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A decomposition technique for reconstructing discrete sets from four projections

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Abstract

The reconstruction of discrete sets from four projections is in general an NP-hard problem. In this paper we study the class of decomposable discrete sets and give an efficient reconstruction algorithm for this class using four projections. It is also shown that an arbitrary discrete set which is Q-convex along the horizontal and vertical directions and consists of several components is decomposable. As a consequence of decomposability we get that in a subclass of *hv*-convex discrete sets the reconstruction from four projections can also be solved in polynomial time. Possible extensions of our method are also discussed.

Keywords: Discrete tomography; Reconstruction algorithm; Decomposable discrete set; Q-convexity; *hv*-Convexity

1. Introduction

One of the most frequently studied problems in discrete tomography is the reconstruction of 2-dimensional discrete sets (the finite subsets of $\mathbb{Z}^2$) from few (usually up to four) projections. Reconstruction algorithms have a wide area of applications (e.g., in electron microscopy, image processing, non-destructive testing, radiology) [1]. Several theoretical questions are also connected with reconstruction such as consistency and uniqueness (as a summary see [2,3]). Since the reconstruction problem is usually underdetermined the number of solutions for a given reconstruction task can be very large. Moreover, the reconstruction under certain circumstances can be NP-hard (see [3,4]). In order to keep the reconstruction process tractable and to reduce the number of solutions a commonly used technique is to suppose having some a priori information of the set to be reconstructed. The most frequently used properties are connectedness, directedness and some kind of discrete versions of the convexity. A lot of work have been done in designing efficient reconstruction algorithms for different classes of discrete sets (e.g., [5–12]). However, up to now only few papers deal with the problem of reconstruction if it is known in advance that the discrete set to be reconstructed consists of several components. While the reconstruction from two projections in the class of *hv*-convex sets is in general NP-hard [4] it turned out that the additional prior knowledge that the set has only one component (i.e., it is 4-connected) leads to a polynomial-time reconstruction algorithm [8]. Later, this algorithm was generalised for the class $\mathcal{S}_8$ of 8-connected *hv*-convex sets, too [5]. Surprisingly, in [13] the authors showed that in this class the prior information that the set to be reconstructed consists of more than one component (i.e., if the set is not 4-connected) leads to a more efficient reconstruction algorithm than the general one developed for the class $\mathcal{S}_8$. These results show the importance of investigating the reconstruction in classes of discrete sets having several components. In this paper we present a class of discrete sets consisting of several components, the class of decomposable discrete sets and study the reconstruction problem in this class.

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This article is an extended version of the conference paper [14]. The structure of this contribution and the main novelties compared to [14] are the followings. First, the necessary definitions are introduced in Section 2. In Section 3 we define the class of decomposable discrete sets and prove some important properties of sets belonging to this class. While in [14] we had the restriction that a decomposable discrete set cannot have an empty row or column in this paper this restriction is omitted. Moreover, in this contribution we introduce the concept of centre by which a more exhausted study of the decomposable discrete sets is possible. According to this the polynomial-time reconstruction algorithm developed in [14] is redesigned. The results of Sections 4 and 5 are totally new, they are not represented in the conference paper. In Section 4 we show that every discrete set which is Q-convex along the horizontal and vertical directions is also decomposable if it has at least two components. Applying the results of Section 3 we get an O(mn) algorithm for the reconstruction problem in this class using four projections. In Section 5 we discuss the possibility to adapt our decomposition technique to *hv*-convex sets to facilitate the reconstruction. Finally, in Section 6 we conclude our results and discuss some possible extensions of our work.

2. Definitions and notation

The finite subsets of $\mathbb{Z}^2$ are called *discrete sets*. Let $\hat{F} = (\hat{f}_{ij})_{m\times n}$ be a binary matrix where $m,n \geqslant 1$ and F denote the set of positions (i,j) where $\hat{f}_{ij} = 1$, i.e., $F = \{(i,j)|\hat{f}_{ij} = 1\}$. Due to the strong relation between binary matrices and discrete sets F is called a discrete set represented by $\hat{F}$, and its elements are called *points* or *positions*. Without loss of generality in the followings we will always assume that there is at least one non-zero element in both the last row and column of $\hat{F}$, i.e., there exist $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ such that $\hat{f}_{i,n} = \hat{f}_{m,j} = 1$. The *kth diagonal/antidiagonal* ($k = 1,\ldots,m+n-1$) of $\hat{F}$ is defined by the set D_k/A_k, respectively, where

$$D_k = \{(i,j) \in \{1,\ldots,m\} \times \{1,\ldots,n\}|i + (n-j) = k\}, \quad (1)$$

$$A_k = \{(i,j) \in \{1,\ldots,m\} \times \{1,\ldots,n\}|i + j = k+1\}. \quad (2)$$

Let $\mathscr{F}$ denote the class of discrete sets. For any discrete set $F \in \mathscr{F}$ we define the functions $\mathscr{H}, \mathscr{V}, \mathscr{D}$, and $\mathscr{A}$ as follows. $\mathscr{H} : \mathscr{F} \to \mathbb{N}_0^m, \mathscr{H}(F) = H = (h_1,\ldots,h_m)$, where

$$h_i = \sum_{j=1}^{n} \hat{f}_{ij}, \qquad i = 1,\ldots,m, \quad (3)$$

$\mathscr{V} : \mathscr{F} \to \mathbb{N}_0^n, \mathscr{V}(F) = V = (v_1,\ldots,v_n)$, where

$$v_j = \sum_{i=1}^{m} \hat{f}_{ij}, \qquad j = 1,\ldots,n, \quad (4)$$

$\mathscr{D} : \mathscr{F} \to \mathbb{N}_0^{m+n-1}, \mathscr{D}(F) = D = (d_1,\ldots,d_{m+n-1})$, where

$$d_k = \sum_{(i,j)\in D_k} \hat{f}_{ij} = |F \cap D_k|, \qquad k = 1,\ldots,m+n-1, \quad (5)$$

and $\mathscr{A} : \mathscr{F} \to \mathbb{N}_0^{m+n-1}, \quad \mathscr{A}(F) = A = (a_1,\ldots,a_{m+n-1})$, where

$$a_k = \sum_{(i,j)\in A_k} \hat{f}_{ij} = |F \cap A_k|, \qquad k = 1,\ldots,m+n-1. \quad (6)$$

The vectors H, V, D, and A are called the *horizontal*, *vertical*, *diagonal*, and *antidiagonal projections* of F, respectively. The *cumulated vectors* of F are denoted by $\widetilde{H} = (\widetilde{h}_1,\ldots,\widetilde{h}_m)$, $\widetilde{V} = (\widetilde{v}_1,\ldots,\widetilde{v}_n)$, $\widetilde{D} = (\widetilde{d}_1,\ldots,\widetilde{d}_{m+n-1})$, and $\widetilde{A} = (\widetilde{a}_1,\ldots,\widetilde{a}_{m+n-1})$, and defined by the following formulas (see Fig. 1).

$$\widetilde{h}_i = \sum_{l=1}^{i} h_l, \qquad i = 1,\ldots,m, \quad (7)$$

$$\widetilde{v}_j = \sum_{l=1}^{j} v_l, \qquad j = 1,\ldots,n, \quad (8)$$

$$\widetilde{d}_k = \sum_{l=1}^{k} d_l, \qquad \widetilde{a}_k = \sum_{l=1}^{k} a_l, \qquad k = 1,\ldots,m+n-1. \quad (9)$$

Given a class $\mathscr{G} \subseteq \mathscr{F}$ of discrete sets we say that the discrete set $F \in \mathscr{G}$ is *unique in the class* $\mathscr{G}$ (with respect to some projections) if there is no different discrete set $F' \in \mathscr{G}$ with the same projections.

Two points $P = (p_1,p_2)$ and $Q = (q_1,q_2)$ in a discrete set F are said to be *4-adjacent* if $|p_1 - q_1| + |p_2 - q_2| = 1$. The points P and Q are said to be *8-adjacent* if they are 4-adjacent or ($|p_1 - q_1| = 1$ and $|p_2 - q_2| = 1$). The sequence of distinct points $P_0,\ldots,P_k$ is a *4/8-path* from point P_0 to point P_k in a discrete set F if each point of the sequence is in F and P_l is 4/8-adjacent, respectively, to P_{l-1} for each $l = 1,\ldots,k$. Two points are 4/8-connected in the discrete set F if there is a 4/8-path, respectively, in F between them. A discrete set F is *4/8-connected* if any two points in F are 4/8-connected, respectively, in F. The 4-connected set is also called as polyomino [15]. The discrete set F is *horizontally convex/vertically convex* (or shortly, *h*-convex/*v*-convex) if its rows/columns are 4-connected, respectively. The *h*- and *v*-convex sets are called *hv*-convex (see Fig. 1). In this paper we are going to study the reconstruction problem

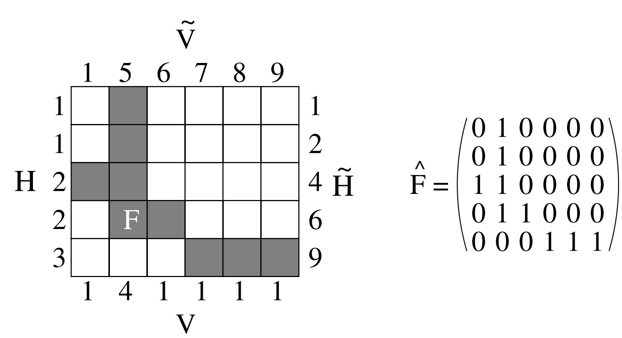

Fig. 1. An *hv*-convex 8- but not 4-connected discrete set F and the corresponding binary matrix $\hat{F}$. The elements of F are marked with grey squares. The projections of F are H, V, $D = (0,0,0,0,2,2,3,2,0,0)$, and $A = (0,1,2,1,1,1,0,1,1,1)$. The cumulated vectors of F are $\widetilde{H}, \widetilde{V}, \widetilde{D} = (0,0,0,0,2,4,7,9,9,9)$, and $\tilde{A} = (0,1,3,4,5,6,6,7,8,9)$.

from four projections in several classes. This task can be formulated as follows.

Problem 1. Given a class $\mathscr{G} \subseteq \mathscr{F}$ and four non-negative vectors $H \in \mathbb{N}_0^m$, $V \in \mathbb{N}_0^n$, $D \in \mathbb{N}_0^{m+n-1}$, and $A \in \mathbb{N}_0^{m+n-1}$ construct a discrete set $F \in \mathscr{G}$ with $\mathscr{H}(F) = H$, $\mathscr{V}(F) = V, \mathscr{D}(F) = D$, and $\mathscr{A}(F) = A$.

3. Reconstruction of decomposable discrete sets

Let F be a discrete set. A maximal 4-connected subset of F is called a *component* of F. For example, the discrete set F in Fig. 1 has two components: $\{(5,4), (5,5), (5,6)\}$ and $\{(1,2), (2,2), (3,1), (3,2), (4,2), (4,3)\}$. Clearly, the components of F give a uniquely determined partition of F. Given two discrete sets C and D represented by the binary matrices $\hat{C} = (\hat{c}_{ij})_{m_1 \times n_1}$ and $\hat{D} = (\hat{d}_{ij})_{m_2 \times n_2}$, respectively, we say that we get the discrete set F represented by the binary matrix $\hat{F} = (\hat{f}_{ij})_{m_3 \times n_3}$ by a *NorthWest-gluing* (or shortly, NW-gluing) from C and D if

$$\hat{F} = \begin{pmatrix} \hat{C} & \mathbf{0} \\ \mathbf{0} & \hat{D} \end{pmatrix} \text{such that } m_3 \geqslant m_1 + m_2 \text{ and } n_3 \geqslant n_1 + n_2. \tag{10}$$

We stress the importance that in the resulting set F there can be empty rows or/and columns between the subsets C and D (namely, if $m_3 > m_1 + m_2$ or/and $n_3 > n_1 + n_2$). If C is a single component then we say that C is the *NW-component of F*. NE-, SE-, SW-gluings and -components are defined similarly. We say that a discrete set F consisting of k ($k \geqslant 2$) components is *decomposable* if

- (α) the components are uniquely reconstructible from their horizontal and vertical projections in polynomial time, and
- (β) the sets of the row/column indices of the components are disjoint, i.e., if $I \times J \subset \{1,\ldots,m\} \times \{1,\ldots,n\}$ is the smallest containing discrete rectangle (SCDR) of a component of the discrete set F, then $\bar{I} \times J \cap F = I \times \bar{J} \cap F = \emptyset$ (where $\bar{I} = \{1,\ldots,m\} \setminus I$ and $\bar{J} = \{1,\ldots,n\} \setminus J$), and
- (γ) if $k > 2$ then we get F by gluing a single component to a decomposable discrete set consisting of $k-1$ components using one of the four gluing operators.

In fact, to satisfy property (α) we usually need to have some a priori information about the components. For example, NW-directed *hv*-convex discrete sets can be used as components since in this class property (α) is fulfilled [16]. Another class of discrete sets where property (α) is satisfied is the class of L-convex polyominoes [17]. As a straight consequence of the definition we get that every discrete set consisting of one component is undecomposable and every discrete set consisting of two or three components and satisfying properties (α) and (β) is decomposable. Fig. 2 shows some decomposable and undecomposable configurations if the set satisfies property (β) and consists of four components. The class of decomposable discrete sets is denoted by $\mathscr{DEC}$. The following lemma gives a description of decomposable discrete sets.

Lemma 2. *A discrete set F is decomposable if and only if it satisfies property* (α) *and there exists a sequence of discrete sets* $F^{(1)},\ldots,F^{(k)}$ *such that* $F^{(1)}$ *consists of one component,* $F^{(k)} = F$, *and for each* $l = 1,\ldots,k-1$ *we get* $F^{(l+1)}$ *by gluing a component to* $F^{(l)}$ *using a gluing operator.*

Proof. It is a straight consequence of the definition of decomposability. □

For a given discrete set $F \in \mathscr{DEC}$ the sequence described in Lemma 2 is not uniquely determined. We will refer to any sequence satisfying the properties of Lemma 2 as a *gluing sequence* of F. For example, the sequences $F^{(1)} = \{(1,1)\}$, $F^{(2)} = \{(1,1), (2,2)\}$ and $G^{(1)} = \{(2,2)\}$, $G^{(2)} = \{(1,1), (2,2)\}$ are both gluing sequences of the same discrete set $F = \{(1,1), (2,2)\}$. Clearly, every gluing sequence of a given decomposable discrete set must be of the same length (namely, the length is equal to the number of the components the discrete set consists of). There are two special subclasses of $\mathscr{DEC}$. If during the building of $F \in \mathscr{DEC}$ one uses only NW- and/or SE-gluings then we say that F is of *type 1*. Similarly, if during the building of $F \in \mathscr{DEC}$ only SW- and/or NE-gluings are used then we say that F is of *type 2*. The class of decomposable discrete sets of type 1/2 is denoted by $\mathscr{S}^*/\mathscr{S}^{**}$, respectively (see Fig. 3). Note, that $\mathscr{S}^*, \mathscr{S}^{**} \subset \mathscr{DEC}$ and $\mathscr{S}^* \cap \mathscr{S}^{**} = \emptyset$. If the discrete set $F \in \mathscr{DEC}$ consists of 2 components then, clearly, either $F \in \mathscr{S}^*$ or $F \in \mathscr{S}^{**}$. Moreover, from property (γ) it follows that if $F^{(1)},\ldots,F^{(k)}$ is a gluing sequence of $F \in \mathscr{DEC}$ then there exists an integer $1 < j \leqslant k$ such that $F^{(j)} \in \mathscr{S}^* \cup \mathscr{S}^{**}$ and $F^{(j+1)} \notin \mathscr{S}^* \cup \mathscr{S}^{**}$ (if $j \neq k$). Now, we can describe the relation between two arbitrary gluing sequences of a given decomposable discrete set.

Lemma 3. *Let* $F^{(1)},\ldots,F^{(k)}$ *and* $G^{(1)},\ldots,G^{(k)}$ *are two different gluing sequences of the same discrete set* $F \in \mathscr{DEC}$. *Moreover let* $1 < j,j' \leqslant k$ *such that* $F^{(j)}, G^{(j')} \in \mathscr{S}^* \cup \mathscr{S}^{**}$ *and* $F^{(j+1)}, G^{(j'+1)} \notin \mathscr{S}^* \cup \mathscr{S}^{**}$ (*if* $F^{(j+1)}$ *and* $G^{(j'+1)}$ *exist*). *Then* $j = j'$ *and* $F^{(j)} = G^{(j')}$.

Proof. If $F \in \mathscr{S}^* \cup \mathscr{S}^{**}$ then the lemma is true with $j = j' = k$. Consider $F \in \mathscr{DEC} \setminus (\mathscr{S}^* \cup \mathscr{S}^{**})$. If $F^{(j)} \neq G^{(j')}$ then there exists a position (p,q) such that $(p,q) \in F^{(j)} \setminus G^{(j')}$ or $(p,q) \in G^{(j')} \setminus F^{(j)}$. Assume that $(p,q) \in F^{(j)} \setminus G^{(j')}$ and the SCDR of $G^{(j')}$ is $I \times J = \{i',\ldots,i''\} \times \{j',\ldots,j''\}$. Then, on the basis of property (β) $(p,q) \in \bar{I} \times \bar{J}$ where $\bar{I} = \{1,\ldots,\mathrm{m}\} \setminus I$ and $\bar{J} = \{1,\ldots,n\} \setminus J$. Since $(p,q) \in F^{(j)}$ and $F^{(j)} \in \mathscr{S}^* \cup \mathscr{S}^{**}$ it follows that $F^{(j)} \subset \bar{I} \times \bar{J}$ therefore $F^{(j)} \cap G^{(j')} = \emptyset$. Since $F^{(j+1)}, G^{(j'+1)} \notin \mathscr{S}^* \cup \mathscr{S}^{**}$ the components of $G^{(j')}$ cannot be glued to a subset of F containing $F^{(j)}$ or vice versa therefore $F^{(j)} \cup G^{(j')} \nsubseteq F$ is a contradiction. The proof is similar if it is assumed that $(p,q) \in G^{(j')} \setminus F^{(j)}$. □

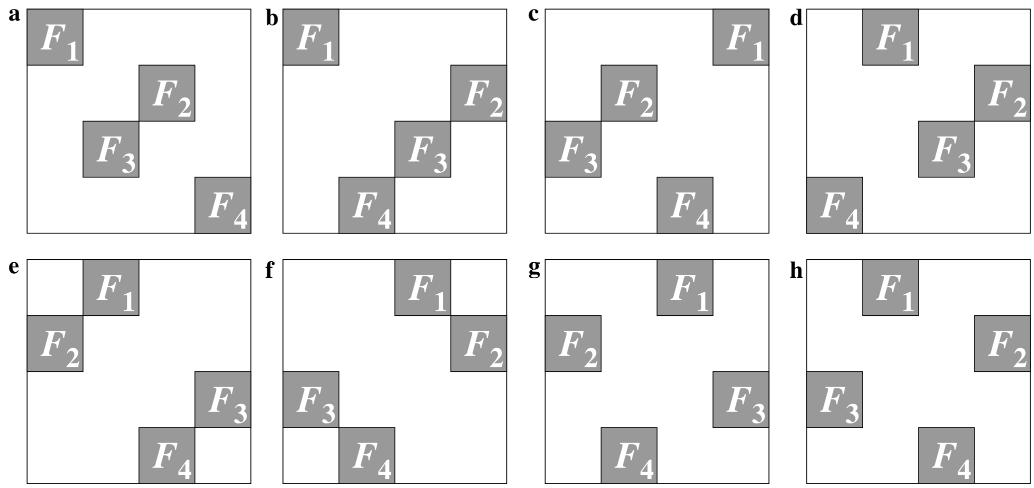

Fig. 2. (a)–(d) Some decomposable and (e)–(h) undecomposable configurations of the components F_1, F_2, F_3, and F_4. The SCDRs of the components are marked with grey squares. The possible empty rows and columns are not indicated.

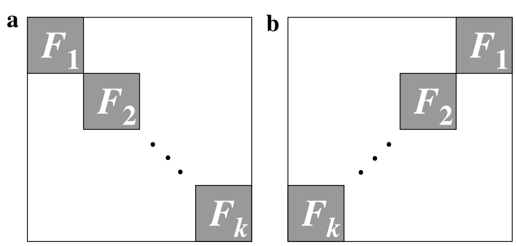

Fig. 3. The configurations of the components $F_1, F_2, \ldots, F_k$ in the class (a) $\mathcal{S}^*$ and (b) $\mathcal{S}^{**}$. The possible empty rows and columns are not indicated.

Based on this lemma we can say that for every set $F \in \mathcal{DEC}$ there exists an integer j such that in every gluing sequence $F^{(1)}, \ldots, F^{(k)} = F$ $F^{(j)}$ is the same, $F^{(j)} \in \mathcal{S}^* \cup \mathcal{S}^{**}$, and $F^{(j+1)} \notin \mathcal{S}^* \cup \mathcal{S}^{**}$ (if $j \neq k$). The uniquely determined set $F^{(j)}$ is called the *centre* of F and is denoted by $C(F)$. For example, if the configuration of the components of the set F is given as in Fig. 2a then $C(F) = F_2 \cup F_3$ while in the case given in Fig. 2b $C(F) = F_2 \cup F_3 \cup F_4$. On the basis of properties (α) and (β) in the reconstruction of a decomposable discrete set it is sufficient to identify the SCDRs of the components. In order to do this we give a necessary condition.

Lemma 4. [NW-version] *Let $F \in \mathcal{DEC}$. If (i,j) is the bottom right position of the SCDR of the NW-component of F then i is the smallest integer for which there exists an integer j with $\widetilde{h}_i = \widetilde{v}_j = \widetilde{a}_{i+j-1} > 0$ and $a_{i+j} = 0$.*

Proof. Define a set E as follows

$$E = (\{1, \ldots, i\} \times \{j+1, \ldots, n\}) \cup (\{i+1, \ldots, m\} \times \{1, \ldots, j\}). \tag{11}$$

If (i,j) is the bottom right position of the SCDR of the NW-component of F then, clearly, $\widetilde{h}_i > 0$, $\widetilde{v}_j > 0$, and $\widetilde{a}_{i+j-1} > 0$. Moreover, $F \cap E = \emptyset$, and so

$$\begin{aligned}\widetilde{h}_i = \sum_{t=1}^{i} h_t &= |F \cap \{1, \ldots, i\} \times \{1, \ldots, n\}| = |F \cap \{1, \ldots, i\} \\ &\quad \times \{1, \ldots, j\}| \\ &= |F \cap \{1, \ldots, m\} \times \{1, \ldots, j\}| = \sum_{t=1}^{j} v_t = \widetilde{v}_j.\end{aligned} \tag{12}$$

Furthermore, $(F \cap A_k) \cap E \subseteq F \cap E = \emptyset$ for every $k = 1, \ldots, m+n-1$. Then, recalling that $a_k = |F \cap A_k|$ we get that

$$\begin{aligned}\widetilde{a}_{i+j-1} &= \sum_{k=1}^{i+j-1} |F \cap A_k| = |F \cap \{1, \ldots, i\} \times \{1, \ldots, j\}| \\ &= \widetilde{h}_i = \widetilde{v}_j.\end{aligned} \tag{13}$$

Moreover, $A_{i+j} \subseteq E$ implies $F \cap A_{i+j} \subseteq F \cap E = \emptyset$ and we get that $a_{i+j} = |F \cap A_{i+j}| \leqslant |F \cap E| = 0$. Finally, assume that an integer $i' < i$ exists for which an integer j' exists such that $\widetilde{h}_{i'} = \widetilde{v}_{j'} = \widetilde{a}_{i'+j'-1} > 0$ and $a_{i'+j'} = 0$. Clearly, in this case $j' < j$. Since (i,j) is the bottom right position of the SCDR of the NW-component and every component is a polyomino we get that the 1st, $\ldots$, $(i+j)$th coordinates of the antidiagonal projection have to be of the form $(0, \ldots, 0, a_{k_1}, \ldots, a_{k_2}, 0, \ldots, 0)$, where $1 \leqslant k_1 \leqslant k_2 < i+j$ and $a_l \neq 0$ for every $k_1 \leqslant l \leqslant k_2$. But then $a_{i'+j'} = 0$ only if $i'+j' < k_1$ or $i'+j' > k_2$. If $i'+j' < k_1$ then $\widetilde{a}_{i'+j'-1} = 0$ is a contradiction. Otherwise, i.e., if $i'+j' > k_2$ then $\widetilde{h}_i \geqslant \widetilde{h}_{i'}$ since $i > i'$. If $\widetilde{h}_i > \widetilde{h}_{i'}$ then $\widetilde{a}_{i'+j'-1} = \widetilde{a}_{i+j-1} = \widetilde{h}_i > \widetilde{h}_{i'}$ is a contradiction. If $\widetilde{h}_i = \widetilde{h}_{i'}$ then $h_i = 0$ which contradicts the assumption that (i,j) is the bottom right position of the SCDR of the NW-component of F. □

The NE-, SE-, and SW-versions of Lemma 4 can be given analogously. Unfortunately, this lemma does not give a sufficient condition for identifying the SCDR of a component of F. For example, if the discrete set F is represented by $\hat{F}_{5\times 3}$ where $\hat{f}_{ij} = 1$ if and only if $(i,j) \in \{(1,3), (2,2),$

$(5,1)\}$ then for the position $(2,2)$ the conditions of Lemma 4 hold (since $\widetilde{h}_2 = \widetilde{v}_2 = \widetilde{a}_3 = 2$ and $a_4 = 0$) but F has no NW-component at all. With the aid of the following theorem it is possible to test whether the decomposable discrete set has a NW- or SE-component.

Theorem 5. [NW-version] *Let $F \in \mathcal{DEC}$, $\mathcal{H}(F) = (h_1, \ldots, h_m)$, $\mathcal{V}(F) = (v_1, \ldots, v_n)$, and $\mathcal{A}(F) = (a_1, \ldots, a_{m+n-1})$. If (i,j) is a position such that i is the smallest integer for which there exists an integer j with $\widetilde{h}_i = \widetilde{v}_j = \widetilde{a}_{i+j-1}$ and $a_{i+j} = 0$ and a polyomino P exists according to the a priori information which guarantees that property (α) is satisfied such that $\mathcal{H}(P) = (h_1, \ldots, h_i)$, $\mathcal{V}(P) = (v_1, \ldots, v_j)$, and $\mathcal{A}(P) = (a_1, \ldots, a_{i+j-1})$ then $F \cap E = \emptyset$ where E is defined by (11) and P is the NW-component of F or/and F has a SE-component. If no such position exists then F has no NW-component.*

Proof. Define a set by $T = \bigcup_{k=1}^{i+j-1} A_k$ and let $Q' = F \cap T$. We have $\mathcal{A}(Q') = \mathcal{A}(P)$ so the sum of the terms of $\mathcal{H}(Q')$ are equal to the sum of the terms in $\mathcal{H}(P)$. Moreover, $\mathcal{H}(Q') \leqslant \mathcal{H}(P)$, so $\mathcal{H}(Q') = \mathcal{H}(P)$. Similarly, $\mathcal{V}(Q') = \mathcal{V}(P)$. Then, $Q' \subseteq \{1, \ldots, i\} \times \{1, \ldots, j\}$ therefore $F \cap E = \emptyset$ where E is defined by (11). Then, F has neither NE- nor SW-components, i.e., it has a NW- or a SE-component. If F has a NW-component then it is the polyomino P based on Lemma 4 and property (α). Then, the first part of the theorem is proven. The second part of the theorem follows from the fact that the position that satisfies the necessary conditions of Lemma 4 is uniquely determined. □

Similar theorems can be given for testing the existence of a NE-, SE-, and SW-component. Note, that if the conditions of the above theorem hold then in some cases the discrete set can have both NW- and SE-components, e.g., if the discrete set belongs to $\mathcal{S}^*$. Although Theorem 5 excludes the existence of NE- and SW-components it does not state that the discrete set has necessarily a NW-component (see, e.g., Fig. 4). However, if the set is of $\mathcal{S}^*/\mathcal{S}^{**}$ then with the aid of the NW/NE-version of Theorem 5 it is possible to find the SCDR of the NW/NE-component of F, respectively. On the basis of the following theorem one can find the NW-component of F (if exists) if $F \in \mathcal{DEC} \setminus (\mathcal{S}^* \cup \mathcal{S}^{**})$, too.

Theorem 6. [NW-version] *Assume that $F \in \mathcal{DEC} \setminus (\mathcal{S}^* \cup \mathcal{S}^{**})$ and (i,j) satisfies the conditions of the NW-version of Theorem 5 with a polyomino P. Moreover, assume that the SCDR of $C(F)$ is $\{i_1, \ldots, i_2\} \times \{j_1, \ldots, j_2\}$. Then, P is the NW-component of F if and only if there exists $i' \in \{i_1, \ldots, i_2\}$ such that $i < i'$ or there exists $j' \in \{j_1, \ldots, j_2\}$ such that $j < j'$.*

Proof. Assume that $C(F) \in \mathcal{S}^*$. Since the NW-version of Theorem 5 holds F cannot have a NE- or SW-component. If F has a NW-component then the gluing sequence of F is of the form $F^{(1)}, \ldots, F^{(l)} = C(F)$, $F^{(l+1)}, \ldots, F^{(k)}$ where $F^{(l+1)}$ must exist (since $F \notin \mathcal{S}^*$) and we get it from $C(F)$ by a NE- or SW-gluing. Clearly, in both cases for the bottom right position (i,j) of the SCDR of the NW-component of F $i < i_1$ and $j < j_1$ must hold (clearly, $i < i'$ and $j < j'$ for any $i' \in \{i_1, \ldots, i_2\}$ and $j' \in \{j_1, \ldots, j_2\}$ is also true). On the other hand, if $i < i'$ for an arbitrary $i' \in \{i_1, \ldots, i_2\}$ or $j < j'$ for an arbitrary $j' \in \{j_1, \ldots, j_2\}$ then the centre is on the southeast of (i,j) and $F \cap E = \emptyset$ where E is defined by (11) (since (i,j) satisfies the conditions of Theorem 5). So F must have a NW-component which so must be P by Theorem 5. The proof is similar if $C(F) \in \mathcal{S}^{**}$. □

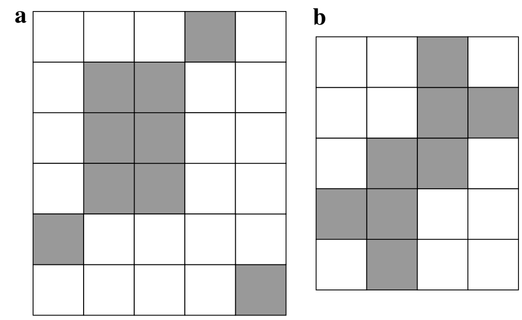

Fig. 4. (a) A decomposable discrete set which has no NW-component although the position $(5,4)$ satisfies the conditions of Theorem 5 with the polyomino in (b).

The NE-, SE-, and SW-versions of the above theorem can be given similarly. Now, an algorithm can be outlined for reconstructing decomposable discrete sets with given horizontal, vertical, diagonal, and antidiagonal projections. We first describe a procedure for decomposing a NW-component.

Algorithm 1. Procedure DecomposeNW

Step 1: find the position (i,j) for which $\widetilde{h}_i = \widetilde{v}_j = \widetilde{a}_{i+j-1} > 0$ and $a_{i+j} = 0$;
if no such position exists **then return** (no component);
Step 2: **if** $(t = m$ and $i \geqslant l > 0)$ or $(t = n$ and $j \geqslant l > 0)$ **then return** (no component);
Step 3: construct a polyomino P according to the prior information with $\mathcal{H}(P) = (h_1, \ldots, h_i)$, $\mathcal{V}(P) = (v_1, \ldots, v_j)$, and $\mathcal{A}(P) = (a_1, \ldots, a_{i+j-1})$;
if no such polyomino exists **then return** (no component);
Step 4: $F := F \cup P$;
Step 5: update H, V, D, and A according to the projections of P;
Step 6: **return**;

The procedures DecomposeNE/DecomposeSE/DecomposeSW for decomposing NE-/SE-/SW-components, respectively, can be outlined similarly. The main algorithm for reconstructing decomposable discrete sets gets four vectors $H \in \mathbb{N}_0^m$, $V \in \mathbb{N}_0^n$, $D \in \mathbb{N}_0^{m+n-1}$, and $A \in \mathbb{N}_0^{m+n-1}$ as input and outputs the set S containing all the decomposable discrete sets with the projections H, V, D, and A. This algorithm reconstructs the solutions component by component calling the procedures for decomposition. The algorithm can be outlined as follows.

Algorithm 2

For reconstructing sets of $\mathcal{DEC}$ from four projections

Step 1: $t := \min\{m,n\}$; $l := -1$; $S = \emptyset$;
Step 2: $F := \emptyset$; $l := l + 1$; **restore** H,V,D, and A;
Step 3: **repeat**
  call DecomposeNW;
  if (no component) **then** call DecomposeNE;
  if (no component) **then** call DecomposeSE;
  if (no component) **then** call DecomposeSW;
  until (no component);
Step 4: try to reconstruct the last component;
Step 5: **if** $D = 0$ **and** $A = 0$ **then** $S := S \cup \{F\}$;
Step 6: **if** $l = t$ **then return** S **else goto** Step 2;

This algorithm works as follows. First, we set $S := \emptyset$ and determine whether the horizontal or the vertical size of the set G to be reconstructed is smaller (Step 1). Then, in Step 2 we define $F := \emptyset$ and we restore the vectors H, V, D, and A to their original values (the importance of this will be explained later). In Step 3 we first try to decompose a NW-component. This is done by Algorithm 1 which seeks the SCDR of the NW-component of G by trying to find the uniquely determined position which satisfies the conditions of Theorem 5 with a polyomino P (Steps 1 and 3 of Algorithm 1). If no such position exists then on the basis of Theorem 5 the assumption that the set to be reconstructed has a NW-component was false and the procedure simply returns. Otherwise, the polyomino P is assumed to be the NW-component of G, we simulate the effect of this component on the projections of G (Steps 4 and 5 of Algorithm 1) and we return to the main algorithm. If we were not able to find a NW-component then we try to decompose a NE-, SE-, and SW-component, similarly, in this order. If we found a component (which can be any of the four kinds) then we go on and try to decompose further components from the remaining set. We repeat this until we cannot decompose a component. In this case the remaining set is undecomposable therefore it must consist of a single component. We try to reconstruct this last component in Step 4. Then, in Step 5 we check whether the reconstructed set F has the given projections, and if so then we set $S := S \cup \{F\}$. There can be two cases when the reconstructed set F does not have the given projections H, V, D, and A.

- There is no decomposable discrete set with these four projections at all.
- During the decomposition we assumed that the reconstructed polyomino P is the component (say, the NW-component) of G although G does not have a NW-component which can occur because Theorem 5 does not give a sufficient condition. In this case we can call Theorem 6 for help. This theorem states that the position (i,j) found in Step 1 of Algorithm 1 must be to the northwest of the SCDR $\{i',\ldots,i''\} \times \{j',\ldots,j''\}$ of $C(G)$. The only problem is that $C(G)$ is not known in advance. Therefore we first assume that the first column of G is also a column of the SCDR of $C(G)$, i.e., $1 \in \{j',\ldots,j''\}$. With this assumption we can repeat Step 3 of Algorithm 2. In this case in Procedure DecomposeNW we will omit every position (i,j) that violates the relation $j < 1$ (Step 2 of Algorithm 1). This can be done similarly if we try to decompose a NE-, SE-, or SW-component. Then, we go on by assuming that the second column of G is also a column of the SCDR of $C(G)$, i.e., $2 \in \{j',\ldots,j''\}$. This time we will omit every position (i,j) in Step 2 of Algorithm 1 that violates $j < 2$. Using the variable l we repeat this (in the lth iteration assuming that $l \in \{j',\ldots,j''\}$) until we reach the last column, i.e., $l = n$. Even if the part of the discrete set which is not still reconstructed is an element of $\mathcal{S}^* \cup \mathcal{S}^{**}$ we can continue to impose condition using the variable l (normally imposed by Theorem 6) because there is always a component which does not contain the lth column. In each iteration we start the reconstruction from blank using the original vectors H, V, D, and A (see Step 2) and we put the reconstructed set F into S if it is a different solution of Problem 1. On the basis of Theorem 6 this strategy can also be applied to the rows of the SCDR of $C(F)$. If $m < n$ then we choose the latter version of our strategy (this decision is made in Step 1).

Turning to the analysis of the algorithm we can say the following.

Theorem 7. *If no a priori information is needed to satisfy property* (α) *then Algorithm 2 solves Problem 1 in the class* $\mathcal{DEC}$ *in polynomial time. If the reconstructed set consists of components* $F_1,\ldots,F_k$ *and the time complexity of reconstructing the ith component* F_i *is of* $O(f_i)$ $(i = 1,\ldots,k)$ *then the worst case time complexity of the algorithm is of* $O(\min\{m,n\} \cdot \max_i f_i)$. *The algorithm finds all sets of* $\mathcal{DEC}$ *with the given projections.*

Proof. Let F be an arbitrary set of S. From the discussion in the previous paragraph it follows that F is decomposable. Due to Step 3 of Algorithm 1 the horizontal and vertical projections of F are equal to the given vectors H and V, respectively. Moreover, Step 5 of Algorithm 2 guarantees that the diagonal and antidiagonal projections of F are also equal to the vectors D and A, respectively. Assuming that the lth $(l = 1,\ldots,k)$ component to be reconstructed is a NW-component it takes $O(m + n)$ time to find the (uniquely determined) position which satisfies the necessary conditions of Lemma 4. We do it simply by scanning the vectors $\widetilde{H}$, $\widetilde{V}$, and $\widetilde{A}$. In order to test whether this position is the bottom right position of the SCDR of the NW-component we try to reconstruct this component based on Theorem 5 which takes $O(f_l)$ time. The same is true if the lth component is a NE-, SE- or SW-component. In the worst case the component is a SW-component, i.e., we try to reconstruct the lth component at most four times and so the time complexity of reconstructing all the components (Steps 3 and 4 of Algorithm 2) is of $O(\max_i f_i)$ which is polynomial on the basis of property (α). However, since Theorem 5 does not give a sufficient condition we iterate Steps 3 and 4 of Algorithm 2 by assuming that the first, second,…,nth column of G is also a column of $C(G)$ (if $n < m$) or by assuming that the first, second,…,mth row of G is also a row of $C(G)$ (if $m \leqslant n$). This means that Steps 3 and 4 of Algorithm 2 must be repeated $\min\{m,n\}$ times. So we get that the total execution time of Algorithm 2 is of $O(\min\{m,n\} \cdot \max_i f_i)$. Since we check every column (row) of F whether it is a column (row) of $C(F)$ it follows that the algorithm reconstructs all sets of $\mathcal{DEC}$ with the given projections. Consequently, if the algorithm returns $S = \emptyset$ then there is no solution of Problem 1 in the class $\mathcal{DEC}$. □

As it was mentioned earlier property (α) can usually be guaranteed only by using some a priori information about the components. This prior knowledge is incorporated into Theorem 5 and Step 3 of Algorithm 1 thus we can state the following consequence.

Corollary 8. *If the components of a discrete set are restricted to belong to a certain class C of sets in order to satisfy property (α) then Algorithm 2 solves Problem 1 in the class of discrete sets which satisfy properties (β) and (γ) and their components are from the class C in polynomial time. If reconstructing an element of class C takes O(f) time then the algorithm finds all solutions in* $O(f \cdot \min\{m,n\})$ *time.*

Fig. 5 shows the first two iterations of the algorithm reconstructing the discrete set in Fig. 4a from the vectors $H = (1,2,2,2,1,1)$, $V = (1,3,3,1,1)$, $D = (0,1,0,1,2,3,1,0,1,0)$, and $A = (0,0,1,3,3,1,0,0,0,1)$. The prior information used this time is that the components are rectangles or of the form of Fig. 4b. First, the algorithm seeks a position (i,j) that satisties $\tilde{h}_i = \tilde{v}_j = \tilde{a}_{i+j-1} > 0$ and $a_{i+j} = 0$ (Fig. 5a) and tries to reconstruct a polyomino in the discrete rectangle defined by the positions $(1,1)$ and (i,j) (Fig. 5b). Reconstructing the last component (Fig. 5c) the resulting set does not have the given projections therefore we go on to the next iteration. Since $m > n$ we will follow the strategy for finding a column of the centre of the set to be reconstructed. We set $l = 1$, i.e., the first column is assumed to be a column of the centre of the set to be reconstructed. On the basis of Theorem 6 $(5,4)$ cannot be the bottom right position of SCDR of the NW-component (Fig. 5d). Moreover, there is no position which satisfies the conditions of

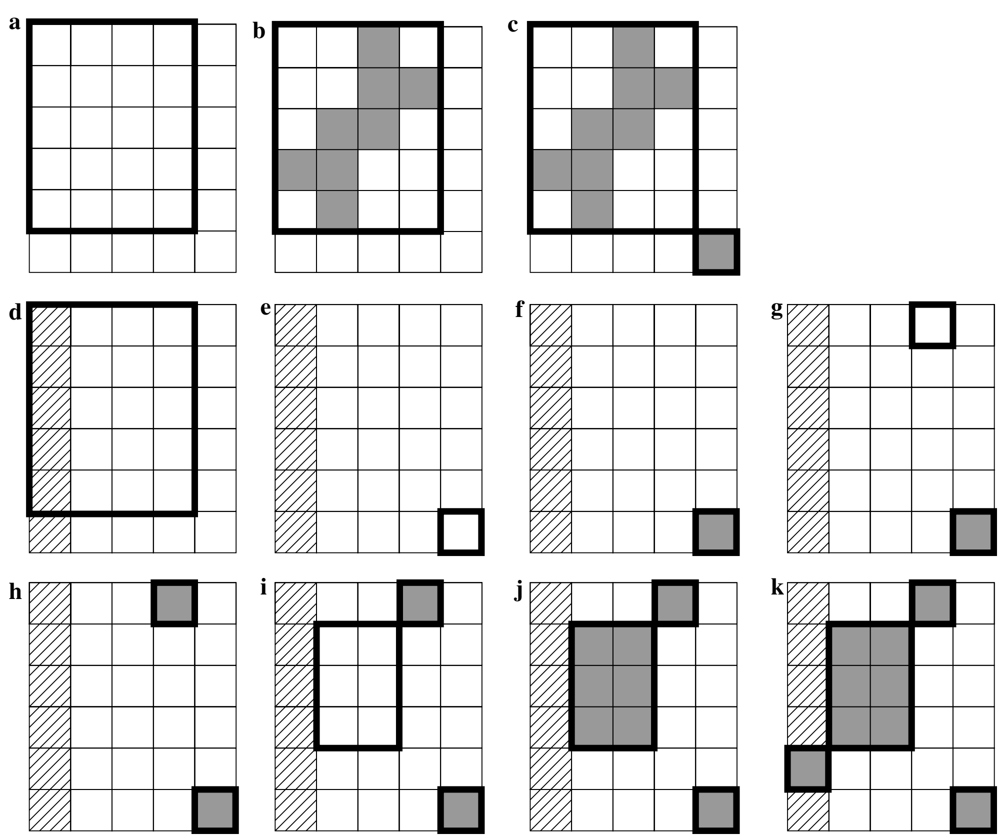

Fig. 5. An example how Algorithm 2 works. The identified discrete rectangles which must contain the next component are marked with bold squares. The column assumed to be a column of $C(F)$ is filled with slanted lines.

the NE-version of Theorem 5. However, there is a position satisfying the conditions of the SW-version of Theorem 5 (Fig. 5e) with a polyomino (Fig. 5f). In the next step the conditions of the NE-version of Theorem 5 are satisfied (Figs. 5g and h). Then, for the remaining set, again, the conditions of the NE-version of Theorem 5 are satisfied (Figs. 5i and j). Finally, we reconstruct the last component (Fig. 5k). Since the reconstructed set F has the given projections we set $S = \{F\}$. The reader can check easily that in this case further iterations do not give different solutions. Note, however, that for some certain inputs there can be several solutions (see, e.g., Fig. 6).

4. Connection with the Q-Convexes

Q-convexity was introduced in [18] as an intermediate property between line convexity and convexity. The class of Q-convexes is one of the most general classes for which a polynomial-time reconstruction algorithm is known. In this section we show that decomposability is a more general property than Q-convexity along the horizontal and vertical directions if the discrete set consists of at least two components. For any point $P = (p_1, p_2) \in \mathbb{Z}^2$ we define the four quadrants around P by

$$R_0(P) = \{Q = (q_1, q_2) \mid q_1 \leqslant p_1 \text{ and } q_2 \leqslant p_2\},$$
$$R_1(P) = \{Q = (q_1, q_2) \mid q_1 \geqslant p_1 \text{ and } q_2 \leqslant p_2\},$$
$$R_2(P) = \{Q = (q_1, q_2) \mid q_1 \geqslant p_1 \text{ and } q_2 \geqslant p_2\},$$
$$R_3(P) = \{Q = (q_1, q_2) \mid q_1 \leqslant p_1 \text{ and } q_2 \geqslant p_2\}.$$

A discrete set F is *Q-convex* if for any point $P \in \mathbb{Z}^2$ $R_k(P) \cap F \neq \emptyset$ for all $k \in \{0,1,2,3\}$ implies $P \in F$ (see Fig. 7). We will denote the class of non-4-connected Q-convexes by $\mathcal{Q}'$.

Remark 9. *More precisely, the above definition corresponds to the Q-convexity along the horizontal and vertical directions as Q-convexity can be defined in a more general manner*

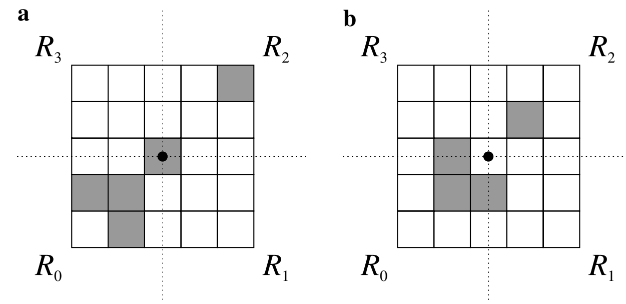

Fig. 7. (a) A discrete set that is Q-convex and (b) a decomposable discrete set that is non-Q-convex.

(see [18]). However, for the sake of simplicity we will use this abbreviated form.

Let F be a non-4-connected Q-convex set having components $F_1, \ldots, F_k$ such that $\{i_l, \ldots, i'_l\} \times \{j_l, \ldots, j'_l\}$ is the SCDR of the lth ($l = 1, \ldots, k$) component of F. Every Q-convex set is *hv*-convex, too, therefore the sets of the row/column indices of the components of F consist of consecutive indices and they are disjoint. Without loss of generality we can assume that

$$1 = i_1 \leqslant i'_1 < i_2 \leqslant i'_2 < \ldots \leqslant i'_k = m. \quad (14)$$

The following lemma shows that the SCDRs of the components of F can be arranged in only two ways.

Lemma 10. *Let $F \in \mathcal{Q}'$ having components $F_1, \ldots, F_k$ ($k \geqslant 2$) with the SCDRs $\{i_l, \ldots, i'_l\} \times \{j_l, \ldots, j'_l\}$ ($l = 1, \ldots, k$) such that (14) holds. Then, exactly one of the following cases is possible*

(1) $1 = j_1 \leqslant j'_1 < j_2 \leqslant j'_2 < \ldots \leqslant j'_k = n$, or
(2) $n = j_1 \geqslant j'_1 > j_2 \geqslant j'_2 > \ldots \geqslant j'_k = 1$.

Proof. Clearly, both Cases (1) and (2) cannot hold together. For $k = 2$ the statement is trivial. Assume that $k > 2$ and neither Case (1) nor Case (2) is satisfied. Then, there exist three components, say F_{l_1}, F_{l_2}, and F_{l_3} such that $i_{l_1} < i_{l_2} < i_{l_3}$ and exactly one of the following relations hold

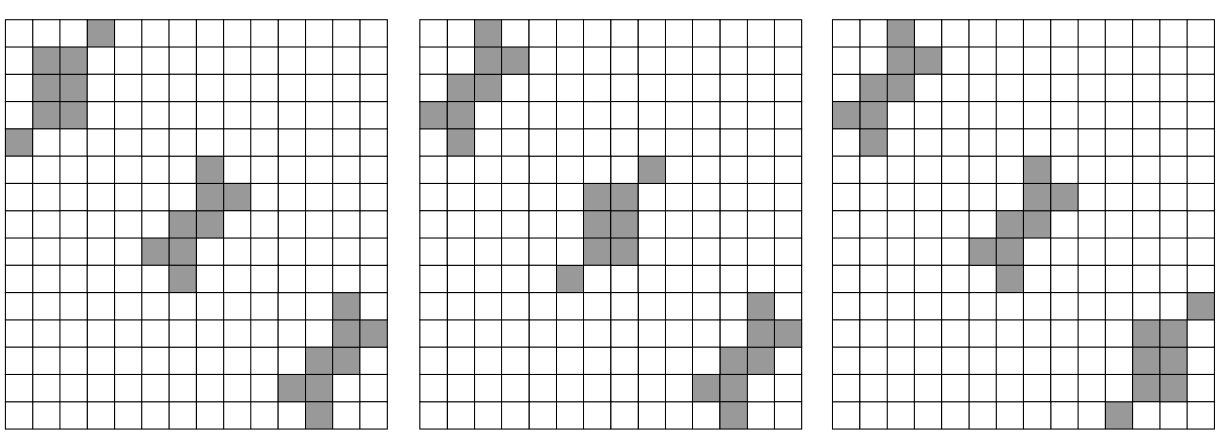

Fig. 6. Three different decomposable discrete sets with the same projections.

(a) $j_{l_2} < j_{l_1} < j_{l_3}$, or
(b) $j_{l_3} < j_{l_1} < j_{l_2}$, or
(c) $j_{l_1} < j_{l_3} < j_{l_2}$, or
(d) $j_{l_2} < j_{l_3} < j_{l_1}$.

Assume that Case (a) is true. Then, for every position $M \in \{i_2, \ldots, i'_2\} \times \{j_1, \ldots, j'_1\}$ $M \notin F$ holds since F is also *hv*-convex. However, $R_k(M) \cap F \neq \emptyset$ for all $k \in \{0,1,2,3\}$ which contradicts the Q-convexity. The proof is similar in the Cases (b), (c), and (d). □

We say that an *hv*-convex polyomino P with the SCDR $\{i, \ldots, i'\} \times \{j, \ldots, j'\}$ is *NW/NE/SE/SW-directed* if $(i', j') \in P/(i', j) \in P/(i, j) \in P/(i, j') \in P$, respectively. The following lemma is about the directedness of the components of an arbitrary set $F \in \mathcal{Q}'$.

Lemma 11. *Let $F \in \mathcal{Q}'$ having components $F_1, \ldots, F_k (k \geqslant 2)$ with the SCDRs $\{i_l, \ldots, i'_l\} \times \{j_l, \ldots, j'_l\}$ $(l = 1, \ldots, k)$ such that (14) holds. If for F Case (1)/(2) of Lemma 10 is satisfied then $F_1, \ldots, F_{k-1}$ are NW/NE-directed and $F_2, \ldots, F_k$ are SE/SW-directed, respectively.*

Proof. Assume that the for the set $F \in \mathcal{Q}'$ Case (1) of Lemma 10 is satisfied and let M denote the bottom right position of the SCDR of F_1. If F_1 is not NW-directed then $M \neq F$. However, $R_k(M) \cap F \neq \emptyset$ for all $k \in \{0,1,2,3\}$ which contradicts the Q-convexity. In the same way we get that $F_2, \ldots, F_{k-1}$ are also NW-directed. The SW-directedness of the components $F_2, \ldots, F_k$ can be proven analogously by investigating the top left positions of the SCDRs of the components. The proof if similar if for F Case (2) of Lemma 10 holds. □

Now, we can prove that every Q-convex set consisting of at least two components is decomposable, too. In fact, a somewhat stronger statement can also be formulated.

Theorem 12

$\mathcal{Q}' \subset \mathcal{S}^* \cup \mathcal{S}^{**}$.

Proof. Let $F \in \mathcal{Q}'$ having components $F_1, \ldots, F_k (k \geqslant 2)$ such that (14) holds. Since F is non-4-connected it has at least two components. Clearly, the components are *hv*-convex and on the basis of Lemma 11 they are also directed therefore they can be reconstructed uniquely from the horizontal and vertical projections in O(mn) time (see Theorem 3 in [19]), i.e., property (α) is satisfied by these informations. Furthermore, on the basis of Lemma 10 the configuration of the components can follow only two cases. In both cases the sequence $F^{(i)} = \bigcup_{l=1}^{i} F_l (i = 1, \ldots, k)$ is a gluing sequence of F therefore $F \in \mathcal{DEC}$ based on Lemma 2. If Case (1) of Lemma 10 holds then we use only SE-gluings to build the set F. Otherwise, i.e., if Case (2) of Lemma 10 is true, we use only SW-gluings to build F. Then, $F \in \mathcal{S}^* \cup \mathcal{S}^{**}$. For $\mathcal{Q}' \neq \mathcal{S}^* \cup \mathcal{S}^{**}$ consider, e.g., Fig. 7b. □

Corollary 13. *Problem 1 can be solved in the class $\mathcal{Q}'$ in O(mn) time. The solution is uniquely determined.*

Proof. Let $F \in \mathcal{Q}'$. From Lemma 11 we know that the components of F belong to the class C of directed *hv*-convex polyominoes. Moreover, reconstructing an element of C takes O(mn) time (Theorem 3 in [19]). On the basis of Theorem 12 $F \in \mathcal{S}^* \cup \mathcal{S}^{**}$. If $F \in \mathcal{S}^* / F \in \mathcal{S}^{**}$ then in each time calling the procedures for decomposition in Step 3 of Algorithm 2 we can decompose a NW/NE-component, respectively. Then, Algorithm 2 reconstructs the solution in the first iteration (i.e., if $l = 0$) and this solution is uniquely determined. Since no more iterations are needed the factor $\min\{m, n\}$ can be eliminated from the formula of the reconstruction complexity given in Corollary 8. Then, the theorem follows. □

Remark 14. *Independently from the author's work, same results as stated in Lemmas 10, 11 and Theorem 12 have been enounced on page 408 of [20] (c.f. [21]).*

The class of *hv*-convex 8- but not 4-connected discrete sets (denoted by $\mathcal{S}'_8$) was studied in [13]. In this paper the authors gave a reconstruction algorithm for this class using four projections. The worst case time complexity of this algorithm is of $O(mn)$ and the solution is uniquely determined. In the following we show that this result is strongly related with the results given above.

Corollary 15. *Problem 1 can be solved in the class $\mathcal{S}'_8$ in $O(mn)$ time. The solution is uniquely determined.*

Proof. Sets of $\mathcal{S}'_8$ have the same properties as sets of $\mathcal{Q}'$ (see [13]). The only difference is that the SCDRs of the components in the class $\mathcal{Q}'$ might be separated, i.e., there can be empty rows or columns between two consecutive components while in $\mathcal{S}'_8$ the SCDRs of the components are always connected. Therefore $\mathcal{S}'_8 \subset \mathcal{Q}'$. Since empty rows and columns do not have an effect on the complexity of Algorithm 2 the statement follows from Corollary 13. □

5. Reconstruction of decomposable *hv*-convex discrete sets

The class of *hv*-convex discrete sets (in the following denoted by $\mathcal{HV}$) is a frequently studied class in discrete tomography. In [10] an algorithm is published for reconstructing sets of $\mathcal{HV}$ from two projections. However, as it turned out later the reconstruction problem in this class is NP-hard [4]. In [22] an evolutionary algorithm is suggested for solving reconstruction problems in several difficult classes. In the same paper the author also analyzes the performance of the algorithm on *hv*-convex sets. The following corollary shows that with the aid of the decomposition technique described in Section 3 it is possible to solve the reconstruction problem in the class of decomposable *hv*-convex discrete sets in polynomial time.

Corollary 16. *Algorithm 2 solves Problem 1 in the class $\mathcal{HV} \cap \mathcal{DEC}$ in $O(mn \cdot \min\{m^3, n^3\})$ time. The algorithm finds all sets of $\mathcal{HV} \cap \mathcal{DEC}$ with the given projections.*

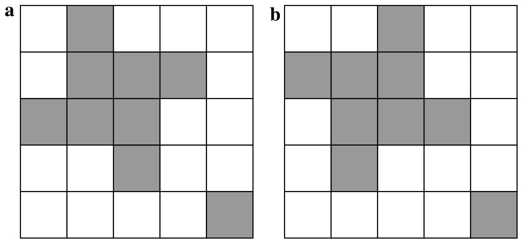

Fig. 8. (a) An *hv*-convex discrete set which is possibly reconstructible by Algorithm 2 and (b) an *hv*-convex discrete set with the same projections but with different components showing that the set in (a) does not satisfies property (α).

Proof. We know that the components of the set F to be reconstructed are uniquely determined by their horizontal and vertical projections (otherwise F would not belong to the class $\mathscr{D}\mathscr{E}\mathscr{C}$). Clearly, they are *hv*-convex, too. Reconstructing a member of the class of *hv*-convex polyominoes from its horizontal and vertical projections takes $O(mn \cdot \min\{m^2, n^2\})$ time (see [5]). Certainly, the same is true if we have the prior information that the polyomino is uniquely determined by these projections. Then, the theorem follows by applying Corollary 8 to the class C of uniquely determined *hv*-convex polyominoes. □

Corollary 16 states that if the *hv*-convex discrete set is decomposable then it can be reconstructed in polynomial time. In this case property (α) was guaranteed simply by assuming that the set is decomposable, i.e., finding a component implies that there is no *hv*-convex polyomino with the same horizontal and vertical projections. Unfortunately, Algorithm 2 is not appropriate to solve the corresponding consistency problem, i.e., to decide whether a set of $\mathscr{H}\mathscr{V} \cap \mathscr{D}\mathscr{E}\mathscr{C}$ exists with the given projections. It can happen that the algorithm reconstructs an *hv*-convex set with the given projections whereas the assumption that the components are uniquely determined is wrong and so property (α) is not satisfied. Clearly, in this case the reconstructed set is not in $\mathscr{H}\mathscr{V} \cap \mathscr{D}\mathscr{E}\mathscr{C}$. It is because in Step 3 of the procedures for decomposition we can construct a polyomino only by using the general algorithm for reconstructing an *hv*-convex polyomino (see [5] for the algorithm). This algorithm reconstructs a polyomino from the horizontal and vertical projections but it is not guaranteed that the reconstructed polyomino is the only one with these projections. In some cases it does not mean that our algorithm fails (see Fig. 8). However, this drawback of the algorithm turns to be an advantage if we want to reconstruct *hv*-convex sets from four projections since Algorithm 2 in certain cases can reconstruct *hv*-convex discrete sets in a somewhat broader class than the decomposables.

6. Conclusions and discussion

In this paper we presented a structural property of discrete sets, the decomposability and studied uniqueness and reconstruction problems in classes of discrete sets having this property. In general, for decomposable discrete sets a polynomial time reconstruction algorithm is given. Then, it is proven that every Q-convex set which consists of at least two components is decomposable. As a consequence we got that Problem 1 in the class $\mathscr{Q}'$ can be solved uniquely in $O(mn)$ time. Since $\mathscr{S}'_8 \subset \mathscr{Q}'$ the reconstruction of *hv*-convex 8- but not 4-connected sets from four projections can also be solved uniquely in $O(mn)$ time. The complexity of our algorithm strongly depends on the assumption that the components are uniquely determined by the horizontal and vertical projections therefore it seems to be important to find classes of discrete sets where the reconstruction problem can be solved uniquely from these two projections.

In this contribution we concentrated on discrete sets consisting of several components which satisfy properties (α), (β), and (γ). More work has to be done on the field whether assuming weaker properties about the components the reconstruction process remains tractable. Further work in this field can lead us towards designing efficient reconstruction algorithms for important classes like the one of *hv*-convex sets. As a first step we showed that the reconstruction of *hv*-convex decomposable discrete sets can be solved in polynomial time.

In this paper we used diagonal and antidiagonal projections for decomposing the components of a discrete set. Notice, that if the set belongs to the class $\mathscr{S}^*/\mathscr{S}^{**}$ then the use of the antidiagonal/diagonal projection, respectively, is sufficient to decompose the set into components, i.e., if the type of the set is known in advance then three projections are sufficient for the reconstruction. Moreover, we also showed that in both classes the reconstruction is faster than in the general class of decomposables. In more general, without further technical details we mention here that instead of the diagonal/antidiagonal projection one can use any projection with angle δ for the decomposition where $(90^\circ < \delta < 180^\circ)/(0^\circ < \delta < 90^\circ)$, respectively.

It is shown that in some cases the discrete set can be decomposed along the diagonal and antidiagonal projections to facilitate the reconstruction. However, in some cases the decomposition into components is impossible. For example, the configuration in Fig. 2e can be decomposed into two parts (one containing F_1 and F_2, and the other containing F_3 and F_4) by the antidiagonal projection but then, the two parts cannot be further decomposed into components since the diagonal projections of the two parts are not independent. In some unfortunate cases the components cannot be separated at all (see, e.g., Figs. 2g and h). Further investigation is needed to answer the question whether the decomposition technique can somehow be applied to certain undecomposable configurations.

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