Available online at www.sciencedirect.com
SCIENCE DIRECT®

Electronic Notes in DISCRETE MATHEMATICS

ELSEVIER

Electronic Notes in Discrete Mathematics 20 (2005) 329–345

www.elsevier.com/locate/endm

Reconstruction of Discrete Sets from Four Projections: Strong Decomposability

Péter Balázs [1,2]

Department of Computer Algorithms and Artificial Intelligence
University of Szeged
Szeged, Hungary

Abstract

In this paper we introduce the class of strongly decomposable discrete sets and give an efficient algorithm for reconstructing discrete sets of this class from four projections. It is also shown that every Q-convex set (along the set of directions $\{x, y\}$) consisting of several components is strongly decomposable. As a consequence of strong decomposability we get that in a subclass of hv-convex discrete sets the reconstruction from four projections can be solved in polynomial time.

Keywords: discrete tomography, reconstruction from projections, decomposable discrete set

1 Introduction

One of the most frequently studied problems in the area of discrete tomography [14] is the reconstruction of 2-dimensional discrete sets (the finite subsets of $\mathbb{Z}^2$) from few (usually up to four) projections. Several theoretical questions are connected with reconstruction such as existence and uniqueness (as

[1] This work was supported by OTKA grant T048476.
[2] Email: pbalazs@inf.u-szeged.hu

doi:10.1016/j.endm.2005.05.072

a summary see [6,13]). Since the reconstruction problem is usually underdetermined the number of solutions for a given reconstruction task can be very large. Moreover, the reconstruction in certain classes can be NP-hard (see [19]). In order to keep the reconstruction process tractable and to reduce the number of solutions a commonly used technique is to suppose having some a priori information of the set to be reconstructed. The most frequently used properties are connectedness, directedness and some kind of discrete versions of the convexity. A lot of work have been done in designing efficient reconstruction algorithms for different classes of discrete sets (e.g., [3,4,7,9,12,15,16,18]). However, up to now only few papers deal with the problem of reconstruction if it is known in advance that the discrete set to be reconstructed consists of several components. For example, in [2] the authors study the class of hv-convex 8- but not 4-connected discrete sets. Then, in [1] the class of decomposable discrete sets is introduced and the reconstruction problem in this class is studied. In this paper we introduce a subclass of decomposable discrete sets called strongly decomposables. Studying uniqueness and reconstruction problems in the class of strongly decomposable discrete sets we state that in this subclass of the decomposable discrete sets a more efficient reconstruction algorithm can be given than the one presented in [1]. Then, it is shown that this class generalizes the class of Q-convexes having several components. We also investigate the possibility to extend our results to the class of hv-convex discrete sets.

This article is structured as follows. First, the necessary definitions are introduced in Section 2. In Section 3 we define the class of decomposable discrete sets and prove some important properties of sets belonging to this class. In Section 4 we introduce the class of strongly decomposable discrete sets and give a polynomial reconstruction algorithm in this class using four projections. In Section 5 we show that every Q-convex set along the set of directions $\{x, y\}$ is also strongly decomposable and applying the results of Section 4 we get an $O(mn)$ algorithm for the reconstruction problem in this class using four projections. In Section 6 we analyse how the results of Section 4 can be adopted to hv-convex sets to facilitate the reconstruction. Finally, in Section 7 we conclude our results.

2 Definitions and Notation

Let $\hat{F} = (\hat{f}_{ij})_{m \times n}$ be a binary matrix where $m, n \geq 1$. Let F denote the set of positions (i, j) where $\hat{f}_{ij} = 1$, i.e., $F = \{(i, j) | \hat{f}_{ij} = 1\}$. Due to the strong relation between binary matrices and discrete sets F is called a *discrete* *set* represented by $\hat{F}$, its elements are called *points* or *positions*. The *kth diagonal/antidiagonal* ($k = 1, \dots, m+n-1$) of $\hat{F}$ are defined by the set D_k/A_k, respectively, where

$$D_k = \{(i,j) \in \{1,\dots,m\} \times \{1,\dots,n\} \mid i + (n-j) = k\} \ , \tag{1}$$

$$A_k = \{(i,j) \in \{1,\dots,m\} \times \{1,\dots,n\} \mid i + j = k+1\} \ . \tag{2}$$

Let $\mathcal{F}$ denote the class of discrete sets. For any discrete set $F \in \mathcal{F}$ we define the functions $\mathcal{H}$, $\mathcal{V}$, $\mathcal{D}$, and $\mathcal{A}$ as follows.
$\mathcal{H} : \mathcal{F} \longrightarrow \mathbb{N}_0^m$, $\mathcal{H}(F) = H = (h_1, \dots, h_m)$, where

$$h_i = \sum_{j=1}^{n} \hat{f}_{ij}, \qquad i = 1, \dots, m \ , \tag{3}$$

$\mathcal{V} : \mathcal{F} \longrightarrow \mathbb{N}_0^n$, $\mathcal{V}(F) = V = (v_1, \dots, v_n)$, where

$$v_j = \sum_{i=1}^{m} \hat{f}_{ij}, \qquad j = 1, \dots, n \ , \tag{4}$$

$\mathcal{D} : \mathcal{F} \longrightarrow \mathbb{N}_0^{m+n-1}$, $\mathcal{D}(F) = D = (d_1, \dots, d_{m+n-1})$, where

$$d_k = \sum_{(i,j) \in D_k} \hat{f}_{ij} = |F \cap D_k|, \qquad k = 1, \dots, m+n-1 \ , \tag{5}$$

and $\mathcal{A} : \mathcal{F} \longrightarrow \mathbb{N}_0^{m+n-1}$, $\mathcal{A}(F) = A = (a_1, \dots, a_{m+n-1})$, where

$$a_k = \sum_{(i,j) \in A_k} \hat{f}_{ij} = |F \cap A_k|, \qquad k = 1, \dots, m+n-1 \ . \tag{6}$$

The vectors H, V, D, and A are called the *row, column, diagonal* and *antidiagonal* projections of F, respectively (see Fig. 1). The *cumulated horizontal/vertical/diagonal/antidiagonal vectors* are denoted by $\widetilde{H} = (\widetilde{h}_1, \dots, \widetilde{h}_m)$, $\widetilde{V} = (\widetilde{v}_1, \dots, \widetilde{v}_n)$, $\widetilde{D} = (\widetilde{d}_1, \dots, \widetilde{d}_{m+n-1})$, and $\widetilde{A} = (\widetilde{a}_1, \dots, \widetilde{a}_{m+n-1})$, respectively, and defined by the following formulas (see Fig. 1)

$$\widetilde{h}_i = \sum_{l=1}^{i} h_l, \qquad i = 1, \dots, m \ , \tag{7}$$

$$\widetilde{v}_j = \sum_{l=1}^{j} v_l, \qquad j = 1, \dots, n \ , \tag{8}$$

$$\widetilde{d}_k = \sum_{l=1}^{k} d_l, \qquad \widetilde{a}_k = \sum_{l=1}^{k} a_l, \qquad k = 1, \dots, m+n-1 \ . \tag{9}$$

Given a class $\mathcal{G} \subseteq \mathcal{F}$ of discrete sets we say that the discrete set $F \in \mathcal{G}$ is *unique in the class* $\mathcal{G}$ (with respect to some projections) if there is no different

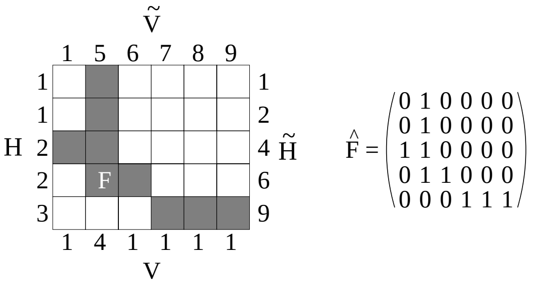


Fig. 1. An hv-convex 8- but not 4-connected discrete set F with the projections H, V, $D = (0,0,0,0,2,2,3,2,0,0)$, and $A = (0,1,2,1,1,1,0,1,1,1)$. The elements of F are marked with grey squares. The cumulated vectors of F are $\widetilde{H}$, $\widetilde{V}$, $\widetilde{D} = (0,0,0,0,2,4,7,9,9,9)$, and $\widetilde{A} = (0,1,3,4,5,6,6,7,8,9)$. The corresponding binary matrix is denoted by $\hat{F}$.

discrete set $F' \in \mathcal{G}$ with the same projections.

Two points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ in a discrete set F are said to be *4-adjacent* if $|p_1 - q_1| + |p_2 - q_2| = 1$. The points P and Q are said to be *8-adjacent* if they are 4-adjacent or $|p_1 - q_1| = 1$ and $|p_2 - q_2| = 1$. The sequence of distinct points $P_0, \ldots, P_k$ is a *4/8-path* from point P_0 to point P_k in a discrete set F if each point of the sequence is in F and P_l is 4/8-adjacent, respectively, to P_{l-1} for each $l = 1, \ldots, k$. Two points are 4/8-connected in the discrete set F if there is a 4/8-path, respectively, in F between them. A discrete set F is *4/8-connected* if any two points in F are 4/8-connected, respectively, in F. The 4-connected set is also called as polyomino. The discrete set F is *horizontally convex/vertically convex* (or shortly, h-convex/v-convex) if its rows/columns are 4-connected, respectively. The h- and v-convex sets are called hv-convex (see Fig. 1). In this paper we are going to study the reconstruction problem from four projections in several classes. For a given class $\mathcal{G} \subseteq \mathcal{F}$ the problem can be formulated as follows

Problem 2.1 *Given four non-negative vectors* $H \in \mathbb{N}_0^m$, $V \in \mathbb{N}_0^n$, $D \in \mathbb{N}_0^{m+n-1}$, *and* $A \in \mathbb{N}_0^{m+n-1}$ *construct a discrete set* $F \in \mathcal{G}$ *with* $\mathcal{H}(F) = H$, $\mathcal{V}(F) = V$, $\mathcal{D}(F) = D$, *and* $\mathcal{A}(F) = A$.

3 Decomposable Discrete Sets

Let F be a discrete set. A maximal 4-connected subset of F is called a *component* of F. For example, the discrete set F in Fig. 1 has two components: $\{(5,4),(5,5),(5,6)\}$ and $\{(1,2),(2,2),(3,1),(3,2),(4,2),(4,3)\}$. Clearly, the components of F give a (uniquely determined) partition of F. The *NorthWest gluing* (or shortly, NW-gluing) is an operator defined by

$$\mathcal{F}^2 \longrightarrow \mathcal{F} \;:\; C \times D \to F, \text{ where } \hat{F} = \begin{pmatrix} \hat{C} & \mathbf{0} \\ \mathbf{0} & \hat{D} \end{pmatrix} . \tag{10}$$

If C is a single component then we say that C is the *NW-component of F*. NE-, SE-, SW-gluings and -components are defined similarly. We say that a discrete set F consisting of k ($k \geq 2$) components is *decomposable* if

(α) the components are uniquely reconstructible from their horizontal and vertical projections in polynomial time, and

(β) the sets of the row/column indices of the components are disjoint, i.e., if $I \times J \subseteq \{1,\ldots,m\} \times \{1,\ldots,n\}$ is the smallest containing discrete rectangle (SCDR) of a component of the discrete set F, then $\bar{I} \times J \;\cap\; F = I \times \bar{J} \;\cap\; F = \emptyset$ (where $\bar{I} = \{1,\ldots,m\} \setminus I$ and $\bar{J} = \{1,\ldots,n\} \setminus J$), and

(γ) if $k > 2$ then we get F by gluing a single component to a decomposable discrete set consisting of $k-1$ components using one of the four gluing operators.

In fact, to satisfy property (α) we need to have some a priori information about the components. For example, NW-directed hv-convex discrete sets can be used as components since in this class property (α) is fulfilled [11]. Another class of discrete sets where property (α) is satisfied is the class of L-convex polyominoes [8]. As a straight consequence of the definition we get that every discrete set consisting of one component is undecomposable and every discrete set consisting of two or three components and satisfying properties (α) and (β) is decomposable. Figure 2 shows some decomposable and undecomposable configurations if the set consists of four components. The class of decomposable discrete sets is denoted by $\mathcal{DEC}$. The following lemma gives a description of decomposable discrete sets.

Lemma 3.1 *A discrete set F is decomposable if and only if it satisfies property (α) and there exists a sequence of discrete sets $F^{(1)},\ldots,F^{(k)}$ such that $F^{(1)}$ consists of one component, $F^{(k)} = F$, and we get $F^{(l+1)}$ by gluing a component to $F^{(l)}$ using a gluing operator for each $l = 1,\ldots,k-1$.*

Proof. It is a straight consequence of the definition of decomposability. □

For a given discrete set $F \in \mathcal{DEC}$ the sequence described in Lemma 3.1 is not uniquely determined. We will refer to any sequence satisfying the properties of Lemma 3.1 as a *gluing sequence* of F. For example, the sequences $F^{(1)} = \{(1,1)\}$, $F^{(2)} = \{(1,1),(2,2)\}$ and $G^{(1)} = \{(2,2)\}$, $G^{(2)} = \{(1,1),(2,2)\}$ are

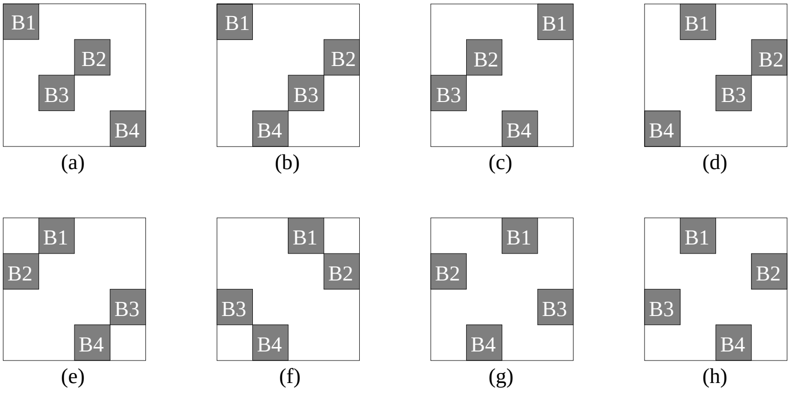


Fig. 2. Some decomposable (first row) and undecomposable (second row) configurations of the components. The SCDRs of the components are denoted by B1, B2, B3, and B4.

both gluing sequences of the same discrete set $F = \{(1,1),(2,2)\}$. On the basis of properties (α) and (β) in the reconstruction of a decomposable discrete set it is sufficient to identify the SCDRs of the components. In order to do this we give a necessary condition.

Theorem 3.2 *Let $F \in \mathcal{DEC}$. If (i,j) is the bottom right position of the SCDR of the NW-component of F then i is the smallest integer for which there exists an integer j with $\widetilde{h}_i = \widetilde{v}_j = \widetilde{a}_{i+j-1} > 0$ and $a_{i+j} = 0$.*

Proof. Define a set E as follows

$$E = (\{1,\dots,i\} \times \{j+1,\dots,n\}) \cup (\{i+1,\dots,m\} \times \{1,\dots,j\}) \ . \tag{11}$$

If (i,j) is the bottom right position of the SCDR of the NW-component of F then, clearly, $\widetilde{h}_i > 0$, $\widetilde{v}_j > 0$, and $\widetilde{a}_{i+j-1} > 0$. Moreover, $F \cap E = \emptyset$ (see Fig. 3), and so

$$\widetilde{h}_i = \sum_{t=1}^{i} h_t = |F \cap \{1,\dots,i\} \times \{1,\dots,n\}| = |F \cap \{1,\dots,i\} \times \{1,\dots,j\}|$$

$$= |F \cap \{1,\dots,m\} \times \{1,\dots,j\}| = \sum_{t=1}^{j} v_t = \widetilde{v}_j \ . \tag{12}$$

Furthermore, $(F \cap A_k) \cap E \subseteq F \cap E = \emptyset$ for every $k = 1,\dots,m+n-1$ (see again Fig. 3). Then, recalling that $a_k = |F \cap A_k|$ for $k = 1,\dots,n+m-1$ we

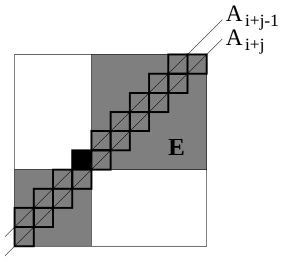


Fig. 3. The relations between the sets A_{i+j-1}, A_{i+j} and E. The position (i, j) is marked with black square. The antidiagonals A_{i+j-1} and A_{i+j} are marked with bold squares. The set E is drawn with grey squares.

get that

$$\widetilde{a}_{i+j-1} = \sum_{k=1}^{i+j-1} |F \cap A_k| = |F \cap \{1, \ldots, i\} \times \{1, \ldots, j\}| = \widetilde{h}_i = \widetilde{v}_j \ . \tag{13}$$

Moreover, $A_{i+j} \subseteq E$ (see Fig. 3). Then, $F \cap A_{i+j} \subseteq F \cap E = \emptyset$ and we get that

$$a_{i+j} = |F \cap A_{i+j}| \leq |F \cap E| = 0 \ . \tag{14}$$

Finally, assume that an integer $i' < i$ exists for which an integer j' exists such that $\widetilde{h}_{i'} = \widetilde{v}_{j'} = \widetilde{a}_{i'+j'-1} > 0$ and $a_{i'+j'} = 0$. Clearly, in this case $j' < j$. Since (i, j) is the bottom right position of the SCDR of the NW-component and every component is a polyomino we get that the 1st, $\ldots, (i+j)$-th coordinates of the antidiagonal projection have to be of the form $(0, \ldots, 0, a_{k_1}, \ldots, a_{k_2}, 0, \ldots, 0)$, where $1 \leq k_1 \leq k_2 < i + j$ and $a_l \neq 0$ for every $k_1 \leq l \leq k_2$. But then $a_{i'+j'} = 0$ only if $i' + j' < k_1$ or $i' + j' > k_2$. If $i' + j' < k_1$ then $\widetilde{a}_{i'+j'-1} = 0$. Since the cumulated horizontal sums of every discrete set satisfy the relation

$$0 < \widetilde{h}_1 \leq \widetilde{h}_2 \leq \ldots \leq \widetilde{h}_m \tag{15}$$

we get that $\widetilde{a}_{i'+j'-1} < \widetilde{h}_{i'}$ which is a contradiction. Otherwise, i.e., if $i'+j' > k_2$ then $\widetilde{h}_i \geq \widetilde{h}_{i'}$ since $i > i'$ and (15) holds. If $\widetilde{h}_i > \widetilde{h}_{i'}$ then $\widetilde{a}_{i'+j'-1} = \widetilde{a}_{i+j-1} = \widetilde{h}_i > \widetilde{h}_{i'}$ is a contradiction. If $\widetilde{h}_i = \widetilde{h}_{i'}$ then $h_i = 0$ which contradicts the assumption that (i, j) is the bottom right position of the SCDR of the NW-component of F. □

Similar theorems can be given for NE-, SE- and SW-components. Unfortunately, Theorem 3.2 does not give a sufficient condition for identifying the SCDR of a component of F. For example, if the discrete set is $F = \{(1, 3), (2, 2), (5, 1)\}$ then for the position $(2, 2)$ the conditions of Theorem 3.2 hold, since $\widetilde{h}_2 = \widetilde{v}_2 = \widetilde{a}_3 = 2$ and $a_4 = 0$ but F has no NW-component at all. Before giving a stronger condition for finding the SCDR of a component of F we introduce some further concepts. Let $F, F' \in \mathcal{F}$ such that $F' \setminus F = \{(p_1, q_1)\}$ and $F \setminus F' = \{(p_2, q_2)\}$. If $(p_1 + k, q_1 + k) = (p_2, q_2)$ for a $k \in \mathbb{Z} \setminus \{0\}$ then we say that we get F' by applying a *slide* on F. Similarly, if $(p_1 + k, q_1 - k) = (p_2, q_2)$ for a $k \in \mathbb{Z} \setminus \{0\}$ then we say that we get F' by applying an *antislide* on F. Clearly, applying slides/antislides on a discrete set, the diagonal/antidiagonal projection does not change, respectively. With the aid of the following theorem it is possible to test whether the decomposable discrete set has a NW- or SE-component.

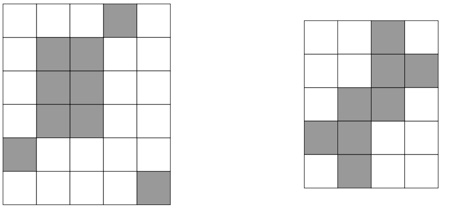
Theorem 3.3 [NW-version] *Let $F \in \mathcal{DEC}$, $\mathcal{H}(F) = (h_1, \ldots, h_m)$, $\mathcal{V}(F) = (v_1, \ldots, v_n)$, and $\mathcal{A}(F) = (a_1, \ldots, a_{m+n-1})$. If (i, j) is a position such that i is the smallest integer for which there exists an integer j with $\widetilde{h}_i = \widetilde{v}_j = \widetilde{a}_{i+j-1}$ and $a_{i+j} = 0$ and a polyomino P exists according to the a priori information which guarantees that property (α) is satisfied such that $\mathcal{H}(P) = (h_1, \ldots, h_i)$, $\mathcal{V}(P) = (v_1, \ldots, v_j)$, and $\mathcal{A}(P) = (a_1, \ldots, a_{i+j-1})$ then P is the NW-component of F or/and F has a SE-component. If no such position exist then F has no NW-component.*

Proof. Define a set by $T = \bigcup_{k=1}^{i+j-1} A_k$ and let $Q' = F \cap T$. Since $\mathcal{A}(P) = (a_1, \ldots, a_{i+j-1})$ we get Q' by applying some (possibly none) antislides on P. Let Q be an arbitrary discrete set with the projections $\mathcal{H}(Q) = (h_1^q, \ldots, h_m^q)$, $\mathcal{V}(Q) = (v_1^q, \ldots, v_n^q)$, and $\mathcal{A}(Q) = (a_1^q, \ldots, a_{m+n-1}^q)$ that we get by applying some antislides on P. Clearly, $a_l^q = a_l$ for each $l = 1, \ldots, i + j - 1$ and $Q \subseteq T$. Moreover, for the horizontal and vertical projections of Q exactly one of the following cases holds

(i) $\exists\ i' \le i$ such that $h_{i'}^q \ne h_{i'}$ or $\exists\ j' \le j$ such that $v_{j'}^q \ne v_{j'}$,

(ii) $h_{i'}^q = h_{i'}$ for each $i' = 1, \ldots, i$ and $v_{j'}^q = v_{j'}$ for each $j' = 1, \ldots, j$.

Assume that Case (i) is true with $h_{i'}^q \ne h_{i'}$ for an $i' \le i$. Then, there also exists an $i'' \le i$ such that $h_{i''}^q > h_{i''}$ or a $j'' \le j$ such that $v_{j''}^q > v_{j''}$. Clearly, in this case there is no discrete set F' with the projections $\mathcal{H}(F') = (h_1, \ldots, h_m)$ and $\mathcal{V}(F') = (v_1, \ldots, v_n)$ such that $F' \cap T = Q$. Assuming that Case (i) is true with $v_{j'}^q \ne v_{j'}$ for a $j' \le j$ we get the same analogously. Therefore F can have the given projections if and only if for Q' Case (ii) is true which is possible only if $Q' \subseteq \{1, \ldots, i\} \times \{1, \ldots, j\}$. Since $\mathcal{H}(Q') = (h_1, \ldots, h_i)$ and $\mathcal{V}(Q') = (v_1, \ldots, v_j)$ it follows that F can have the prescribed projections if and only if $F \cap E = \emptyset$ where E is defined by (11). Then, F has neither NE- nor SW-components, i.e., it has a NW- or a SE-component. If F has a NW-



(a) (b)

Fig. 4. (a) A decomposable discrete set which has no NW-component in spite of that the position $(5, 4)$ satisfies the conditions of Theorem 3.3 with the polyomino in (b).

component then it is the polyomino P based on Theorem 3.2 and property (α). Then, the first part of the theorem is proven. The second part of the theorem follows from the fact that the position that satisfies the necessary conditions of Theorem 3.2 is uniquely determined. $\square$

Similar theorems can be given for testing the existence of a NE-, SE-, and SW-component. Note, that if the conditions of Theorem 3.3 hold then in some cases the discrete set can have both NW- and SE-components. For example if the discrete set is $F = \{(1, 1), (2, 2)\}$ then the position $(1, 1)$ satisfies the conditions of Theorem 3.3 and F has both NW- and SE-components.

4 Reconstruction of Strongly Decomposable Discrete Sets

Although Theorem 3.3 excludes the existence of NE- and SW-components it does not guarantee that the discrete set has a NW-compoent (see Fig. 4). Therefore we introduce a subclass of decomposable discrete sets where Theorem 3.3 gives a sufficient condition for the existence of a NW-component. A decomposable discrete set is called *strongly decomposable* if it satisfies the following extra property

(δ) if $F^{(1)}, \ldots, F^{(k)}$ is a gluing sequence of F then for every $F^{(l)}$ $(l = 2, \ldots, k)$ if a position (i, j) satisfies the conditions of the NW/NE/SE/SW-version of Theorem 3.3 then $F^{(l)}$ has a NW/NE/SE/SW-component, respectively.

We will denote the class of strongly decomposable discrete sets by $\mathcal{DEC}^+$. Clearly, $\mathcal{DEC}^+ \subset \mathcal{DEC}$ (see, again, Fig. 4 for the real inclusion). Now, an algorithm can be outlined for reconstructing strongly decomposable discrete sets with given horizontal, vertical, diagonal, and antidiagonal projections.

We first describe a procedure for decomposing a NW-component based on Theorem 3.3.

Algorithm 1 Procedure DecomposeNW_(H, V, D, A)_

Step 1: *find the position* (i, j) *for which* $\widetilde{h}_i = \widetilde{v}_j = \widetilde{a}_{i+j-1} > 0$ *and* $a_{i+j} = 0$;
if *no such position exists* **then return**_(no component);_

Step 2: *construct a polyomino* P *according to the a priori information with* $\mathcal{H}(P) = (h_1, \ldots, h_i)$, $\mathcal{V}(P) = (v_1, \ldots, v_j)$, *and* $\mathcal{A}(P) = (a_1, \ldots, a_{i+j-1})$;
if *no such polyomino exists* **then return**_(no component);_

Step 3: *update* H, V, D, *and* A *according to the projections of* P;

Step 4: $F = F \cup P$;

Step 5: **return**;

The procedures DecomposeNE/DecomposeSE/DecomposeSW for decomposing NE-/SE-/SW-components, respectively, can be outlined similarly. The main algorithm for reconstructing strongly decomposable discrete sets gets four vectors $H \in \mathbb{N}_0^m$, $V \in \mathbb{N}_0^n$, $D \in \mathbb{N}_0^{m+n-1}$, and $A \in \mathbb{N}_0^{m+n-1}$ as input and outputs the uniquely determined strongly decomposable discrete set F with the projections H, V, D, and A or FAIL (if no such set exists). This algorithm reconstructs the set component by component calling the procedures for decomposing a component. The algorithm can be outlined as follows

Algorithm 2

Step 1: $F = \emptyset$;

Step 2: **repeat**
call DecomposeNW(H, V, D, A);
if *(no component)* **then** *call DecomposeNE*(H, V, D, A);
if *(no component)* **then** *call DecomposeSE*(H, V, D, A);
if *(no component)* **then** *call DecomposeSW*(H, V, D, A);
until *(no component);*

Step 3: *try to reconstruct the last component;*

Step 4: **if** $D = 0$ **and** $A = 0$ **then return** F **else** *FAIL (no solution);*

Turning to the analysis of the algorithm we can say the following

Theorem 4.1 *Algorithm 2 solves Problem 2.1 in the class* $\mathcal{DEC}^+$ *in polynomial time. If the reconstructed set consists of components* $F_1, \ldots, F_k$ *and the time complexity of reconstructing the ith component* F_i $(i = 1, \ldots, k)$ *is of* C_i *then the worst case time complexity of the algorithm is of* $\max_{1 \le i \le k} C_i$. *The solution is uniquely determined.*

Proof. As a straight consequence of the algorithm we get that the reconstructed set F is decomposable. Due to Step 2 of Algorithm 1 the horizontal and vertical projections of F are equal to the given vectors H and V, respectively. Moreover, Step 4 of Algorithm 2 guarantees that the diagonal and antidiagonal projections of F are also equal to the vectors D and A, respectively. Assuming that the lth ($l = 1, \ldots, k$) component to be reconstructed is a NW-component it takes $O(m + n)$ time to find the (uniquely determined) position which satisfies the necessary conditions of Theorem 3.2. We do it simply by scanning the vectors $\widetilde{H}$, $\widetilde{V}$, and $\widetilde{A}$. In order to test whether this position is the bottom right position of the SCDR of the NW-component we try to reconstruct this component based on Theorem 3.3 which takes C_l time. The same is true if the lth component is a NE-, SE- or SW-component. In the worst case the component is a SW-component, i.e., we try to reconstruct the l-th component at most four times and so the reconstruction complexity is $\max_{1 \le i \le k} C_i$ which is polynomial on the basis of property (α). The uniqueness of the solution follows also from property (α). □

5 Connection with the Q-Convexes

Q-convexity was introduced in [10] as an intermediate property between line convexity and convexity. In this section we show that strong decomposability is a more general property than Q-convexity along the set of directions $\{x, y\}$ if the discrete set consist of at least two components. For any point $P = (p_1, p_2)$ we define the four quadrants around P by

$$\begin{aligned}
R_0(P) &= \{Q = (q_1, q_2) \mid q_1 \le p_1 \text{ and } q_2 \le p_2\},\\
R_1(P) &= \{Q = (q_1, q_2) \mid q_1 \ge p_1 \text{ and } q_2 \le p_2\},\\
R_2(P) &= \{Q = (q_1, q_2) \mid q_1 \ge p_1 \text{ and } q_2 \ge p_2\},\\
R_3(P) &= \{Q = (q_1, q_2) \mid q_1 \le p_1 \text{ and } q_2 \ge p_2\}.
\end{aligned}$$

A discrete set F is *Q-convex* if $R_k(P) \cap F \ne \emptyset$ for all $k \in \{0, 1, 2, 3\}$ implies $P \in F$. We will denote the class of non-4-connected Q-convexes by $\mathcal{Q}'$.

Remark 5.1 More precisely, the above definition corresponds to the Q-convexity along the set of directions $\{x, y\}$ as Q-convexity can be defined in a more general manner (see [10]). However, for the sake of simplicity we will use this abbreviated form.

Let F be a non-4-connected Q-convex set having components $F_1, \ldots, F_k$ such that $\{i_l, \ldots, i'_l\} \times \{j_l, \ldots, j'_l\}$ is the SCDR of the lth ($l = 1, \ldots, k$) component of F. Every Q-convex set is hv-convex, too, therefore the sets of the row/column indices of the components of F consist of consecutive indices and they are disjoint. Without loss of generality we can assume that

$$1 = i_1 \le i'_1 < i_2 \le i'_2 < \ldots \le i'_k = m \; . \tag{16}$$

The following lemma shows that the SCDRs of the components of F can be arranged in only two ways.

Lemma 5.2 *Let $F \in \mathcal{Q}'$ having components $F_1, \ldots, F_k$ $(k \ge 2)$ with the SCDRs $\{i_l, \ldots, i'_l\} \times \{j_l, \ldots, j'_l\}$ such that (16) holds. Then, exactly one of the following cases is possible*

(1) $1 = j_1 \le j'_1 < j_2 \le j'_2 < \ldots \le j'_k = n$, *or*

(2) $n = j_1 \ge j'_1 > j_2 \ge j'_2 > \ldots \ge j'_k = 1$.

Proof. Clearly, both Cases (1) and (2) cannot hold together. For $k = 2$ the statement is trivial. Assume that $k > 2$ and neither (1) nor (2) is satisfied. Then, there exist three components, say F_{l_1}, F_{l_2}, and F_{l_3} such that $i_{l_1} < i_{l_2} < i_{l_3}$ and exactly one of the following relations hold

(a) $j_{l_2} < j_{l_1} < j_{l_3}$, or

(b) $j_{l_3} < j_{l_1} < j_{l_2}$, or

(c) $j_{l_1} < j_{l_3} < j_{l_2}$, or

(d) $j_{l_2} < j_{l_3} < j_{l_1}$.

Assume that Case (a) is true. Then for every position $M \in \{i_2, \ldots, i'_2\} \times \{j_1, \ldots, j'_1\}$ $M \notin F$ holds since F is also hv-convex. However, $R_k(M) \cap F \neq \emptyset$ for all $k \in \{0, 1, 2, 3\}$ which contradicts the Q-convexity. The proof is similar in the Cases (b), (c), and (d). □

If Case 1/2 of Lemma 5.2 is satisfied then we say that F is of type 1/2, respectively (see Fig. 5). We say that an hv-convex polyomino P with the SCDR $\{i, \ldots, i'\} \times \{j, \ldots, j'\}$ is *NW/NE/SE/SW-directed* if $(i', j') \in P/(i', j) \in P/(i, j) \in P/(i, j') \in P$, respectively. The following lemma is about the directedness of the components of an arbitrary set $F \in \mathcal{Q}'$.

Lemma 5.3 *Let $F \in \mathcal{Q}'$ having components $F_1, \ldots, F_k$ $(k \ge 2)$. If F is of type 1 then $F_1, \ldots, F_{k-1}$ are NW-directed and $F_2, \ldots, F_k$ are SE-directed. If F is of type 2 then $F_1, \ldots, F_{k-1}$ are NE-directed and $F_2, \ldots, F_k$ are SW-directed.*

Proof. Assume that the set $F \in \mathcal{Q}'$ is of type 1 and let M denote the bottom right position of the SCDR of F_1. If F_1 is not NW-directed then $M \notin F$. However, $R_k(M) \cap F \neq \emptyset$ for all $k \in \{0, 1, 2, 3\}$ which contradicts the Q-convexity. The directedness of the components $F_2, \ldots, F_k$ can be proven analogously.

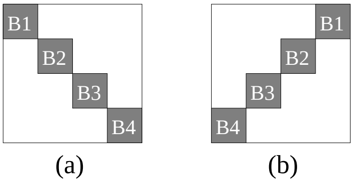


Fig. 5. The two possible configurations of the components in the class $\mathcal{Q}'$. (a) A set of type 1 and (b) a set of type 2. The possible empty rows and columns are not indiciated.

The proof if similar if the set is of type 2. □

Now, we can prove that every Q-convex set consisting of at least two components is strongly decomposable, too.

Theorem 5.4 $\mathcal{Q}' \subset \mathcal{D}EC^+$.

Proof. Let $F \in \mathcal{Q}'$ having components $F_1, \ldots, F_k$ $(k \geq 2)$ such that (16) holds. Since F is non-4-connected it has at least two components. On the basis of Lemma 5.3 and Theorem 3 in [17] the components can be reconstructed uniquely from the horizontal and vertical projections in $O(mn)$ time, i.e., property (α) is satisfied. Furthermore, on the basis of Lemma 5.2 the configuration of the components can follow only two cases. In both cases the sequence $F^{(i)} = \bigcup_{l=1}^{i} F_l$ $(i = 1, \ldots, k)$ is a gluing sequence of F therefore $F \in \mathcal{D}EC$ based on Lemma 3.1. It remains to prove that F satisfies the extra condition (δ) of strong decomposability. Let $F^{(1)}, \ldots, F^{(k)}$ be an arbitrary gluing sequence of F. If F is of type 1 then for every $F^{(l)}$ $(l = 2, \ldots, k)$ there is no position (i, j) which satisfies the conditions of the NE- or SW-versions of Theorem 3.3. Moreover, every $F^{(l)}$ $(l = 2, \ldots, k)$ has both NW- and SE-components and so property (δ) is satisfied. Similarly, if F is of type 2 then for every $F^{(l)}$ $(l = 2, \ldots, k)$ there is no position (i, j) which satisfies the conditions of the NW- or SE-versions of Theorem 3.3. Moreover, every $F^{(l)}$ $(l = 2, \ldots, k)$ has both NE- and SW-components, i.e., property (δ) is also satisfied. □

Then, applying Theorem 4.1 we get the following

Corollary 5.5 *Algorithm 2 solves Problem 2.1 in the class $\mathcal{Q}'$ in $O(mn)$ time. The reconstructed set is uniquely determined.*

The class of *hv*-convex 8- but not 4-connected discrete sets (denoted by $\mathcal{S}_8'$) was studied in [2]. In this paper the authors gave a reconstruction algorithm in this class using four projections. The worst case time complexity of this algorithm is of $O(mn)$ and the solution is uniquely determined. In the following we show that this result is strongly related with the results of this paper.

Lemma 5.6 $\mathcal{S}'_8 \subset \mathcal{Q}'$.

Proof. Sets of $\mathcal{S}'_8$ have the same properties as sets of $\mathcal{Q}'$ (see [2]). The only difference is that the SCDRs of the components in the class $\mathcal{Q}'$ might be separated, i.e., there can be empty rows or columns between two consecutive components while in $\mathcal{S}'_8$ the SCDRs of the components are always connected. □

Since empty rows and columns do not affect on the complexity of the reconstruction we get

Corollary 5.7 *Algorithm 2 solves Problem 2.1 in the class $\mathcal{S}'_8$ in $O(mn)$ time. The reconstructed set is uniquely determined.*

6 Reconstruction of Strongly Decomposable *hv*-Convex Discrete Sets

The class of hv-convex discrete sets (in the following denoted by $\mathcal{HV}$) is a frequently studied class in discrete tomography. In [15] an algorithm is published for reconstructing sets of $\mathcal{HV}$ from two projections. However, as it turned out later the reconstruction problem in this class is NP-hard [19]. In [5] an evolutionary algorithm is suggested for solving reconstruction problems in several difficult classes. In the same paper the author also analyzes the performance of the algorithm on hv-convex sets. The following theorem shows that with the aid of the decomposition technique described in Section 4 it is possible to solve the reconstruction problem efficiently in the class of strong decomposable hv-convex discrete sets.

Theorem 6.1 *Algorithm 2 solves Problem 2.1 in the class $\mathcal{HV} \cap \mathcal{DEC}^+$ in $O(mn \cdot \min\{m^2, n^2\})$ time. The reconstructed set is uniquely determined.*

Proof. Correctness and uniqueness follows from Theorem 4.1. The reconstruction complexity follows from the fact that the components are hv-convex polyominoes and they can be reconstructed from their horizontal and vertical projections in $O(mn \cdot \min\{m^2, n^2\})$ time (see [3]). □

Although this latter theorem states that if the hv-convex discrete set is strongly decomposable then it can be reconstructed in polynomial time Algorithm 2 is not appropriate to decide whether a set of $\mathcal{HV} \cap \mathcal{DEC}^+$ exists with the given projections. If we do not have any a priori information with

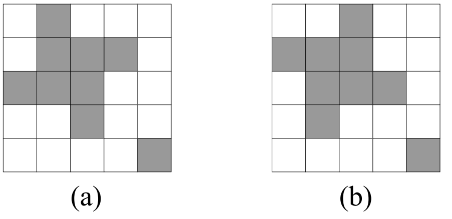


Fig. 6. (a) An hv-convex discrete set which is possibly reconstructible by Algorithm 2 and (b) an hv-convex discrete set with the same projections but with different components showing that the set in (a) does not satisfies property (α).

which property (α) is guaranteed then it can happen that the algorithm reconstructs an hv-convex set with the given projections whereas there is no set of $\mathcal{HV} \cap \mathcal{DEC}^+$ with these projections at all. It is because in this case in Step 2 of the procedures for decomposition we construct a polyomino simply by calling the general algorithm for reconstructing an hv-convex polyomino (see [3] for the algorithm). This algorithm reconstructs a polyomino from the horizontal and vertical projections but it is not guaranteed that the reconstructed polyomino is the only one with the given horizontal and vertical projections. In some cases this does not lead to failing our algorithm (see Fig. 6). However, this drawback of the algorithm turns to be an advantage if we want to reconstruct hv-convex sets from four projections since Algorithm 2 in certain cases can reconstruct hv-convex discrete sets in somewhat broader class than the strongly decomposables.

7 Conclusions and Further Work

In this paper we have introduced a new property of discrete sets, the strong decomposability and begun to study uniqueness and reconstruction problems in classes of discrete sets having this property. In general, for strongly decomposable discrete sets a polynomial time reconstruction algorithm is given. Then, it is proven that every Q-convex set which consists of at least two components is strongly decomposable. As a consequence we got that Problem 2.1 in the class $\mathcal{Q}'$ can be solved uniquely in $O(mn)$ time. Since $\mathcal{S}'_8 \subset \mathcal{Q}'$ the reconstruction of hv-convex 8- but not 4-connected sets from four projections can also be solved uniquely in $O(mn)$ time. The complexity of our algorithm strongly depends on the fact that the components are uniquely determined by the horizontal and vertical projections therefore it seems to be important to find classes of discrete sets where the reconstruction problem can be solved uniquely from these two projections.

It is shown that in some cases the discrete set can be decomposed along the diagonal and antidiagonal projections to facilitate the reconstruction. However, in some cases the decomposition into components is impossible. For example, the configuration in Fig. 2e can be decomposed into two parts (one containing $B1$ and $B2$, and the other containing $B3$ and $B4$) by the antidiagonal projection but then, the two parts cannot be further decomposed into components since the diagonal projections of the two parts are not independent. In some unfortunate cases the components cannot be separated at all (see, e.g., Fig. 2g and 2h). Further investigation is needed to answer the question whether the decomposition technique can somehow be applied to certain undecomposable configurations.

In this contribution we concentrated on discrete sets consisting of several components which satisfy properties (α), (β), (γ), and (δ). More work has to be done on the field whether assuming weaker properties about the components the reconstruction process remains tractable. Further work in this field can lead us towards designing efficient reconstruction algorithms for important classes like the one of *hv*-convex sets.

Acknowledgement

The author of this paper would like to thank to Attila Kuba for suggesting to investigate the connection between the classes of decomposable and Q-convex discrete sets.

References

[1] Balázs, P., *Reconstruction of decomposable discrete sets from four projections*, Lecture Notes in Computer Science **3429** (2005), 104–114.

[2] Balázs, P., E. Balogh, A. Kuba, *Reconstruction of 8-connected but not 4-connected hv-convex discrete sets*, Discrete Applied Mathematics **147** (2005), 149–168.

[3] Balogh, E., A. Kuba, Cs. Dévényi, A. Del Lungo, *Comparison of algorithms for reconstructing hv-convex discrete sets*, Lin. Alg. and Its Appl. **339** (2001), 23–35.

[4] Barcucci, E., A. Del Lungo, M. Nivat, R. Pinzani, *Reconstructing convex polyominoes from horizontal and vertical projections*, Theor. Comput. Sci. **155** (1996), 321–347.

[5] Batenburg, K.J., *An evolutionary approach for discrete tomography* Discrete Applied Mathematics, to appear.

[6] Brualdi, R.A., *Matrices of zeros and ones with fixed row and column sum vectors*, Lin. Algebra and Its Appl. **33** (1980), 159–231.

[7] Brunetti, S., A. Daurat, *Reconstruction of discrete sets from two or more X-rays in any direction*, Proceedings of the seventh International Workshop on Combinatorial Image Analysis (2000), 241–258.

[8] Castiglione, G., A. Frosini, A. Restivo, S. Rinaldi, *A tomographical characterization of L-convex polyominoes*, Lecture Notes in Computer Science **3429** (2005), 115–125.

[9] Chrobak, M., Ch. Dürr, *Reconstructing hv-convex polyominoes from orthogonal projections*, Information Processing Letters **69(6)** (1999), 283–289.

[10] Daurat, A., *Determination of Q-convex Sets by X-rays*, Theor. Comput. Sci. **332(1-3)** (2005), 19–45.

[11] Del Lungo, A., *Polyominoes defined by two vectors*, Theor. Comput. Sci. **127** (1994), 187–198.

[12] Del Lungo, A., M. Nivat, R. Pinzani, *The number of convex polyominoes reconstructible from their orthogonal projections*, Discrete Math. **157** (1996), 65–78.

[13] Gardner, R.J., P. Gritzmann, *Uniqueness and complexity in discrete tomography*, In [14] (1999), 85–113.

[14] Herman, G.T., A. Kuba (Eds.), "Discrete Tomography: Foundations, Algorithms and Applications", Birkhäuser, Boston, 1999.

[15] Kuba, A., *The reconstruction of two-directionally connected binary patterns from their two orthogonal projections*, Comp. Vision, Graphics, and Image Proc. **27** (1984), 249–265.

[16] Kuba, A., *Reconstruction in different classes of 2D discrete sets*, Lecture Notes on Computer Sciences **1568** (1999), 153–163.

[17] Kuba, A., E. Balogh, *Reconstruction of convex 2D discrete sets in polynomial time*, Theor. Comput. Sci. **283** (2002), 223–242.

[18] Ryser, H.J., *Combinatorial properties of matrices of zeros and ones*, Canad. J. Math. **9** (1957), 371–377.

[19] Woeginger, G.W., *The reconstruction of polyominoes from their orthogonal projections*, Inform. Process. Lett. **77** (2001), 225–229.