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Reconstruction of 8-connected but not 4-connected hv-convex discrete sets

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Abstract

The reconstruction of 8-connected but not 4-connected hv-convex discrete sets from few projections is considered. An algorithm is given with worst case complexity of $\mathrm{O}(mn\min\{m,n\})$ to reconstruct all sets with given horizontal and vertical projections. Experimental results are also presented. It is shown, that using also the diagonal projections the algorithm can be speeded up having complexity of $\mathrm{O}(mn)$ and in this case the solution is uniquely determined. Finally, we consider the possible generalizations of our results to solve the problem in more general classes.

Keywords: Discrete tomography; Reconstruction; Connected discrete set

1. Introduction

One of the most frequently studied problems in the area of discrete tomography [15,16] is the reconstruction of two-dimensional (2D) discrete sets from few (usually upto four) projections. Several theoretical questions are connected with reconstruction such as existence and uniqueness (as a summary see [5,11,14]). There are also reconstruction algorithms for different classes of discrete sets (e.g., [3,6,8,9,13,17,18,22]). However, the reconstruction in certain classes can be NP-hard (see [24]). Since applications like electron microscopy [10], image processing [23], and radiology [21] require fast algorithms, it is important to

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find reconstruction algorithms in those classes of 2D discrete sets where the reconstruction can be performed in polynomial time. We always suppose having some a priori information of the set to be reconstructed. The most frequently used properties are connectedness, directedness and some kind of discrete versions of the convexity.

One important class where the reconstruction problem from two given projections can be solved in polynomial time is the class of hv-convex 8-connected sets. Several algorithms have been developed for solving this problem [7,18], among them the fastest has worst case complexity of $\mathrm{O}(mn\min\{m^2, n^2\})$ [2]. In this paper we study uniqueness and reconstruction problems in the case when the 8-connected set is not 4-connected.

This article is an extended version of paper [1] and is structured as follows. First, the necessary definitions are introduced in Section 2. In Section 3.1 we define $\mathscr{S}_4$-components of an 8-connected but not 4-connected hv-convex set and prove some properties of them, then, in Section 3.2 we investigate the directedness of these components. $\mathscr{S}_4$-components can be identified from two given projections, as it is shown in Section 3.3. In Section 3.4 an algorithm for reconstructing hv-convex 8-connected but not 4-connected sets from two projections is presented, and the experimental results are given in Section 3.5. In Section 4 we show how to speed up the algorithm using two more projections. Then, in Section 5 we discuss the possible generalizations of our results to adapt the algorithm to work for broader classes. Finally, in Section 6 we conclude our results.

2. Definitions and notation

Let $\hat{F} = (\hat{f}_{ij})_{m\times n}$ be a binary matrix where $m, n \geqslant 1$. Let F denote the set of positions (i, j) where $\hat{f}_{ij} = 1$, i.e. $F = \{(i, j) \mid \hat{f}_{ij} = 1\}$. F is called a *discrete set*, its elements are called *points* or *positions*. We define the *kth negative/positive diagonal* ($k = 1, \ldots, m + n - 1$) by the set A_k/B_k, respectively, where

$$A_k = \{(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\} \mid i + j = k + 1\}, \tag{1}$$

$$B_k = \{(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\} \mid i + (n - j) = k\}. \tag{2}$$

Let $\mathscr{F}$ denote the class of discrete sets. For any discrete set $F \in \mathscr{F}$ we define its *projections* by the functions $\mathscr{H}$, $\mathscr{V}$, $\mathscr{D}^-$ and $\mathscr{D}^+$ as follows:

$$\mathscr{H} : \mathscr{F} \longrightarrow \mathbb{N}_0^m, \quad \mathscr{H}(F) = H = (h_1, \ldots, h_m),$$

where

$$h_i = \sum_{j=1}^{n} \hat{f}_{ij}, \quad i = 1, \ldots, m, \tag{3}$$

$$\mathscr{V} : \mathscr{F} \longrightarrow \mathbb{N}_0^n, \quad \mathscr{V}(F) = V = (v_1, \ldots, v_n),$$

where

$$v_j = \sum_{i=1}^{m} \hat{f}_{ij}, \quad j = 1, \ldots, n, \tag{4}$$

$$\mathscr{D}^- : \mathscr{F} \longrightarrow \mathbb{N}_0^{m+n-1}, \quad \mathscr{D}^-(F) = D^- = (d_1^-, \ldots, d_{m+n-1}^-),$$

where

$$d_k^- = \sum_{(i,j)\in A_k} \hat{f}_{ij} = |F \cap A_k|, \quad k = 1, \ldots, m+n-1, \tag{5}$$

$$\mathscr{D}^+ : \mathscr{F} \longrightarrow \mathbb{N}_0^{m+n-1}, \quad \mathscr{D}^+(F) = D^+ = (d_1^+, \ldots, d_{m+n-1}^+),$$

where

$$d_k^+ = \sum_{(i,j)\in B_k} \hat{f}_{ij} = |F \cap B_k|, \quad k = 1, \ldots, m+n-1. \tag{6}$$

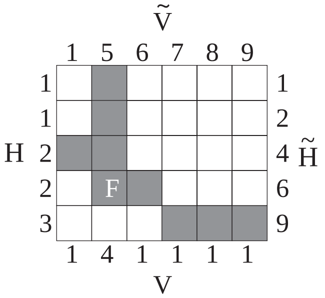
The vectors H, V, D^- and D^+ are called the *row*, *column*, *negative diagonal* and *positive diagonal sums* of F, respectively. H, V, D^- and D^+ are also called the *projections* of F (see Fig. 1). The discrete sets containing 0 row- or column sums are not interesting in this study. In the following we suppose that $h_i > 0$ and $v_j > 0$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. The *cumulated horizontal/vertical vectors* are denoted by $\widetilde{H} = (\widetilde{h}_1, \ldots, \widetilde{h}_m)$ and $\widetilde{V} = (\widetilde{v}_1, \ldots, \widetilde{v}_n)$, respectively, and defined by the following recursive formulas (see Fig. 1):

$$\widetilde{h}_1 = h_1, \quad \widetilde{h}_i = \widetilde{h}_{i-1} + h_i, \quad i = 2, \ldots, m, \tag{7}$$

$$\widetilde{v}_1 = v_1, \quad \widetilde{v}_j = \widetilde{v}_{j-1} + v_j, \quad j = 2, \ldots, n. \tag{8}$$

Given a class $\mathscr{G} \subseteq \mathscr{F}$ of discrete sets, we say that the discrete set $F \in \mathscr{G}$ is *unique in the class* $\mathscr{G}$ (with respect to some projections) if there is no different discrete set $F' \in \mathscr{G}$ with the same projections.

Two points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ in a discrete set F are said to be 4-*adjacent* if $|p_1 - q_1| + |p_2 - q_2| = 1$. The points P and Q are said to be 8-*adjacent* if they are 4-adjacent or $|p_1 - q_1| = 1$ and $|p_2 - q_2| = 1$. The sequence of distinct points $(i^{(0)}, j^{(0)}), \ldots, (i^{(k)}, j^{(k)})$ is a 4/8-*path* from point $(i^{(0)}, j^{(0)})$ to point $(i^{(k)}, j^{(k)})$ in a discrete set F if each point of the sequence is in F and $(i^{(l)}, j^{(l)})$ is 4/8-adjacent, respectively, to $(i^{(l-1)}, j^{(l-1)})$ for each



$$\hat{F} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Fig. 1. An hv-convex 8- but not 4-connected discrete set F and the corresponding binary matrix $\hat{F}$. The elements of F are marked with grey squares. The projections of F are the vectors H, V, $D^+ = (0, 0, 0, 0, 2, 2, 3, 2, 0, 0)$ and $D^- = (0, 1, 2, 1, 1, 1, 0, 1, 1, 1)$. The cumulated vectors are denoted by $\widetilde{H}$ and $\widetilde{V}$.

$l = 1, \ldots, k$. Two points are 4/8-connected in the discrete set F if there is a 4/8-path, respectively, in F between them. A discrete set F is 4/8-*connected* if any two points in F are 4/8-connected, respectively, in F. The 4-connected discrete set is also called *polyomino*. The discrete set F is *horizontally convex/vertically convex* (or shortly, h-convex/v-convex) if its rows/columns are 4-connected, respectively. The h- and v-convex sets are called hv-convex (see Fig. 1). We denote the class of hv-convex 8-connected and hv-convex 4-connected discrete sets by $\mathscr{S}_8$ and $\mathscr{S}_4$, respectively. Clearly, $\mathscr{S}_8 \supset \mathscr{S}_4$ (see, e.g. Fig. 1) and so $\mathscr{S}_8 \backslash \mathscr{S}_4 \neq \emptyset$. Let $\mathscr{S}'_8$ denote the class of hv-convex 8- but not 4-connected discrete sets, i.e. $\mathscr{S}'_8 = \mathscr{S}_8 \backslash \mathscr{S}_4$. We are going to study the reconstruction of discrete sets in $\mathscr{S}'_8$.

3. Reconstruction of sets of $\mathscr{S}'_8$ from two projections

In [17] an algorithm is published to reconstruct hv-convex discrete sets from two projections. As it turned out later the reconstruction problem in this class is NP-complete [24], therefore several efforts have been made for finding sub-classes of the class of hv-convex sets where the reconstruction can be solved in polynomial time. An algorithm for reconstructing hv-convex polyominoes was presented in [3,4]. Then, the method was improved to reconstruct discrete sets of $\mathscr{S}_8$, too [7]. The worst case computational complexity of this algorithm is of $\mathrm{O}(mn \log(mn) \min\{m^2, n^2\})$. In [9] another reconstruction algorithm was published for the class of hv-convex polyominoes having worst case time complexity of $\mathrm{O}(mn \min\{m^2, n^2\})$. With a small modification this algorithm is also suitable to reconstruct hv-convex 8-connected discrete sets [18]. After implementing the two methods for reconstructing sets of $\mathscr{S}_8$ [2] it turned out that the first algorithm [7] reconstructs the solutions generally faster than the other one [18] in almost every studied case. During the testing of the programs a third algorithm, a combination of the previous ones, was developed, which has the same worst case computational complexity as the second algorithm but it remains as fast as the first one in average case. In this sense this is the best algorithm known so far for the reconstruction problem of hv-convex 8-connected sets. In this section we give a faster algorithm for the case when the hv-convex 8-connected discrete set F is not 4-connected, i.e. when $F \in \mathscr{S}'_8$. We can formulate the problem as follows:

2-Reconstruction($\mathscr{S}'_8$)

Instance: *Two non-negative vectors* $H \in \mathbb{N}^m$ *and* $V \in \mathbb{N}^n$.

Task: *Construct a discrete set* $F \in \mathscr{S}'_8$ *such that* $\mathscr{H}(F) = H$ *and* $\mathscr{V}(F) = V$.

3.1. $\mathscr{S}_4$-components

Let $F \in \mathscr{S}'_8$. A maximal hv-convex 4-connected subset of F is called an hv-convex 4-*connected component* (shortly, an $\mathscr{S}_4$-component) of F. Clearly, the $\mathscr{S}_4$-components $F_1, \ldots, F_k$ of F give a partition of F. This $\mathscr{S}_4$-*partition* is uniquely determined, as it will be proven in the sequel. The number of $\mathscr{S}_4$-components of F is at least 2 (e.g. in Fig. 1 there are two $\mathscr{S}_4$-components: $\{(1,2), (2,2), (3,1), (3,2), (4,2), (4,3)\}$ and $\{(5,4), (5,5), (5,6)\}$).

Since F is hv-convex, the sets of the row/column indices of the elements of $F_1, \ldots, F_k$ consist of consecutive integers and they are disjoint. Then, the $\mathscr{S}_4$-components can be arranged as follows. There is an $\mathcal{S}_4$-component of F, say F_1 with the smallest containing discrete rectangle (SCDR) $R_1 = I_1 \times J_1$ such that $I_1 = \{1, \dots, i_1\}$ for some $i_1 \geqslant 1$. Similarly, we get that there is another $\mathcal{S}_4$-component of F, say F_2 with the SCDR $R_2 = I_2 \times J_2$ such that $I_2 = \{i_1 + 1, \dots, i_2\}$ for some $i_2 > i_1$ and so on. Generally, there are integers $0 = i_0 < i_1 < \cdots < i_{k-1} < i_k = m$ $(k \geqslant 2)$ such that $I_l = \{i_{l-1} + 1, \dots, i_l\}$ contains the row indices of the lth $\mathcal{S}_4$-component of F for each l $(1 \leqslant l \leqslant k)$. As a summary we can write that

$$F = F_1 \cup \cdots \cup F_k \subseteq (I_1 \times J_1) \cup \cdots \cup (I_k \times J_k). \tag{9}$$

Among $I_1, \dots, I_k$ we define a relation "$<$" as follows. Let $I, I' \in \{I_1, \dots, I_k\}$. We say that $I < I'$ if each element of I is less than any element of I'. Then, we can write shortly that

$$I_1 < I_2 < \cdots < I_k. \tag{10}$$

We define the same relation among $J_1, \dots, J_k$. In order to give a description of the relative positions of the $\mathcal{S}_4$-components of F consider

Lemma 1. *Let $F \in \mathcal{S}_8'$ and G be an $\mathcal{S}_4$-component of F with the SCDR $I \times J$. Then exactly one of the following cases is possible.*

(1) $$F \backslash G \subseteq (I' \times J') \cup (I'' \times J''), \tag{11}$$

(2) $$F \backslash G \subseteq (I' \times J'') \cup (I'' \times J'), \tag{12}$$

where I'/I'' denote the set of row indices being lesser/greater than the row indices in I, and J'/J'' denote the set of column indices being lesser/greater than the column indices in J, respectively. Possibly at most one of I' or I'' (J' or J'') is the empty set.

Proof. (11) and (12) follow from the properties of F discussed in this section and from the 8-connectedness. $I' = \emptyset$ if and only if $G = F_1$, $I'' = \emptyset$ if and only if $G = F_k$. $J' = \emptyset$ or $J'' = \emptyset$ if and only if $G = F_1$ or $G = F_k$. Both I' and I'' (J' and J'') cannot be the empty set, since F has at least two $\mathcal{S}_4$-components. □

As a consequence we get that the possible configurations of the $\mathcal{S}_4$-components follow one of the two cases given by

Theorem 2. *Let $F \in \mathcal{S}_8'$ having $\mathcal{S}_4$-components $F_1, \dots, F_k$ with the SCDRs $I_1 \times J_1, \dots, I_k \times J_k$ $(k \geqslant 2)$ such that* (10) *is satisfied. Then exactly one of the following cases is possible.*

(1) $$J_1 < J_2 < \cdots < J_k, \tag{13}$$

(2) $$J_1 > J_2 > \cdots > J_k. \tag{14}$$

Proof. We can start the proof from the fact that there are only two possible relations between J_1 and J_2: $J_1 < J_2$ or $J_1 > J_2$ (since J_1 and J_2 are disjoint sets of consecutive column indices).

First, let us suppose that $J_1 < J_2$. Apply Lemma 1 for $G = F_1$. Then $I' = \emptyset$, $I'' = I_2 \cup \cdots \cup I_k$, $J' \not\supseteq J_2$ and $J'' \supseteq J_2$. Since (12) is impossible ($F_2 \subseteq F \backslash F_1 = F \backslash G$, but $(I' \times J'') \cup (I'' \times J') = \emptyset \cup (I'' \times J') \not\supseteq I_2 \times J_2 \supseteq F_2$), (11) is true now, i.e.

$$\begin{aligned} F_2 \cup \cdots \cup F_k = F \backslash F_1 &\subseteq (I' \times J') \cup (I'' \times J'') \\ &= (I_2 \cup \cdots \cup I_k) \times J'' \end{aligned} \tag{15}$$

from which we have that $J_2 \cup \cdots \cup J_k \subseteq J''$, i.e.

$$J_1 < J_2, \ldots, J_1 < J_k. \tag{16}$$

Apply now Lemma 1 for $G = F_2$. Then $I' = I_1$, $I'' = I_3 \cup \cdots \cup I_k$, $J' \supseteq J_1$ and $J'' \not\supseteq J_1$. Since (12) is impossible ($F_1 \subseteq F \backslash F_2 = F \backslash G$, but $(I' \times J'') \cup (I'' \times J') = (I_1 \times J'') \cup ((I_3 \cup \cdots \cup I_k) \times J') \not\supseteq (I_1 \times J_1) \supseteq F_1$), (11) is true now, i.e.

$$\begin{aligned} F_1 \cup F_3 \cup \cdots \cup F_k \subseteq F \backslash F_2 &\subseteq (I' \times J') \cup (I'' \times J'') \\ &= (I_1 \times J') \cup ((I_3 \cup \cdots \cup I_k) \times J'') \end{aligned} \tag{17}$$

from which we have that $J_3 \cup \cdots \cup J_k \subset J''$, i.e.

$$J_2 < J_3, \ldots, J_2 < J_k \tag{18}$$

and so on. We get that if $J_1 < J_2$ then we have to apply (11) several times, getting (16), (18), ..., and so the result (13). Since F is 8-connected, J_2 starts in the column next to J_1, J_3 starts in the column next to J_2, and so on.

Now let us suppose that the other relation is true, i.e. $J_1 > J_2$. Then we get (14) in a similar way as in the previous case. Since F is 8-connected, J_1 starts in the column next to J_2, J_2 starts in the column next to J_3, and so on. □

In the following we say that $F \in \mathscr{S}'_8$ has *type* 1 if (13) is satisfied, otherwise, i.e. if (14) is satisfied, it has *type* 2. As an example see Fig. 2.

As a consequence of Theorem 2 we can say that the positions of the SCDRs are uniquely determined for both types.

Corollary 3. *Let $F \in \mathscr{S}'_8$. Then there are uniquely determined row indices $0 = i_0 < i_1 < \cdots < i_k = m$ and column indices $0 = j_0 < j_1 < \cdots < j_k = n$ such that $I_l \times J_l$ is the SCDR of the $\mathscr{S}_4$-component F_l of F for each $l = 1, \ldots, k$ $(k \geqslant 2)$, where*

$$I_l = \{i_{l-1} + 1, \ldots, i_l\}$$

and

$$J_l = \begin{cases} \{j_{l-1} + 1, \ldots, j_l\} & \textit{if } F \textit{ has type } 1, \\ \{j_{k-l} + 1, \ldots, j_{k-l+1}\} & \textit{if } F \textit{ has type } 2. \end{cases}$$

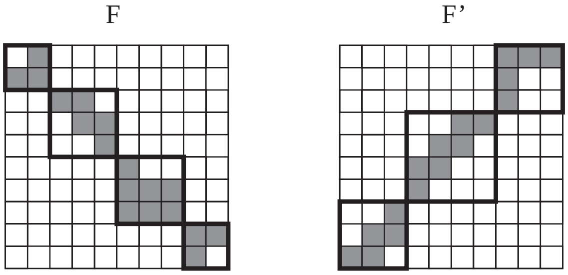


Fig. 2. A discrete set F of type 1 and a discrete set F' of type 2. The borders of the SCDRs are drawn with bold lines. $C_F = \{(2, 2), (5, 5), (8, 8)\}$, $C_{F'} = \{(3, 8), (7, 4)\}$.

3.2. Directed discrete sets

The directedness of discrete sets (in discrete tomography) was introduced in [12], the reconstruction of certain classes of directed discrete sets was studied in [19].

An 8-path in a discrete set F is an *NE-path* from point $(i^{(0)}, j^{(0)})$ to point $(i^{(t)}, j^{(t)})$ if each point $(i^{(l)}, j^{(l)})$ of the path is in north or east or northeast to $(i^{(l-1)}, j^{(l-1)})$ for each $l = 1, \ldots, t$. *SW*-, *SE*-, *NW*-paths can be defined similarly. The discrete set F is *NE-directed* if there is a particular point of F, called *source*, such that there is an *NE*-path in F from the source to any other point of F. It follows from the definition that the source point of an *NE*-directed set is necessarily the point $(m, 1)$. Similar definitions can be given for *SW*-, *SE*-, and *NW*-directedness. Sometimes we simply say that the discrete set is *directed* if it is *NE*-, *SW*-, *SE*-, or *NW*-directed (see also [12]).

On the basis of the following lemma, it is easy to check the directedness of discrete sets in the class $\mathscr{S}_4$.

Lemma 4. *Let* $G \in \mathscr{S}_4$ *and* $\{i' + 1, \ldots, i''\} \times \{j' + 1, \ldots, j''\}(i' < i'', j' < j'')$ *be its SCDR.*

(1) *G is SE-directed if and only if* $\hat{g}_{i'+1,j'+1} = 1$;
(2) *G is NW-directed if and only if* $\hat{g}_{i'',j''} = 1$;
(3) *G is SW-directed if and only if* $\hat{g}_{i'+1,j''} = 1$;
(4) *G is NE-directed if and only if* $\hat{g}_{i'',j'+1} = 1$.

Proof. It follows from the definitions directly. □

Now, we can describe the directedness of the $\mathscr{S}_4$-components depending on the type of the discrete set.

Theorem 5. *Let* $F \in \mathscr{S}'_8$ *having* $\mathscr{S}_4$*-components* $F_1, \ldots, F_k$ $(k \geqslant 2)$. *If F has type* 1 *then* $F_1, \ldots, F_{k-1}$ *are NW-directed and* $F_2, \ldots, F_k$ *are SE-directed. If F has type* 2 *then* $F_1, \ldots, F_{k-1}$ *are NE-directed and* $F_2, \ldots, F_k$ *are SW-directed.*

Proof. First, let us suppose that F has type 1. Since F is 8-connected $F_1 \cup F_2, F_2 \cup F_3, \ldots, F_{k-1} \cup F_k$ are also 8-connected. Knowing the relative positions of the SCDRs $I_1 \times J_1$ and $I_2 \times J_2$ (see Corollary 3), we can say that $F_1 \cup F_2$ is 8-connected if and only if

$$\hat{f}_{i_1,j_1} = \hat{f}_{i_1+1,j_1+1} = 1. \tag{19}$$

Similarly, we get from the 8-connectedness of F that

$$\begin{aligned} \hat{f}_{i_2,j_2} &= \hat{f}_{i_2+1,j_2+1} = 1, \\ &\vdots \\ \hat{f}_{i_{k-1},j_{k-1}} &= \hat{f}_{i_{k-1}+1,j_{k-1}+1} = 1. \end{aligned} \tag{20}$$

On the basis of Lemma 4 we know that (19) and (20) are equivalent to the *NW*-directedness of $F_1, \ldots, F_{k-1}$ and to the *SE*-directedness of $F_2, \ldots, F_k$.

Analogously, we can prove the second part of the theorem if F has type 2. In this case

$$\begin{aligned} \hat{f}_{i_1,j_{k-1}+1} &= \hat{f}_{i_1+1,j_{k-1}} = 1, \\ \hat{f}_{i_2,j_{k-2}+1} &= \hat{f}_{i_2+1,j_{k-2}} = 1, \\ &\vdots \\ \hat{f}_{i_{k-1},j_1+1} &= \hat{f}_{i_{k-1}+1,j_1} = 1. \end{aligned} \tag{21}$$

It is easy to recognize that the positions listed in (19), (20), and (21) are the source points of the corresponding $\mathscr{S}_4$-components of F. □

Depending on the type of F let us define

$$C_F = \begin{cases} \{(i_l, j_l) \mid l = 1, \ldots, k-1\} & \text{if } F \text{ has type 1}, \\ \{(i_l, j_{k-l}+1) \mid l = 1, \ldots, k-1\} & \text{if } F \text{ has type 2}, \end{cases} \tag{22}$$

where $i_1, \ldots, i_{k-1}$ and $j_1, \ldots, j_{k-1}$ denote the uniquely determined indices mentioned in Corollary 3, i.e. C_F consists of the source points of the *NW*-/*NE*-directed $\mathscr{S}_4$-components $F_1, \ldots, F_{k-1}$ if F has type 1/2, respectively (see Fig. 2). The knowledge of any element of C_F is useful in the reconstruction of an $F \in \mathscr{S}'_8$, as we can see on the basis of the following theorem.

Theorem 6. *Any $F \in \mathscr{S}'_8$ is uniquely determined by its horizontal and vertical projections, its type, and an arbitrary element of C_F.*

Proof. First, let us suppose that F has type 1 and $(i_l, j_l) \in C_F$ is given for some $l \in \{1, \ldots, k-1\}$. On the basis of Theorem 5 we know that (i_l, j_l) is the source point of the *NW*-directed $\mathscr{S}_4$-component F_l and (i_l+1, j_l+1) is the source point of the *SE*-directed $\mathscr{S}_4$-component F_{l+1}. From Theorem 5 we also know that $F_1 \cup \cdots \cup F_l$ is also *NW*-directed

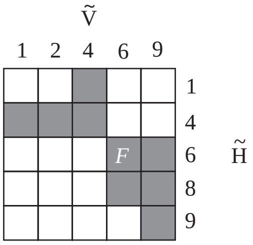


Fig. 3. A discrete set F with cumulated vectors $\widetilde{H}$ and $\widetilde{V}$. $(1,1)$, $(2,3)$ and $(3,4)$ are equality positions of type 1. However, only $(2,3)$ is in C_F. $(4,2)$ is the only equality position of type 2 but it is not in C_F since F has type 1.

hv-convex with the source point (i_l, j_l) and $F_{l+1} \cup \cdots \cup F_k$ is also *SE*-directed hv-convex with the source point $(i_l + 1, j_l + 1)$. Since hv-convex directed sets can be reconstructed from their projections uniquely (see [19, Theorem 3]), $F_1 \cup \cdots \cup F_l$ and $F_{l+1} \cup \cdots \cup F_k$, and so F can be reconstructed uniquely. The uniqueness of F can be proved analogously, if F has type 2. □

As a direct consequence, we get from Theorem 6 that different solutions of 2-Reconstruction($\mathscr{S}'_8$) with same type have different source points.

Corollary 7. *If $F, F' \in \mathscr{S}'_8$ are different solutions of the same reconstruction problem and they have the same type then $C_F \cap C_{F'} = \emptyset$.*

3.3. Equality positions

Let $\widetilde{H}$ and $\widetilde{V}$ be the cumulated vectors of the projections of $F \in \mathscr{S}'_8$. We say that $(i,j) \in \{1,\ldots,m\} \times \{1,\ldots,n\}$ is an *equality position of type* 1 if $\widetilde{h}_i = \widetilde{v}_j$.$(m,n)$ is a trivial equality position of type 1 and in the following we omit it. We say that $(i,j) \in \{1,\ldots,m\} \times \{2,\ldots,n+1\}$ is an *equality position of type* 2 if $\widetilde{h}_i = \widetilde{v}_n - \widetilde{v}_{j-1}$. Not every equality position is in C_F but they are useful to find the elements of C_F (see Fig. 3).

Lemma 8. *Let $F \in \mathscr{S}'_8$ and C_F be defined by* (22). *Then the elements of C_F are all equality positions of the same type as of F.*

Proof. Let us suppose that F has type 1 and define a set E_1 as follows:

$$E_1 = (\{1,\ldots,i\} \times \{j+1,\ldots,n\}) \cup (\{i+1,\ldots,m\} \times \{1,\ldots,j\}). \tag{23}$$

If $(i,j) \in C_F$ then $F \cap E_1 = \emptyset$, and so

$$\begin{aligned}\widetilde{h}_i &= \sum_{t=1}^{i} h_t = |F \cap \{1,\ldots,i\} \times \{1,\ldots,n\}| = |F \cap \{1,\ldots,i\} \times \{1,\ldots,j\}| \\ &= |F \cap \{1,\ldots,m\} \times \{1,\ldots,j\}| = \sum_{t=1}^{j} v_t = \widetilde{v}_j.\end{aligned} \tag{24}$$

If F has type 2 then define E_2 as follows

$$E_2 = (\{1, \ldots, i\} \times \{1, \ldots, j-1\}) \cup (\{i+1, \ldots, m\} \times \{j, \ldots, n\}). \tag{25}$$

If $(i, j) \in C_F$ then $F \cap E_2 = \emptyset$, and so

$$\begin{aligned} \widetilde{h}_i &= \sum_{t=1}^{i} h_t = |F \cap \{1, \ldots, i\} \times \{1, \ldots, n\}| = |F \cap \{1, \ldots, i\} \times \{j, \ldots, n\}| \\ &= |F \cap \{1, \ldots, m\} \times \{j, \ldots, n\}| = \sum_{t=j}^{n} v_t = \widetilde{v}_n - \widetilde{v}_{j-1}. \qquad \square \end{aligned} \tag{26}$$

3.4. The reconstruction algorithm

Our algorithm is called Algorithm 2-REC8’ and works as follows. We first assume that the set $F \in \mathscr{S}'_8$ to be reconstructed has type 1. On the basis of Theorem 6 it is sufficient to find an arbitrary element of C_F to reconstruct F from its horizontal and vertical projections uniquely. The elements of C_F are equality positions of type 1 on the basis of Lemma 8. So, in order to find all solutions of the reconstruction problem, we start to check every equality position of type 1 whether it is an element of C_F and if it is then we find a solution. The set L_1 of equality positions of type 1 can be found by the comparison of the cumulated row and column sums. This algorithm is called Algorithm L_1 and it is similar to the procedure used for reconstructing the spine of hv-convex polyominoes [19].

Since the knowledge of an arbitrary element of C_F is sufficient, again on the basis of Theorem 6, without losing any solution, we can assume that if an investigated equality position (i, j) of type 1 is in C_F then it is the source of the first $\mathscr{S}_4$-component F_1, i.e. the one with the SCDR $\{1, \ldots, i_1\} \times \{1, \ldots, j_1\}$, i.e. $(i, j) = (i_1, j_1)$. On the base of Theorem 5, this $\mathscr{S}_4$-component is *NW*-directed. Now, in order to decide if (i, j) is the source of F_1 we try to reconstruct an hv-convex *NW*-directed polyomino with source (i, j). This can be done using Algorithm RecNW which is a simple modification of the algorithm for reconstructing directed discrete sets given in [19]. Algorithm RecNW tries to reconstruct an $m \times n$ binary matrix $\hat{G}$ from the input data H, V and (i, j) such that the 1’s of $\hat{G}$ constitute an hv-convex *NW*-directed polyomino G having source (i, j) and the row and column sums of $\hat{G}$ in the non-zero rows and columns are equal to the corresponding elements of H and V, respectively. If RecNW can reconstruct such a $\hat{G}$ then it returns also the upper left position (i', j') of the SCDR of G. If RecNW fails, there is no such binary matrix $\hat{G}$. Now, there are two cases:

Case 1: RecNW fails. Clearly, in this case (i, j) cannot be the source of F_1. We continue with the investigation of the next equality position from L_1.

Case 2: RecNW gives a (unique) solution, i.e. it is possible to reconstruct an hv-convex *NW*-directed polyomino G with source (i, j) and with the SCDR $\{i', \ldots, i\} \times \{j', \ldots, j\}$, where $1 \leqslant i' \leqslant i = i_1$ and $1 \leqslant j' \leqslant j = j_1$. If $(i', j') \neq (1, 1)$ then, clearly G cannot be the first $\mathscr{S}_4$-component of F, i.e. $F_1 \neq G$ and we continue with the investigation of the next equality position from L_1. Otherwise, i.e. when $(i', j') = (1, 1)$, we can assume that $F_1 = G$ and we try to reconstruct the 2nd, 3rd, … $\mathscr{S}_4$-components iteratively. Reconstruction of the *SE*-directed kth $\mathscr{S}_4$-component F_k $(k = 2, \ldots)$ can be done using Algorithm RecSE.

Algorithm RecSE tries to reconstruct an $m \times n$ binary matrix $\hat{G}$ from the input data H, V and (i, j) such that the 1's of $\hat{G}$ constitute an hv-convex *SE*-directed polyomino G having source (i, j) and the row and column sums of $\hat{G}$ in the non-zero rows and columns are equal to the corresponding elements of H and V, respectively. If RecSE can reconstruct such a $\hat{G}$ then it returns also the lower right position (i', j') of the SCDR of G. If RecSE fails, there is no such binary matrix $\hat{G}$. On the basis of Theorem 5, F_k must be *SE*-directed with source $(i_{k-1}+1, j_{k-1}+1)$. We call RecSE to reconstruct such a polyomino. Again, there are two cases:

Case a: RecSE fails. Clearly, in this case $(i_{k-1}+1, j_{k-1}+1)$ cannot be the source of F_k which contradicts the assumption that (i, j) is the source of F_1. We continue with the investigation of the next equality position from L_1.

Case b: RecSE gives a (unique) solution, i.e. it is possible to reconstruct an hv-convex *SE*-directed polyomino G with source $(i_{k-1}+1, j_{k-1}+1)$ and with the SCDR $\{i_{k-1}+1, \ldots, i'\} \times \{j_{k-1}+1, \ldots, j'\}$, where $i_{k-1}+1 \leqslant i' \leqslant m$ and $j_{k-1}+1 \leqslant j' \leqslant n$. Depending on the properties of G we have two cases, again

Case i: If $(i', j') \neq (m, n)$ then F_k cannot be the last $\mathscr{S}_4$-component. Then, on the base of Theorem 5, F_k is *NW*-directed and therefore, $\hat{f}_{i_k, j_k} = 1$ (on the basis of Lemma 4). If $\hat{g}_{i',j'} \neq 1$ then, clearly, $F_k \neq G$ which contradicts the assumption that (i, j) is the source of F_1. We continue with the investigation of the next equality position from L_1. Otherwise, i.e. when $\hat{g}_{i',j'} = 1$, we can assume that $F_k = G$. On the basis of Corollary 7, G cannot be the first $\mathscr{S}_4$-component of any other solution of the same type therefore, (i', j') can be deleted from L_1 and we continue with the next iteration.

Case ii: If $(i', j') = (m, n)$ then $F_k = G$ and $F = F_1 \cup \cdots \cup F_k$. We found a solution and we continue with the investigation of the next equality position from L_1 in order to find another solution.

The second part of the algorithm, i.e. when it is assumed that F has type 2, is similar to the first part. We first identify equality positions of type 2 with an algorithm called Algorithm L_2. Then, we try to build *NE*- and *SW*-directed $\mathscr{S}_4$-components from the corresponding sources using the algorithms RecNE and RecSW (see Fig. 4, e.g. how the main algorithm works). If no solutions are found after investigating all equality positions of both types then the assumption that $F \in \mathscr{S}'_8$ is not met, i.e. there is no discrete set with the given projections which is hv-convex, 8- but not 4-connected. However, in some cases there can be several solutions (see Fig. 5).

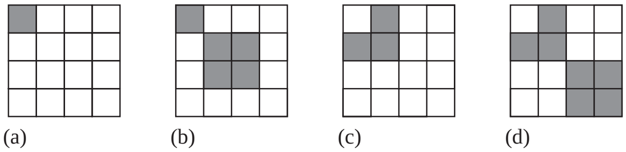

(a) (b) (c) (d)

Fig. 4. Reconstruction of sets of $\mathscr{S}'_8$ with projections $H = (1, 2, 2, 2)$ and $V = (1, 2, 2, 2)$. $L_1 = \{(1, 1), (2, 2), (3, 3)\}$, $L_2 = \emptyset$. (a) Let us suppose that (1, 1) is the source of F_1, (b) the algorithm fails after building the second component because of no place for building F_3. Then, the position (3, 3) can be deleted from L_1 (see Case i of the algorithm). (c) Assuming that (2, 2) is the source of F_1 (d) the algorithm gives a solution of type 1 and ends.

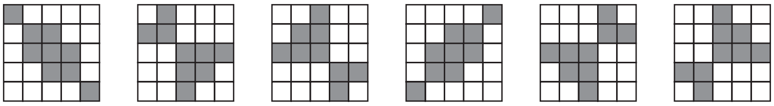


Fig. 5. Six sets of $\mathscr{S}'_8$ with the same row and column sums: $H = (1, 2, 3, 2, 1)$ and $V = (1, 2, 3, 2, 1)$.

Theorem 9. *The worst case computational complexity of Algorithm* 2-*REC*8’ *is of* $O(mn \min\{m, n\})$. *The algorithm finds all sets of* $\mathscr{S}'_8$ *with the given projections.*

Proof. Every row and column index can be in an equality position of both types at most once. This means that we have at most $\min\{m, n\}$ equality positions of type 1 and at most $\min\{m, n\}$ equality positions of type 2. Moreover, equality positions can be found in time $O(m + n)$ by Algorithms L_1 and L_2. Building the $\mathscr{S}_4$-components of F assuming that an equality position (i, j) is in C_F takes $O(mn)$ time. We have to examine every equality position if it is in C_F, so we get the execution time $O(mn \min\{m, n\})$ in the worst case.

On the basis of Theorems 5 and 6 the reconstructed sets are hv-convex, 8-connected and have the given projections H and V. On the basis of Theorem 6 any element of C_F together with the projections and the knowledge of the type of F is sufficient to reconstruct F uniquely. Elements of C_F are equality positions, too, on the basis of Lemma 8. Since Algorithm 2-REC8’ examines every equality position whether it is in C_F, the second part of the theorem follows. $\square$

3.5. Experimental results

In [2] an algorithm, called Algorithm C is presented having worst case complexity of $O(mn \min\{m^2, n^2\})$, which has so far the best average time complexity for reconstructing hv-convex 8-connected discrete sets from two projections. In order to compare the average execution times of this algorithm and Algorithm 2-REC8’ we need to generate sets of $\mathscr{S}'_8$ at random with uniform distribution. In [2] an algorithm is also given to generate sets of $\mathscr{S}_8$ having fixed row and column numbers with uniform distribution. The method is also suitable to generate sets of $\mathscr{S}'_8$ with uniform distribution (if the generated set is 4-connected then we simply omit it). We have generated discrete sets of $\mathscr{S}'_8$ with different sizes. Then, we have reconstructed them with both algorithms. The average execution times in seconds for obtaining all the solutions of different test sets are presented in Table 1 . The results show that not only the worst case complexity of our algorithm is better (see Theorem 9) but also its average execution time was much better using any of the five test sets.

4. Reconstruction of sets of $\mathscr{S}'_8$ from four projections

In this section we speed up our algorithm using two more projections. We show how to identify the type of F and find an element of C_F with the aid of the diagonal projections. Then, on the basis of Theorem 6, we can reconstruct F from its horizontal and vertical projections. We first formulate the problem.

Table 1
Average execution times in seconds of Algorithm 2-REC8' and Algorithm C in [2] depending on the size of the matrix

Size $n \times n$	2-REC8'	C in [2]
20×20	0.000272	0.011511
40×40	0.001064	0.032524
60×60	0.002597	0.065897
80×80	0.004746	0.116505
100×100	0.007831	0.178633

Each set of test data consists of 1000 hv-convex 8-connected but not 4-connected discrete sets.

4-Reconstruction($\mathscr{S}'_8$)

Instance: *Four non-negative vectors* $H \in \mathbb{N}^m$, $V \in \mathbb{N}^n$, $D^+ \in \mathbb{N}_0^{m+n-1}$ *and* $D^- \in \mathbb{N}_0^{m+n-1}$.

Task: *Construct a discrete set* $F \in \mathscr{S}'_8$ *such that* $\mathscr{H}(F) = H$, $\mathscr{V}(F) = V$, $\mathscr{D}^+(F) = D^+$ *and* $\mathscr{D}^-(F) = D^-$.

4.1. Sets of type 1

Throughout this subsection we always assume that the set $F \in \mathscr{S}'_8$ is of type 1, L_1 denotes the set of its equality positions of type 1 and C_F is defined by (22). In order to find the elements of C_F we first need two lemmas.

Lemma 10. *Let* $F \in \mathscr{S}'_8$. *For each* $k \in \{1, \ldots, m+n-1\}$ *the kth negative diagonal* A_k *contains at most one element of* L_1.

Proof. The cumulated horizontal and vertical sums of an 8-connected set always satisfy the following relations:

$$\widetilde{h}_1 < \widetilde{h}_2 < \cdots < \widetilde{h}_m \quad \text{and} \quad \widetilde{v}_1 < \widetilde{v}_2 < \cdots < \widetilde{v}_n. \tag{27}$$

Assume that the kth negative diagonal A_k contains two equality positions of F, (i, j) and (i', j'), so that $(i, j) \neq (i', j')$. Then, $\widetilde{h}_i = \widetilde{v}_j$, $\widetilde{h}_{i'} = \widetilde{v}_{j'}$, moreover, $i + j = i' + j' = k + 1$. If $i' < i$ and $j' > j$ then $\widetilde{h}_{i'} < \widetilde{h}_i = \widetilde{v}_j < \widetilde{v}_{j'}$, i.e. $\widetilde{h}_{i'} \neq \widetilde{v}_{j'}$, therefore $(i', j') \notin L_1$, which contradicts the assumption. Similarly, if $i' > i$ and $j' < j$, then $\widetilde{h}_{i'} > \widetilde{h}_i = \widetilde{v}_j > \widetilde{v}_{j'}$, i.e. $\widetilde{h}_{i'} \neq \widetilde{v}_{j'}$, therefore $(i', j') \notin L_1$, which, again, contradicts the assumption. □

Lemma 11. *Let* $F \in \mathscr{S}'_8$, $\mathscr{D}^-(F) = (d_1^-, \ldots, d_{m+n-1}^-)$ *and* $c \in \{2, \ldots, m+n-2\}$. *If* $d_{c-1}^- = 1$ *and* $d_c^- = 0$ *then the* $c-1$*th negative diagonal* A_{c-1} *contains exactly one element of* C_F.

Proof. Since $d_{c-1}^- = 1$, the $c-1$th negative diagonal A_{c-1} contains exactly one position of F, say (i, j), so that $i + j = c$. But then, $(i, j) \in C_F$, since the set is hv-convex, 8-connected and $d_c^- = 0$. □

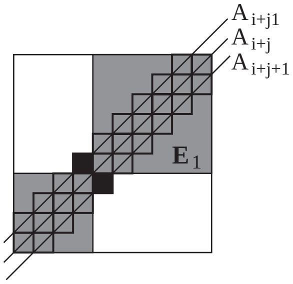


Fig. 6. The relations between the sets A_{i+j-1}, A_{i+j}, A_{i+j+1} and E_1. The positions (i, j) and $(i+1, j+1)$ are marked with black squares. The sets A_{i+j-1}, A_{i+j} and A_{i+j+1} are marked with bold squares. The set E_1 is drawn with grey squares.

We are now able to find the elements of C_F on the basis of

Theorem 12. *Let* $F \in \mathscr{S}'_8$, $\mathscr{D}^-(F) = (d_1^-, \ldots, d_{m+n-1}^-)$, $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$. $(i, j) \in C_F$ *if and only if* $(i, j) \in L_1$, $d_{i+j-1}^- = d_{i+j+1}^- = 1$ *and* $d_{i+j}^- = 0$.

Proof. In order to prove the sufficient part suppose that $(i, j) \in L_1$, $d_{i+j}^- = 0$ and $d_{i+j-1}^- = 1$. Apply Lemma 11 for $c = i + j$. Then, there exists exactly one (i', j') so that $i' + j' = i + j$ and $(i', j') \in C_F$. But then, $(i', j') \in L_1$ on the basis of Lemma 8. Since $(i, j) \in L_1$, on the basis of Lemma 10, $(i, j) = (i', j')$, i.e. $(i, j) \in C_F$ (and clearly, $d_{i+j+1}^- = 1$ in this case).

Turning to the proof of the necessary part let $(i, j) \in C_F$ and E_1 defined by (23). Clearly, $(i, j) \in L_1$, on the basis of Lemma 8. The position (i, j) is in the $i + j - 1$th negative diagonal A_{i+j-1}. Recall that $d_k^- = |A_k \cap F|$ for $k = 1, \ldots, n + m - 1$. Moreover, $A_{i+j} \subset E_1$, $A_{i+j-1} \backslash \{(i, j)\} \subset E_1$ and $A_{i+j+1} \backslash \{(i+1, j+1)\} \subseteq E_1$ (equality holds if and only if $i + j + 1 = n + m - 1$ and in that case $A_{i+j+1} \backslash \{(i+1, j+1)\} = E_1 = \emptyset$) (see Fig. 6). Since $F \cap A_{i+j} \subseteq F \cap E_1 = \emptyset$, we get that

$$d_{i+j}^- = |F \cap A_{i+j}| \leqslant |F \cap E_1| = 0. \tag{28}$$

Moreover, since $F \cap (A_{i+j-1} \backslash \{(i, j)\}) \subseteq F \cap E_1 = \emptyset$, we get that

$$\begin{aligned} F \cap A_{i+j-1} &= F \cap ((A_{i+j-1} \backslash \{(i, j)\}) \cup \{(i, j)\}) \\ &= (F \cap (A_{i+j-1} \backslash \{(i, j)\})) \cup (F \cap \{(i, j)\}) \\ &= \emptyset \cup (F \cap \{(i, j)\}) = F \cap \{(i, j)\} \end{aligned} \tag{29}$$

and $d_{i+j-1}^- = |F \cap A_{i+j-1}| = |F \cap \{(i, j)\}| = 1$, since $(i, j) \in C_F \subset F$.

Finally, since $F \cap (A_{i+j+1} \backslash \{(i+1, j+1)\}) \subseteq F \cap E_1 = \emptyset$, we get that

$$\begin{aligned} F \cap A_{i+j+1} &= F \cap ((A_{i+j+1} \backslash \{(i+1, j+1)\}) \cup \{(i+1, j+1)\}) \\ &= (F \cap (A_{i+j+1} \backslash \{(i+1, j+1)\})) \cup (F \cap \{(i+1, j+1)\}) \\ &= \emptyset \cup (F \cap \{(i+1, j+1)\}) = F \cap \{(i+1, j+1)\} \end{aligned} \tag{30}$$

and $d^-_{i+j+1} = |F \cap A_{i+j+1}| = |F \cap \{(i+1, j+1)\}| = 1$, since $(i+1, j+1) \in F$ (recall that $(i, j) \in C_F$). □

4.2. Sets of type 2

Let $F \in \mathscr{F}$. We say that $F^M \in \mathscr{F}$ with the same SCDR is the *mirror of F* if for each $i \in \{1, \ldots, n\}$ the ith column of F is the $(n-i+1)$th column of F^M. Using the following lemma the reconstruction of a set $F \in \mathscr{S}'_8$ of type 2 can be reduced to the reconstruction of a set of type 1, namely to its mirror.

Lemma 13. *Let $F \in \mathscr{S}'_8$ and F^M be its mirror. The following properties hold*:

(1) *if F is of type* 1/2 *then $F^M \in \mathscr{S}'_8$ and is of type* 2/1, *respectively*;
(2) $\mathscr{H}(F) = \mathscr{H}(F^M)$;
(3) $\mathscr{D}^+(F) = \mathscr{D}^-(F^M)$;
(4) $\mathscr{D}^-(F) = \mathscr{D}^+(F^M)$;
(5) *if $\mathscr{V}(F) = (v_1, \ldots, v_n)$ then $\mathscr{V}(F^M) = (v_n, \ldots, v_1)$*;
(6) $(F^M)^M = F$.

Proof. The properties follow from the definition. □

4.3. Identifying the type of F

We are now able to reconstruct any set $F \in \mathscr{S}'_8$ knowing its type. Using the following lemma the type of F can also be easily detected.

Lemma 14. *Let $F \in \mathscr{S}'_8$ and $\mathscr{D}^-(F) = (d^-_1, \ldots, d^-_{m+n-1})$. F is of type* 1 *if and only if there exists $c \in \{2, \ldots, m+n-2\}$ for which $d^-_{c-1} = d^-_{c+1} = 1$ and $d^-_c = 0$.*

Proof. The necessary part is trivial on the basis of Theorem 12. In order to prove the sufficient part of the theorem let $F \in \mathscr{S}'_8$, $\mathscr{D}^-(F) = (d^-_1, \ldots, d^-_{m+n-1})$ and $\mathscr{D}^+(F) = (d^+_1, \ldots, d^+_{m+n-1})$ and assume that there exists $c \in \{2, \ldots, m+n-2\}$ for which $d^-_{c-1} = d^-_{c+1} = 1$ and $d^-_c = 0$, and contrary assume that the set is of type 2. Mirror the set to get F^M. The negative/positive diagonal projections of F^M are the positive/negative diagonal projections of F, respectively, furthermore F^M is of type 1 (see Lemma 13). But then, there exists a $c' \in \{2, \ldots, m+n-2\}$ for which $d^+_{c'-1} = d^+_{c'+1} = 1$ and $d^+_{c'} = 0$, on the basis of Theorem 12. But this cannot hold together with $d^-_{c-1} = d^-_{c+1} = 1$ and $d^-_c = 0$ when the set is hv-convex and 8-connected. □

4.4. The reconstruction algorithm

We modify Algorithm 2-REC8' in order to solve the reconstruction problem from four projections. This algorithm is called 4-REC8' and works as follows. We first identify the type of the set $F \in \mathscr{S}'_8$ to be reconstructed with the aid of Lemma 14 and if the set has type 2 then we try to reconstruct the mirror image F^M using the properties mentioned in Lemma 13. Now, on the basis of Theorem 6 it is sufficient to find an arbitrary element of C_F to reconstruct F from its projections uniquely. To find the (uniquely determined) solution we scan D^- in order to find the smallest c for which $d^-_c = 1$ and $d^-_{c+1} = 0$. Then, the source point of the first $\mathscr{S}_4$-component F_1 is in the cth negative diagonal A_c. Moreover, this source must be an equality position of F, too, on the basis of Theorem 12. We scan A_c in order to identify the equality position (i, j) in this diagonal, which is uniquely determined on the basis of Lemma 10. We check similarly as in Algorithm 2-REC8' whether (i, j) is really the source of F_1 by calling the algorithms RecNW and RecSE with the corresponding parameters. If a solution is found and the mirror image is reconstructed then we mirror it in order to get the solution of the original problem (see property (6) of Lemma 13). Finally, it also has to be checked whether the diagonal sums of the reconstructed set are equal to the given vectors D^+ and D^-.

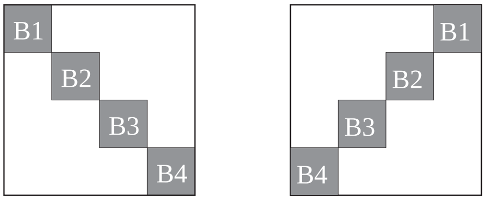
Theorem 15. *Algorithm* 4-*REC*8' *solves problem* 4-Reconstruction$(\mathscr{S}'_8)$ *in time* $\mathrm{O}(mn)$. *The reconstructed set is uniquely determined.*

Proof. Searching for the first element of C_F can be done in time $\mathrm{O}(m+n)$ scanning the diagonal vectors and comparing the cumulated column and row sums. Building the $\mathscr{S}_4$-components of F from an element of C_F takes $\mathrm{O}(mn)$ time. So, we get the execution time $\mathrm{O}(mn)$. The uniqueness of the reconstructed set follows from Theorems 6 and 12. □

5. Generalizations to broader classes

We got our results by recognizing the very important fact, that every $F \in \mathscr{S}'_8$ has at least two $\mathscr{S}_4$-components. The SCDRs of the components are connected to each other with their bottom right and upper left (sets of type 1) or with their bottom left and upper right (sets of type 2) positions. With the aid of the equality positions and the diagonal projections the SCDRs can be determined (see Theorem 12). By this knowledge the directed components can be reconstructed independently. In fact, knowing the type of the discrete set, three projections are sufficient to determine the SCDRs of the components.

Turning to the possible generalizations of our results in the case of *three projections* (the horizontal, vertical, and one of the diagonal projections) we can say that introducing the cumulated diagonal vectors $\widetilde{D}^+$ and $\widetilde{D}^-$ it is possible to find the SCDRs even if weaker hypothesis on the components is assumed (e.g. directed h-convex, directed v-convex or hv-



(a) (b)

Fig. 7. The relative positions of the components in the class (a) $\mathscr{S}^*$ and (b) $\mathscr{S}^{**}$.

convex components). Let F be a discrete set. Consider a subset of F in $I \times J \subseteq \{1,\dots,m\} \times \{1,\dots,n\}$ having the properties

(1) $F \cap I \times J$ is a polyomino, and it is uniquely reconstructible from its row and column sums,
(2) $\bar{I} \times J \cap F = \emptyset$ (where $\bar{I} = \{1,\dots,m\} \backslash I$), and
(3) $I \times \bar{J} \cap F = \emptyset$ (where $\bar{J} = \{1,\dots,n\} \backslash J$).

This relation induces a uniquely determined partition of F. Let us call the subsets created by this partition as components of F (this is a generalization of the concept of $\mathscr{S}_4$-components). Let $\mathscr{S}^*/\mathscr{S}^{**}$ denote the class of discrete sets consisting of components related to each other similarly as in the case of sets of type 1/2, respectively (see, Fig. 7). Clearly, $\mathscr{S}'_8 \subset \mathscr{S}^* \cup \mathscr{S}^{**}$. Without proof we mention here that if a set belongs to $\mathscr{S}^*$ then the bottom right position of the SCDR of a component is the equality position (i, j) so that $d^-_{i+j} = 0$ and $\widetilde{d}^-_{i+j-1} = \widetilde{h}_i = \widetilde{v}_j$. Then, Algorithm 4-REC8' can be generalized to solve the reconstruction problem in $\mathscr{S}^*$ by decomposing the set into components along negative diagonal directions. Clearly, in this case the reconstruction complexity is the maximum of the complexities of the algorithms used to reconstruct the components. For example, if we know that each component is *h*-convex and *NE*-directed then the whole discrete set can be reconstructed component by component uniquely. The same is true in the class $\mathscr{S}^{**}$.

Furthermore, the decomposition into components *using the horizontal*, *vertical and also both diagonal projections* is possible in some more general cases, when we have a priori knowledge about the relative positions of the components. For an instance suppose that we know that the discrete set (Fig. 8b) consists of four (from the horizontal and vertical projections uniquely determined) components $B1$, $B2$, $B3$, $B4$ related to each other in the way as it is shown in Fig. 8a. In this case $B1$ and $B4$ can be decomposed from the set using the negative diagonal projection. Then, after the reconstruction of the components $B1$ and $B4$ the components $B2$ and $B3$ can be split along the positive diagonal projection and reconstructed independently in a second turn. Unfortunately, the decomposition into components is impossible in some cases. For example, in Fig. 9 the configuration can be decomposed into two parts (one containing $B1$ and $B2$, and the other containing $B3$ and $B4$) by the negative diagonal projection and the equality positions but then, the two parts cannot be further decomposed into components since the positive diagonal projections of the two parts are not independent. It needs a further examination to describe configurations which are decompos-

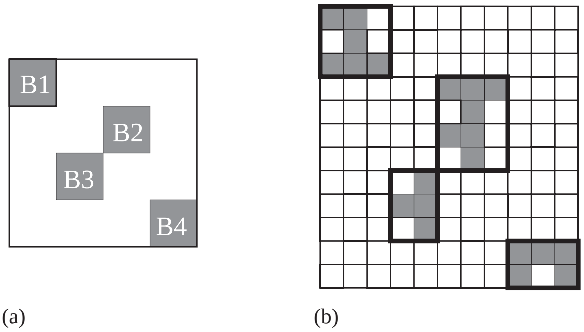


Fig. 8. (b) A more general discrete set which can be reconstructed in polynomial time (a) by the a priori knowledge about the relative positions of the components.

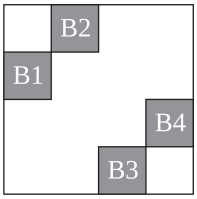


Fig. 9. An undecomposable configuration of the components.

able. Of course, in most cases we do not have information about the relative positions of the components, therefore more work has to be done on getting this information from the given projections (if is possible). Moreover, it seems to be also important to find classes of discrete sets where the reconstruction problem can be solved uniquely, since the complexity of our algorithm strongly depends on the fact that the components are uniquely determined by the projections.

6. Conclusions

We have introduced a subclass of hv-convex 8-connected discrete sets, the class of hv-convex 8- but not 4-connected sets and investigated the reconstruction problem in this class. We have shown that sets belonging to this class can be decomposed into the so-called $\mathscr{S}_4$-components which can be uniquely reconstructed from the horizontal and vertical projections. We also introduced the concept of equality positions in order to determine these components.

An algorithm for reconstructing sets of $\mathscr{S}'_8$ *using two projections* has been given with worst case complexity of $O(mn\min\{m,n\})$ and compared to a previous (more general) one given in [2]. Experimental results show that the algorithm is quite effective also from the viewpoint of average execution time. It is an interesting question if there is a way to determine from two projections beforehand if a binary matrix belongs to $\mathscr{S}'_8$. Unfortunately, this problem seems to be similarly difficult as in other classes (e.g. $\mathscr{S}_4$, $\mathscr{S}_8$) and we can answer it simply by using this algorithm. Of course, if the algorithm does not find a solution in $\mathscr{S}'_8$ then it does not mean that there is no solution in $\mathscr{S}_4$.

Moreover, it has been shown that *using two more projections* the complexity of the algorithm can be improved to $\mathrm{O}(mn)$ and in this case uniqueness also holds. In fact, knowing the type of the discrete set three projections are sufficient to achieve this complexity. However, it is interesting that the assumption on a set being 8- but not 4-connected makes an improvement in the reconstruction complexity, at the same time another algorithm based on topological properties (like, e.g. thinning algorithms) can be more difficult in this class than in the class of hv-convex sets or $\mathscr{S}_4$ (c.f. [20]).

We also considered the possibility to generalize our results to adapt the algorithm to work for broader classes. The concept of $\mathscr{S}_4$-components is generalized. Then, it is shown that the equality positions together with the diagonal projections can be a quite useful tool to decompose discrete sets into components to facilitate the reconstruction. We hope that further work in this field can lead us towards the understanding of the difficulties of the reconstruction problems and the design of efficient reconstruction algorithms for important classes like the one of hv-convex sets.

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