Skeletonization Based on Metrical Neighborhood Sequences

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Abstract. Skeleton is a shape descriptor which summarizes the general form of objects. It can be expressed in terms of the fundamental morphological operations. The limitation of that characterization is that its construction based on digital disks such that cannot provide good approximation to the Euclidean disks. In this paper we define a new type of skeleton based on neighborhood sequences that is much closer to the Euclidean skeleton. A novel method for quantitative comparison of skeletonization algorithms is also proposed.

Keywords: shape representation, skeletonization, neighborhood sequences, mathematical morphology.

1 Introduction

Skeleton is a region-based shape descriptor which summarizes the general form of objects/shapes [2]. There are three major definitions for the skeleton of an Euclidean set/object:

1. **Medial Axis Transformation (MAT).** For each point in an object, we find the closest border point. If a point has more than one such closest point, then it is said to belong to the skeleton of that object.
2. **Prairie-fire analogy or wavefront propagation.** The object boundary is set on fire and the skeleton is formed by the loci where the isotropic fire fronts meet and extinguish each other.
3. **Maximal inscribed disks/hyperspheres.** The centers of all maximal inscribed disks/hyperspheres comprise the skeleton. (A disk/hypersphere is maximal inscribed in an object if there exists no other disk/hypersphere within that object properly containing it.)

The extensions of those definitions to discrete sets lead to various skeletons and skeletonization methods [8].

Mathematical morphology is a powerful tool for image processing and image analysis [7]. Its operators can extract relevant topological and geometrical information from binary (and grey-scale) images by using structuring elements

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with various sizes and shapes. The discrete skeletons can be characterized morphologically: the homotopic wavefront propagation can be defined in terms of morphological hit-or-miss transforms and the centers of all maximal inscribed disks/hyperspheres can be expressed in terms of erosions and dilations.

In their classical paper, Rosenfeld and Pfaltz [6] investigated two types of motions in the two-dimensional digital space. The cityblock motion allows horizontal and vertical movements only, while in the case of chessboard motion one can diagonal movements, as well. The octagonal distances can be obtained by the mixed use of these motions.

This concept has been investigated and extended by many authors, in several directions. Here for the general history and the basic properties of these and related concepts of digital topology, we only refer to the survey paper [5] and the book [9].

Das, Chakrabarti and Chatterji [4] considered arbitrary periodic sequences of cityblock and chessboard motions, called periodic neighborhood sequences, and also their corresponding generalizations in $\mathbb{Z}^n$. Moreover, they established a formula for calculating the distance $d(p, q; A)$ of any two points $p, q \in \mathbb{Z}^n$, determined by such a neighborhood sequence A. Th ey introduced a natural partial ordering relation for periodic neighborhood sequences: if for two periodic neighborhood sequences A, B we have $d(p, q; A) \leq d(p, q; B)$ for all $p, q \in \mathbb{Z}^n$ then A is "faster" than B. Das [3] investigated the lattice properties of the set of periodic neighborhood sequences and some of its subsets under this relation, in 2D. He obtained some positive, but also some negative results. Note that similar results were obtained in 3D case [10].

Later, the investigation were extended to arbitrary, not necessarily periodic neighborhood sequences (see [11]). Such sequences have important applications. For example, it turns out that neighborhood sequences which provide the best approximations to the Euclidean distance in $\mathbb{Z}^2$ in some sense, are not periodic (see [13]).

Those neighborhood sequences which generate metrics on the digital space $\mathbb{Z}^n$ naturally play a special role in several problems, for example in skeletonization. Hence it is important to analyze the structural properties of these sequences. Such an investigation was performed in [12]. It turns out that in 2D the set of such sequences has a nice algebraic structure under the above mentioned natural partial ordering relation.

The structure of this paper is the following. After the Introduction to skeletonization and neighborhood sequences, we summarize the most inportant definitions about neighbourhood sequences (Section 2), and morphological skeleton (Section 3). We define a new type of skeleton based on neighborhood sequences in Section 4.The novel method for quantitative comparison of skeletons can be found in Section 5. In the last section we report our first experimental results.

2 Neighborhood Sequences

In this section we introduce some standard notation concerning neighborhood sequences.

Let $n, m \in \mathbb{N}$ with $m \leq n$. The points $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ in $\mathbb{Z}^n$ are m-neighbors, if the following two conditions hold:

- $|p_i - q_i| \leq 1 \quad (1 \leq i \leq n)$,
- $\sum_{i=1}^{n} |p_i - q_i| \leq m$.

The sequence $A = (A(i))_{i=1}^{\infty}$, where $A(i) \in \{1, \ldots, n\}$ for all $i \in \mathbb{N}$, is called an n-dimensional (shortly nD) neighborhood sequence. If for some $l \in \mathbb{N}$ we have $A(i + l) = A(i)$ for $i \in \mathbb{N}$ then A is called periodic with period l. The set of the nD-neighborhood sequences will be denoted by S_n, while the set of periodic ones by P_n.

Let $p,\ q \in \mathbb{Z}^n$ and $A \in S_n$. The point sequence $p = p_0, p_1, \ldots, p_t = q$, where p_{i-1} and p_i are $A(i)$-neighbors in $\mathbb{Z}^n$ $(1 \leq i \leq t)$, is called an A-path from p to q of length t. The A-distance $d(p, q; A)$ of p and q is defined as the length of the shortest A-path(s) between them. As a brief notation, we also use $d(A)$ for the A-distance.

3 Morphological Skeleton

When using a morphological approach, skeleton is related to the set of centres of all the maximal insribed disks which can be expressed in terms of erosions and dilations [7]. Fist, some concepts of mathematical morphology will be given below.

The *dilation* of set $X \subseteq \mathbb{Z}^n$ by *structuring element* $Y \subseteq \mathbb{Z}^n$ is defined by

$$X \oplus Y = \{\ p \mid (\hat{Y})_p \cap X \neq \emptyset\ \}\ ,$$

where $\hat{Y}$ denotes the *reflection* of set Y defined as $\hat{Y} = \{\ -y \mid y \in Y\ \}$, and $(Y)_p$ denotes the translation of set Y by point $p \in \mathbb{Z}^n$ defined as $(Y)_p = \{\ y + p \mid y \in Y\ \}$.

The *erosion* of X by structuring element Y is defined by

$$X \ominus Y = \{\ p \mid (Y)_p \subseteq X\ \} = (X^c \oplus \hat{Y})^c\ ,$$

where $(X)^c$ denotes the set-theoretic complement of X.

The *morphological skeleton* of set X by structuring element Y is defined by

$$S(X, Y) = \bigcup_{k=0}^{K} S_k(X, Y) = \bigcup_{k=0}^{K} (X \ominus Y^k) - ((X \ominus Y^{k+1}) \oplus Y)\ ,$$

where

$$Y^k = \begin{cases} \{\mathcal{O}\} \ \ \text{(simply the origin)} & \text{if } k = 0 \\ \{\mathcal{O}\} \oplus Y = Y & \text{if } k = 1 \\ Y^{k-1} \oplus Y & \text{otherwise} \end{cases}\ ,$$

and K is the last step before X is eroded completely:

$$K = \max\{\ k \mid X \ominus Y^k \neq \emptyset\ \}.$$

The formulation states that $S(X,Y)$ is obtained as the union of the *skeletal subsets* $S_k(X,Y)$. It can be readily be seen that the set $S_k(X,Y)$ contains all points $p \in X$ such that x is the center of a maximal "disk" included in X. Note that the limitation of the morphological skeleton is that its construction is based on "disks" of the form Y^k. Hence the morphological skeleton does not provides a good approximation to the Euclidean skeleton.

4 Sequence Skeleton

In order to cut the shortage of the morphological skeleton, we propose a new type of skeleton that is based on neighborhood sequences.

Let $A = (A(i))_{i=1}^{\infty}$ be an nD neighborhood sequence and let $\mathcal{Y} = (Y(i))_{i=1}^{\infty}$ be the sequence of structuring elements in which $Y(i)$ corresponds to $A(i)$ ($i = 1, 2, \ldots$). For example, if $n = 2$ and $A(i) = 1$, then

$$Y(i) = \{(0,0), (-1,0), (1,0), (0,-1), (0,1)\}\ .$$

The *sequence skeleton* of set X by sequence of structuring elements $\mathcal{Y}$ is defined by

$$\mathcal{S}(X,\mathcal{Y}) = \bigcup_{k=0}^{K} (X \ominus \mathcal{Y}^k) - ((X \ominus \mathcal{Y}^{k+1}) \oplus Y(k+1))\ ,$$

where

$$\mathcal{Y}^k = \begin{cases} \{\mathcal{O}\} & \text{if } k = 0 \\ \{\mathcal{O}\} \oplus Y(1) = Y(1) & \text{if } k = 1 \\ \mathcal{Y}^{k-1} \oplus Y(k) & \text{otherwise} \end{cases}$$

and

$$K = \max\{\ k \mid X \ominus \mathcal{Y}^k \neq \emptyset\ \}.$$

It is easy to see that

$$\mathcal{S}(X,\mathcal{Y}) = S(X,Y)$$

if $\mathcal{Y} = (Y, Y, \ldots)$. Hence the conventional morphological skeleton is a special case of sequence skeletons.

5 Quantitative Comparison of Skeletons

In this Section, a new and fairly general method is presented for quantitative comparison of different skeletonization algorithms/methods.

The proposed method consists of the following steps for each selected base (binary) image BI:

1. Reference skeleton RS is extracted from BI by a topologically correct skeletonization algorithm.
2. Euclidean distance map DM_{BI} is calculated from object boundary in base image BI [1].
3. Reference image RI is created by replacing each skeletal point p in RS by an Euclidean disk with radius $DM_{BI}(p)$.
   In this way we get a reference image from a base image with its known, connected, and exact reference skeleton. A base image, its reference skeleton, its distance map, and the reference image (as union of Euclidean disks) are shown in Fig. 1.
4. Euclidean distance map DM_{RS} is computed from reference skeleton RS.
5. A skeleton S is extracted from reference image RI by an arbitrary skeletonization algorithm.
   Fig. 2 presents some illustrative examples for sequence skeletons extracted from a reference image.
6. Euclidean distance map DM_S is computed from skeleton S.
   Examples of skeletal distance maps are presented in Fig. 3
7. $d(RS, DM_S)$ is determined, where

$$d(X,Y) = \sum_{p \in X, X(p)=1} Y(p).$$

   In other words, $d(X.Y)$ is the sum of distances in the distance map Y at each skeletal points (having value 1) in sekelton X. It is obvious that $d(RS, DM_{RS}) = 0$.
8. $d(S, DM_{RS})$ is calculated as well.
   We use $d(RS, DM_S)$ and $d(S, DM_{RS})$ to measure the goodness of the investigated skeleton S.

6 Experiments

We made an experiment, using 10 different binary images with different complexity (number of components, shape of components, size). We generated the sequence skeletons of the images using the classical $d_4 = (4)_1^\infty$, $d_8 = (8)_1^\infty$ neighborhood sequences that consist of only one kind of neighborhood examination repeatedly. Furthermore we generated the sequence skeletons with the $d_{\{84\}} = (84)_1^\infty$, called octogonal neighborhood sequence, (which is verified to give a finer approximation of the Euclidean distance then the d_4 or d_8 [3]), and the $d_{opt} = (8444848844844)_1^\infty$ sequence, which consists of the prenex of the best 2 dimensional Euclidean distance approximating non-periodical neighborhood sequence [13]. The length of this prenex is related to the image content, it is about half of the diameter of the objects on most image. There can be several sequences approximating the Euclidean distance well, but the infinite neighborhood sequence described in [13] is theoretically prooved to give the best approximation.

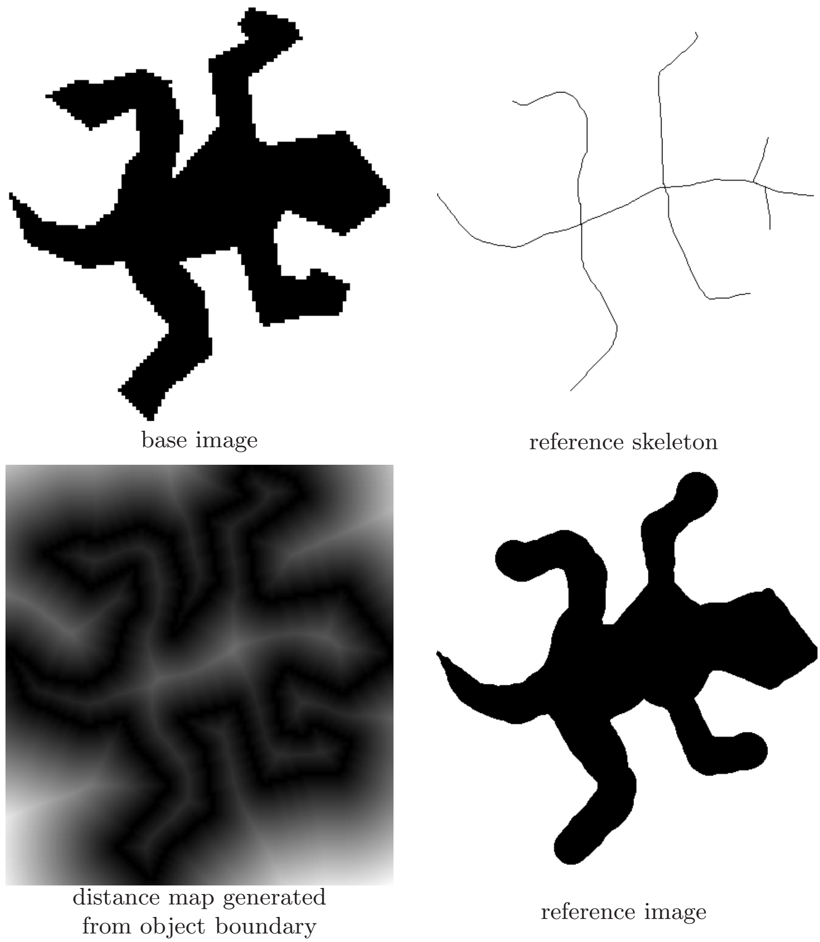


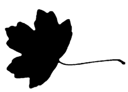
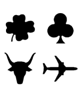
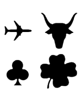
Fig. 1. Base image #1 (of size 428×462), its reference skeleton, its Euclidean distance maps, and the derived reference image #1

According to the comparison method above to measure the goodness of a skeleton the $D1 = d(S, DM_{RS})$ and the $D2 = d(RS, DM_S)$ values have been computed. We computed the following four measures:

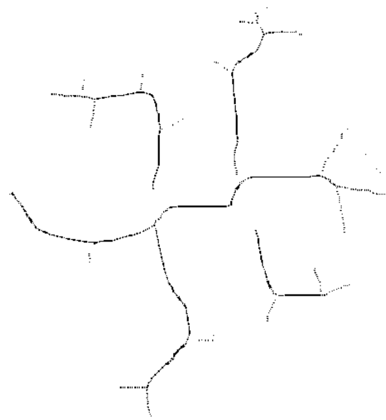
$$\begin{aligned}
MAX &= max\,\{D1, D2\}\ , \\
AVG &= (D1 + D2)/2\ , \\
|D1| &= 100 * D1/(number\ of\ pixels\ in\ S)\ , \\
|D2| &= 100 * D2/(number\ of\ pixels\ in\ RS)\ .
\end{aligned}$$

In Table 1, for all measures the smaller value means the better similarity of the sequence skeleton and the reference skeleton.

Table 1. Comparison for the 10 base images

Image	Size	Goodness	d_4	d_8	$d_{\{84\}}$	d_{opt}
	428 × 464	D1	2039	3653	1802	1481
		D2	4061	2523	1469	1471
		MAX	4061	3653	1802	1481
		AVG	3050	3088	1635	1476
		|D1|	184	231	126	114
		|D2|	320	199	116	116
	118 × 135	D1	131	293	240	165
		D2	3820	3772	2348	2129
		MAX	3820	3772	2348	2129
		AVG	1975	2032	1294	1147
		|D1|	66	129	94	73
		|D2|	529	523	325	295
	160 × 183	D1	131	293	240	165
		D1	200	261	198	173
		D2	164	236	150	142
		MAX	200	261	198	173
		AVG	182	248	174	157
		|D1|	50	62	48	44
		|D2|	45	65	41	39
	586 × 425	D1	1902	3831	2356	1825
		D2	7332	8943	4560	4006
		MAX	7332	8943	4560	4006
		AVG	4617	6387	2458	2915
		|D1|	189	349	200	160
		|D2|	484	591	301	265
	696 × 731	D1	3636	8985	5152	3894
		D2	10049	13416	5920	4834
		MAX	10049	13416	5920	4834
		AVG	6842	11200	4036	4364
		|D1|	126	255	142	110
		|D2|	245	363	160	131
	174 × 224	D1	184	441	335	229
		D2	1101	1148	709	620
		MAX	1101	1148	709	620
		AVG	642	794	522	424
		|D1|	45	87	62	49
		|D2|	198	206	127	111
	160 × 224	D1	129	488	371	237
		D2	1015	1316	646	556
		MAX	1015	1316	646	556
		AVG	572	902	508	396
		|D1|	47	147	107	72
		|D2|	230	298	146	126
	512 × 512	D1	765	2924	1551	967
		D2	3217	5151	1863	1601
		MAX	3217	5151	1863	1601
		AVG	1991	4037	1707	1284
		|D1|	59	187	102	70
		|D2|	191	306	111	95
	512 × 512	D1	846	3157	1754	1046
		D2	3444	5269	2204	1857
		MAX	3444	5269	2204	1857
		AVG	2145	4213	1979	1451
		|D1|	64	197	109	74
		|D2|	203	310	130	110
	512 × 512	D1	1792	2358	1588	1270
		D2	4462	4530	2610	2692
		MAX	4462	4530	2610	2692
		AVG	3127	3444	2099	1981
		|D1|	137	146	94	84
		|D2|	253	257	148	153

sequence skeleton
$A = (4)_1^\infty$



sequence skeleton
$A = (8)_1^\infty$

sequence skeleton
$A = (84)_1^\infty$

sequence skeleton
$A = (8444848444844)_1^\infty$

Fig. 2. Examples of sequence skeletons of reference image #1

Considering the $D1$, $D2$ and MAX measures we can state, that in most cases the sequence skeleton generated by the d_{opt} neighborhood sequence is the best one, and the sequence skeleton by $d_{\{84\}}$ is the second best one of the four sequence skeletons. That matches the theoretical expectations and the observations can be made comparing the sequence skeletons visually in Fig. 2.

In some cases, when the input image contains thin components, the d_4 can give better results for $D1$ then d_{opt}. In these cases the $D2$ of d_4 is worse then $D2$ of d_{opt}. The reason of this is the torn sequence skeleton (and so the small number of skeleton pixels) of thin objects using d_4. Computing $D1$, fewer values from the distance map will be added resulting smaller sum (higher similarity). On the contrary, due to the torn sequence skeleton, in the distance map of S in the positions of RS pixels are not zeros, but positive numbers, meaning how far

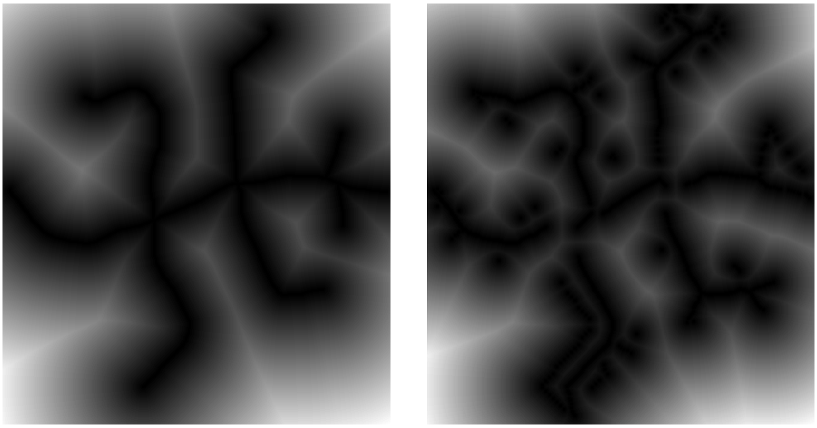


Fig. 3. Distance maps generated from the reference skeleton #1 (left) and the sequence skeleton $A = (4)$ extracted from reference image #1 (right)

the closest skeleton pixel is. That is why the $D2$ sum will be greater and giving worse results in the same time. To solve this problem, we introduced the average of $D1$ and $D2$ to measure the similarity of S to the RS. The results of AVG fit the expectations in all cases.

We introduced two more measures, the $|D1|$ and $|D2|$ are in order the $D1$ and $D2$ normalized by the number of skeleton pixels in the S and RS, respectively. These values are free from the error due to the different number of skeleton pixels in S and RS, but still contain the error originated from the torns in the sequence skeletons.

In this experiment we found the d_{opt} neighborhood sequence to result the best approximation of the RS. Also the $d_{\{84\}}$ sequence gives a better approximation (according to the 6 measures we introduced) of the RS, then the d_4 or d_8 sequences. That matches the theoretical expectations and the visual observations which can be made on the images.

The comparison algorithm for skeletonization methods gave also the expected results, the derived measures are robust, the best one (AVG) gave the theoretical results in all cases. In virtue of these, the comparison algorithm described and applied above can be used to compare skeletoniztation algorithms.

That was the first experiment we have made. To draw stronger conclusions, we make a more exhaustive experiment on a greater set containing binary images of various size and complexity. Another aim is to do similar examination of the sequence based skeletonization algorithm in 3 dimensions.

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