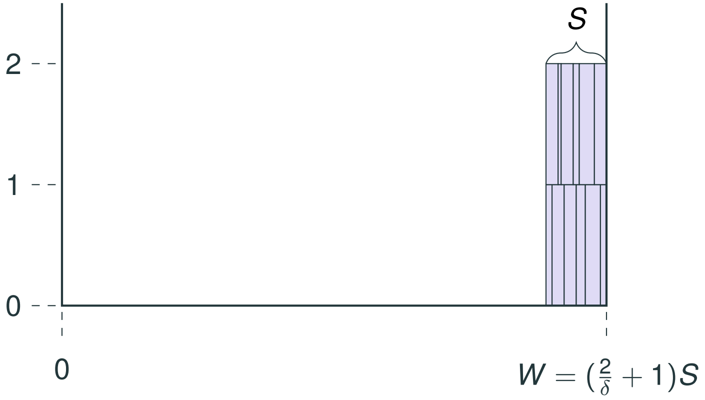
Reduction from Partition

- Classical reduction from Partition: $2n$ rectangles of height 1 and widths $a_1, a_2, \ldots, a_{2n}$ satisfying $a_1 + \cdots + a_{2n} = 2S$.



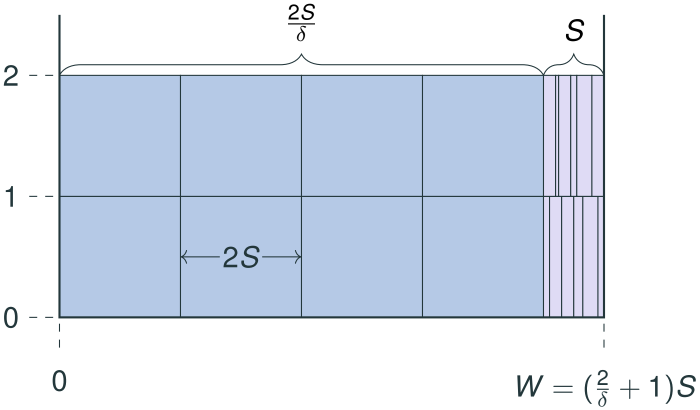
Reduction from Partition

- Let us increase the width of the strip by a factor $2/\delta + 1$; this way the rectangles become δ-skewed.



Reduction from Partition

- Let us add δ-skewed dummy items of width $2S$ and height 1 (they are δ-skewed).



Algorithm

General approach of previous algorithms

Search for well-structured solutions

Decompose the solution into *simple* regions where items are packed *by means of polynomial time algorithms*.

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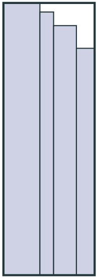

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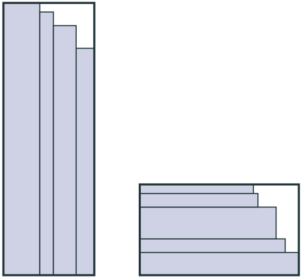

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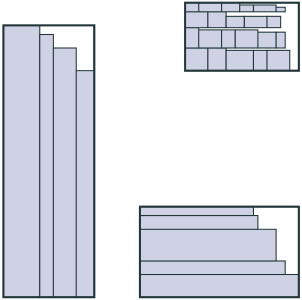

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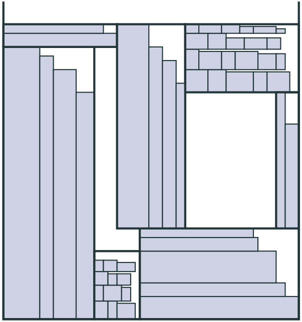
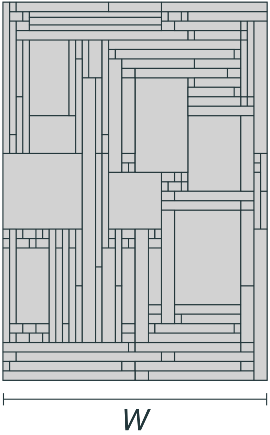
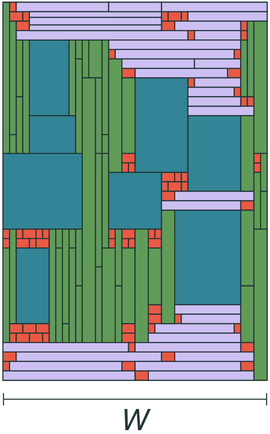


Harren et al.'s algorithm



1. Consider the optimal solution and derive a well-structured solution.



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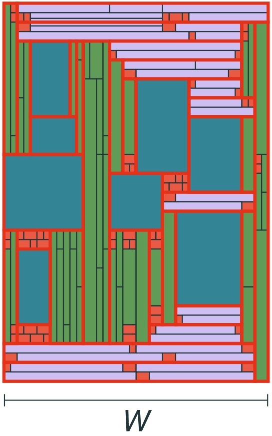

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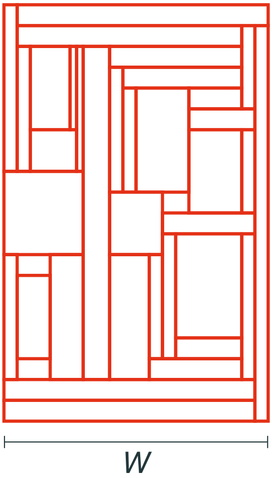

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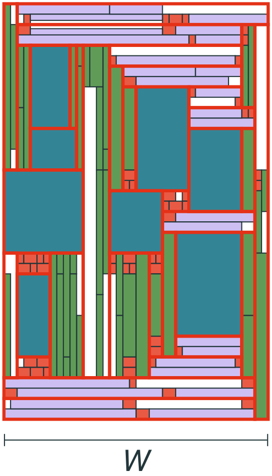

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3. Add almost all the items via linear programming

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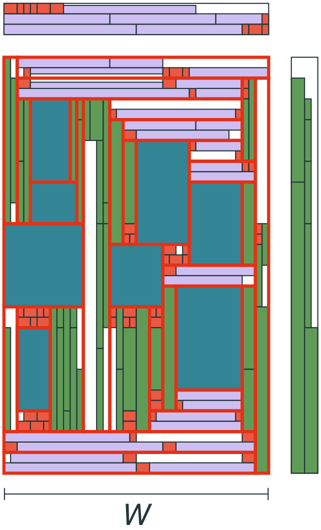

1. Consider the optimal solution and derive a well-structured solution.
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4. Create two extra thin regions for the residual rectangles.

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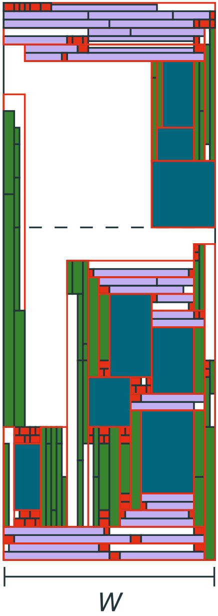

1. Consider the optimal solution and derive a well-structured solution.
2. Iterate all structures to find the correct one
3. Add almost all the items via linear programming
4. Create two extra thin regions for the residual rectangles.
   These containers can be included by using extra $2/3 \cdot \mathrm{OPT}$ height.

Strategy

Idea

Place **all** items with height larger than $\frac{1}{2}\mathrm{OPT}$ inside the structured solution.

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Structural lemma

If $\mathcal{I}$ is an instance of δ-skewed Strip Packing, there exists a well-structured solution of height at most $\left(\frac{3}{2} + \varepsilon\right) OPT$ for $\mathcal{I}$ such that:

- The rectangles of height larger than $\frac{1}{2}OPT$ can be packed exactly in polynomial time into the solution,

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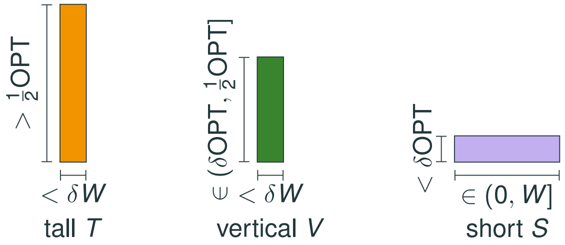
Structural lemma

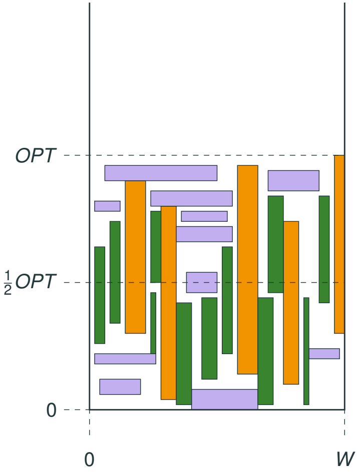
If $\mathcal{I}$ is an instance of δ-skewed Strip Packing, there exists a well-structured solution of height at most $\left(\frac{3}{2} + \varepsilon\right)$ *OPT* for $\mathcal{I}$ such that:

- The rectangles of height larger than $\frac{1}{2}$*OPT* can be packed exactly in polynomial time into the solution, and
- There is a free rectangular region of height $\frac{1}{2}$*OPT* and width ε*W* inside the solution.

New item classification



Generating the Structured Packing



- Classification:
  - T: $h(i) > \frac{1}{2}OPT$;
  - V: $\delta OPT \leq h(i) \leq \frac{1}{2}OPT$;
  - S: $h(i) < \delta OPT$.

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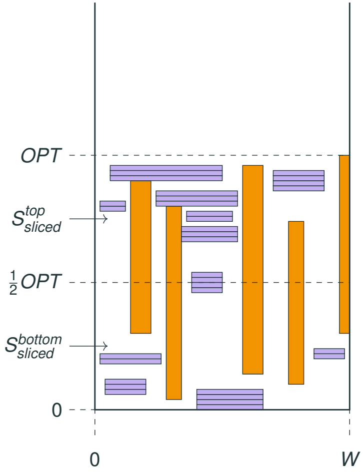

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- Consider only T and S in OPT.

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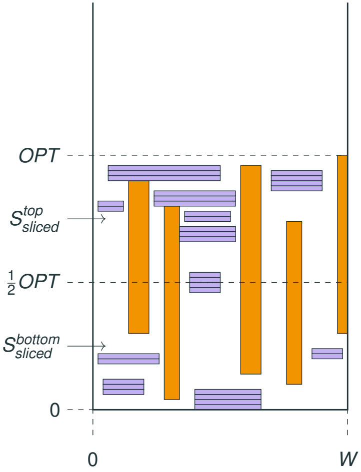
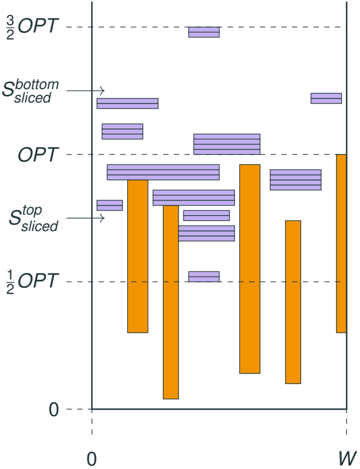


- Consider only T and S in OPT.
- Slice short items horizontally.

Generating the Structured Packing



- Consider only T and S in OPT.
- Slice short items horizontally.
- Shift the bottom sliced short items up by OPT unit.

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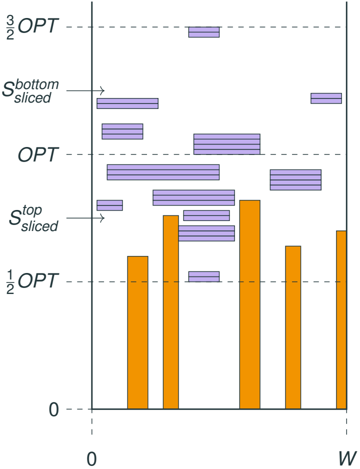
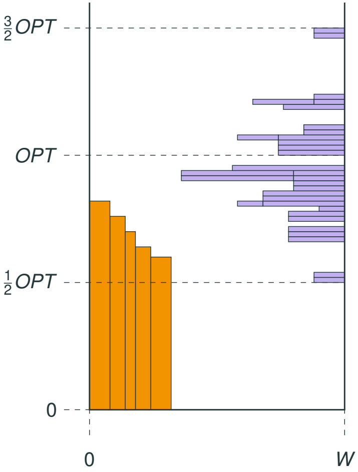


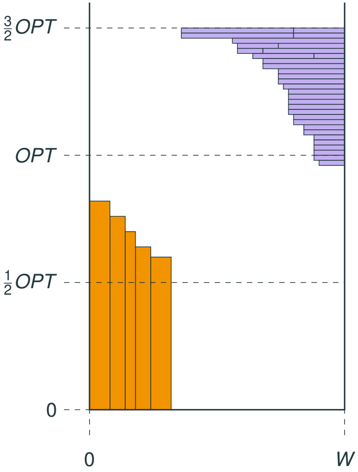
- Consider only T and S in OPT.
- Slice short items horizontally.
- Shift the bottom sliced short items up by OPT unit.
- Shift down tall items.

Generating the Structured Packing



- Slice short items horizontally.
- Shift the bottom sliced short items up by OPT unit.
- Shift down tall items.
- Shift the tall items to the left and sort them according to their height, then shift the slices to the right.

Generating the Structured Packing



- Shift the bottom sliced short items up by OPT unit.
- Shift down tall items.
- Shift the tall items to the left and sort them according to their height, then shift the slices to the right.
- Shift up and sort stripes of short items.

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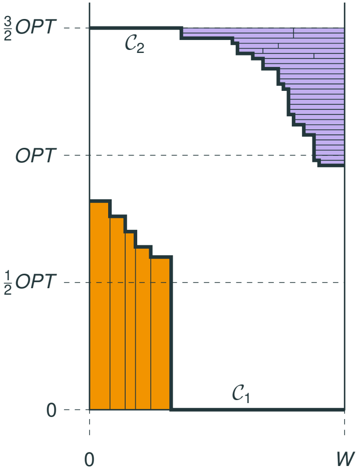

- Shift down tall items.
- Shift the tall items to the left and sort them according to their height, then shift the slices to the right.
- Shift up and sort stripes of short items.
- Silhouette of both item sets has step form

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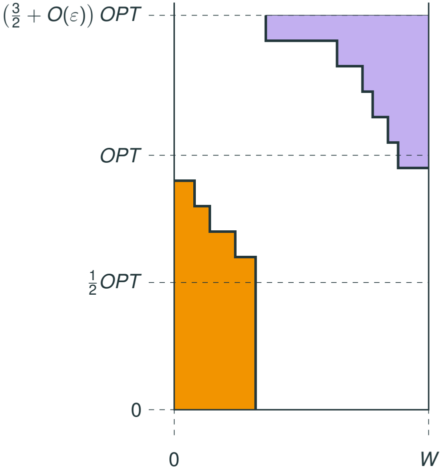
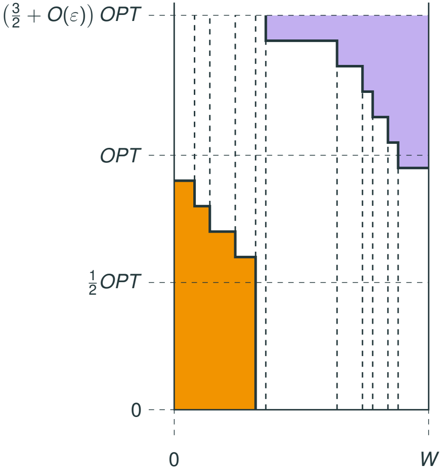


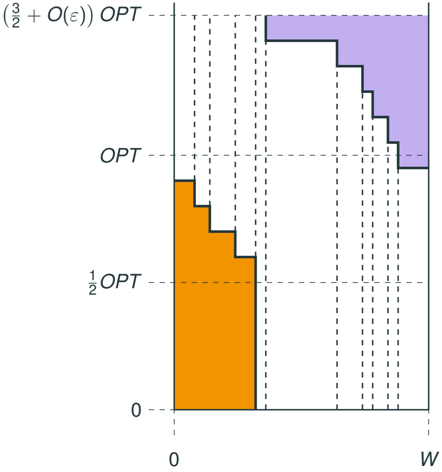
- Shift up and sort stripes of short items.
- Silhouette of both item sets has step form
- Shift the short items up by $\mathcal{O}(\varepsilon)OPT$; round up the silhouettes so as to have $O_\varepsilon(1)$ jumps.

Generating the Structured Packing



- Introduce containers for vertical items.

Generating the Structured Packing



- Introduce containers for vertical items.
- Total area in containers is at least $a(V) + \left(\frac{1}{2} + \varepsilon\right) W \cdot OPT$

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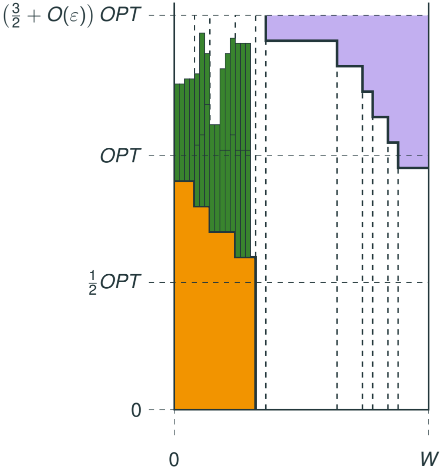
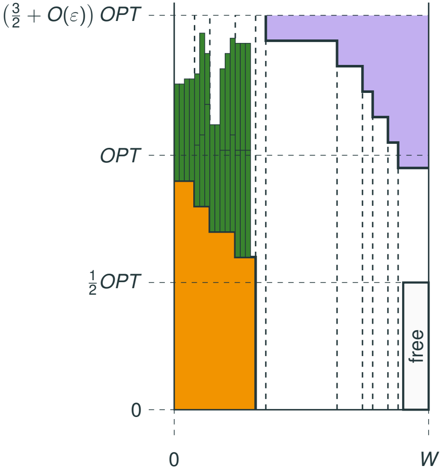


- Introduce containers for vertical items.
- Total area in containers is at least $a(V)+\left(\frac{1}{2}+\varepsilon\right) W \cdot OPT$
- Sliced vertical items can be filled inside containers greedily because they have a height of at most $\mathrm{OPT}/2$

Generating the Structured Packing



- Introduce containers for vertical items.
- Total area in containers is at least $a(V) + \left(\frac{1}{2} + \varepsilon\right) W \cdot OPT$
- Sliced vertical items can be filled inside containers greedily because they have a height of at most OPT$/2$
- Guaranteed existence of a free area with width $\Omega(\varepsilon W)$ and height OPT$/2$

Placing items inside containers

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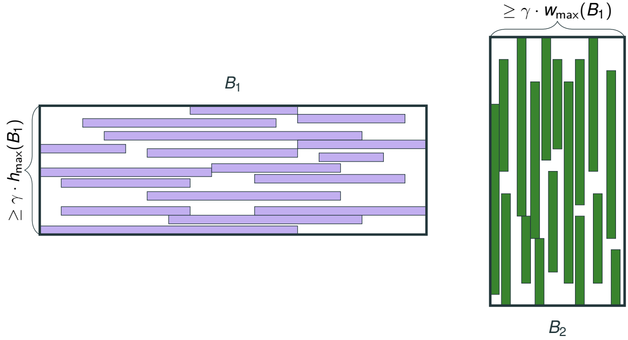
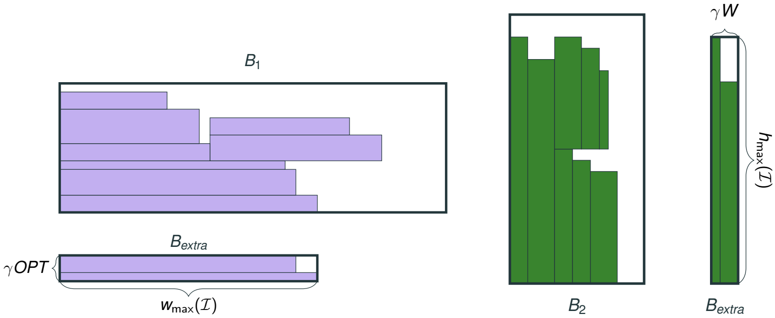
Placing items inside containers



Placing items inside containers

Unpacked items can be placed into thin extra boxes.



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- Place the residual items in one box with width W and height $\mathcal{O}(\varepsilon)\mathrm{OPT}$ at the top and inside the free area of height $\mathrm{OPT}/2$ and width $\mathcal{O}(\varepsilon)W$.

Summary

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For any $\delta > 0$ *and* $\varepsilon > 0$*, there is no polynomial time* $(3/2 - \varepsilon)$*-approximation for* δ*-skewed Strip Packing unless* $P = NP$*.*

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For any $\varepsilon > 0$ *there exists* $\delta > 0$ *such that there is a* $(3/2 + \varepsilon)$*-approximation for* δ*-skewed Strip Packing.*

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- Strip Packing with constant number of types?

Thank You for
Your Attention