ANOTHER PROOF OF THE MARKOV—POST THEOREM*

By

LÁSZLÓ KALMÁR (Szeged), corresponding member of the Academy

The word problem for associative systems (i. e. semigroups without postulating the cancellation law) has been proved to be unsolvable by any (finite, general recursive) algorithm independently and simultaneously by POST [1] and MARKOV.[2] MARKOV's proof is simpler than that of POST's; [3] moreover, MARKOV obtained by means of his method some further results about the non-existence of some algorithms in the theory of associative systems.[4]

In the present paper, I shall give another proof for the non-existence of an algorithm to the effect of solving the word problem for associative systems. My proof seems to show some advantages over both POST's and

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[1] E. L. POST, Recursive unsolvability of a problem of Thue, *The Journal of Symbolic Logic,* **12** (1947), pp. 1—11.

[2] А. Марков, Невозможность некоторых алгорифмов в теории ассоциативных систем, Доклады Академии Наук СССР, **55** (1947), pp. 587—590.

[3] Indeed, POST relies besides (1) on a theorem of CHURCH's stating the non-existence of a decision algorithm for λ-convertability (see A. CHURCH, An unsolvable problem of elementary number theory, *American Journal of Math.,* **58** (1936), pp. 345—363), (2) on ROSSER's combinatorial equivalent of the calculus of λ-conversion (see J. B. ROSSER, A mathematical logic without variables, *Annals of Math.,* (2) **36** (1935), pp. 127—150 and *Duke Math. Journal,* **1** (1935), pp. 328—355), (3) on POST's transformation method of logical systems in canonical form to those in normal form (see E. L. POST, Formal reductions of the general combinatorial decision problem, *American Journal of Math.,* **65** (1943), pp. 197—215), which have been used also by MARKOV, (4) on POST's (very easy) transformation method of logical systems in normal form to those in normal form and containing but two primitive letters (see E. L. POST, Recursively enumerable sets of positive integers and their decision problems, *Bulletin of the American Math. Society,* **50** (1944), pp. 284—316, especially footnote [5]), and (5) on TURING's theory of computing machines (see A. M. TURING, On computable numbers, with an application to the Entscheidungsproblem, *Proceedings of the London Math. Society,* (2) **42** (1937), pp. 230—265 and **43** (1937), pp. 544—546).

[4] See, besides loc. cit. [2], А. Марков, Невозможность некоторых алгорифмов в теории ассоциативных систем II, Доклады Академии Наук СССР, **58** (1947), pp. 353—356, as well as А. Марков, Невозможность некоторых алгоритмов в теории ассоциативных систем, Доклады Академии Наук СССР, **77** (1951), pp. 19—20.

MARKOV's proofs especially for readers who are not acquainted with CHURCH's theory of λ-convertability[5] nor with POST's reduction theory of logical systems.[6]

1. An *associative system* is a set in which an associative binary operation, called *multiplication*, has been defined. We denote the result of this operation, applied to the elements a and b, by ab; we call it the *product* of a and b. On account of the associative law $(ab)c=a(bc)$, a product $a_1a_2\ldots a_l$ of several factors has an obvious sense (a_1 for $l=1$). We confine ourselves to associative systems containing a unit, i. e. an element e for which we have $ae=ea=a$ for every element a of the system. For $l=0$, we define the "empty product" $a_1a_2\ldots a_l$ to denote e.

An equation of the form

$$(1)\qquad a_{i_1}a_{i_2}\ldots a_{i_r}=a_{j_1}a_{j_2}\ldots a_{j_s}$$

is called a *relation;* moreover, a relation in $a_1, a_2, \ldots, a_l$ if each of $i_1, i_2, \ldots, i_r, j_1, j_2, \ldots, j_s$ is one of $1, 2, \ldots, l$. If $a_{i_1}, a_{i_2}, \ldots, a_{i_r}, a_{j_1}, a_{j_2}, \ldots, a_{j_s}$ denote some elements of an associative system $\mathfrak{A}$, we say, the relation (1) holds in $\mathfrak{A}$, or $\mathfrak{A}$ is satisfying (1), if $a_{i_1}a_{i_2}\ldots a_{i_r}$ is the same element of $\mathfrak{A}$ as $a_{j_1}a_{j_2}\ldots a_{j_s}$.

As an instance of an associative system, form the finite sequences or *"words"*, each element or *"letter"* of which is a member of a given finite set or *"alphabet"*. Two words are identical, by definition, if and only if they contain the same letters with the same multiplicity and in the same order. The product of two words is defined as the word formed of them by juxtaposition. In this associative system, called the *free associative system* on the given alphabet, the relation (1) holds if and only if it is "trivial", i. e. if $a_{i_1}a_{i_2}\ldots a_{i_r}$ and $a_{j_1}a_{j_2}\ldots a_{j_s}$ are identical words. However, there are associative systems in which some non-trivial relations hold; e. g. the associative systems $\mathfrak{A}_S$ defined below.

Given a finite system S of relations in $a_1, a_2, \ldots, a_l$, a further relation in $a_1, a_2, \ldots, a_l$ is called a *consequence* of S if and only if it holds in every associative system containing $a_1, a_2, \ldots, a_l$ and satisfying each of the relations

[5] See, A. CHURCH, loc. cit. [3]; A set of postulates for the foundation of logic, *Annals of Math.*, (2) **33** (1932), pp. 346—366 and **34** (1933), pp. 839—864; A proof of freedom of contradiction, *Proceedings of the National Academy of Sciences* (Washington), **21** (1935), pp. 275—281; *Mathematical logic* (Princeton, N. J., 1936); *The calculi of lambda-conversion* (Princeton, N. J., 1941); A. CHURCH and J. B. ROSSER, Some properties of conversion, *Transactions of the American Math. Society,* **39** (1936), pp. 472—482; S. C. KLEENE, Proof by cases in formal logic, *Annals of Math.*, (2) **35** (1934), pp. 529—544; A theory of positive integers in formal logic, *American Journal of Math.,* **57** (1935), pp. 153—173 and 219—244; λ-definiability and recursiveness, *Duke Math. Journal,* **2** (1936), pp. 340—353; J. B. ROSSER, loc. cit. [3].

[6] See E. L. POST, loc. cit. [3]; A variant of a recursively unsolvable problem, *Bulletin of the American Math. Society,* **52** (1946), pp. 264—268; and loc. cit. [1].

of S. The *word problem for associative systems, relative to a given system* S *of relations,* consists in asking for an algorithm by means of which, given any relation in $a_1, a_2, \ldots, a_l$, we could decide if it is a consequence of S.

Obviously, a relation in $a_1, a_2, \ldots, a_l$ is certainly a consequence of a system S of relations in $a_1, a_2, \ldots, a_l$ if it can be obtained from the relation $e=e$ and from the relations of S by means of a finite number of multiplications by one of $a_1, a_2, \ldots, a_l$ on the left or on the right as well as applications of the transitivity law; i. e. if it is a theorem of the following formal system Φ_S. *Symbols* of Φ_S are $a_1, a_2, \ldots, a_l$, $=$. *Terms* of Φ_S are the words formed of the letters $a_1, a_2, \ldots, a_l$. *Formulae* of Φ_S are the relations between such words, i. e. $\mathbf{t}=\mathbf{u}$, where $\mathbf{t}$ and $\mathbf{u}$ are terms of Φ_S. *Axioms* of Φ_S are $e=e$ (e denoting the empty word) as well as the relations of S. *Theorems* of Φ_S are (i) the axioms of Φ_S; (ii) $a_1\mathbf{t}=a_1\mathbf{u}$, $a_2\mathbf{t}=a_2\mathbf{u}$, $\ldots$, $a_l\mathbf{t}=a_l\mathbf{u}$, $\mathbf{t}a_1=\mathbf{u}a_1$, $\mathbf{t}a_2=\mathbf{u}a_2$, $\ldots$, $\mathbf{t}a_l=\mathbf{u}a_l$ if $\mathbf{t}=\mathbf{u}$ is a theorem of Φ_S; (iii) $\mathbf{u}=\mathbf{v}$ if $\mathbf{t}=\mathbf{u}$ and $\mathbf{t}=\mathbf{v}$ are theorems of Φ_S; (iv) nothing else.[7] Conversely, each consequence of the system S is a theorem of the formal system Φ_S. Indeed, the predicate[8] "$\mathbf{t}=\mathbf{u}$ is a theorem of Φ_S" between the terms (of Φ_S) is obviously an equivalence predicate, and, as easily seen, the equivalence classes of the free associative system on the alphabet $\{a_1, a_2, \ldots, a_l\}$ form an associative system[9] $\mathfrak{A}_S$ such that a relation in $a_1, a_2, \ldots, a_l$ holds in $\mathfrak{A}_S$ if and only if it is a theorem of Φ_S. Hence, the relations of the system S hold in $\mathfrak{A}_S$ and a relation which is not a theorem of Φ_S, not holding in $\mathfrak{A}_S$, is no consequence of S. Thus, the word problem relative to a system S of relations can be formulated alternatively as asking for an algorithm by means of which, given two terms $\mathbf{t}$ and $\mathbf{u}$ of Φ_S, we could decide if $\mathbf{t}=\mathbf{u}$ is a theorem of Φ_S. This formulation of the word problem is more appropriate for researches of a logical character.[10]

[7] I. e. the set of theorems of Φ_S is the smallest set (the intersection of all sets) having the axioms of Φ_S, further, together with $\mathbf{t}=\mathbf{u}$, the formulae $a_1\mathbf{t}=a_1\mathbf{u}$, $a_2\mathbf{t}=a_2\mathbf{u}$, $\ldots$, $a_l\mathbf{t}=a_l\mathbf{u}$, $\mathbf{t}a_1=\mathbf{u}a_1$, $\mathbf{t}a_2=\mathbf{u}a_2$, $\ldots$, $\mathbf{t}a_l=\mathbf{u}a_l$, finally, together with $\mathbf{t}=\mathbf{u}$ and $\mathbf{t}=\mathbf{v}$, the formula $\mathbf{u}=\mathbf{v}$ as elements. Hence, we can prove a property of the theorems of Φ_S by proving it for the axioms of Φ_S; then, supposing that $\mathbf{t}=\mathbf{u}$ has the property in question, proving that the same holds for $a_1\mathbf{t}=a_1\mathbf{u}$, $a_2\mathbf{t}=a_2\mathbf{u}$, $\ldots$, $a_l\mathbf{t}=a_l\mathbf{u}$, $\mathbf{t}a_1=\mathbf{u}a_1$, $\mathbf{t}a_2=\mathbf{u}a_2$, $\ldots$, $\mathbf{t}a_l=\mathbf{u}a_l$ too, and, supposing that $\mathbf{t}=\mathbf{u}$ and $\mathbf{t}=\mathbf{v}$ have the property in question, proving that the same holds for $\mathbf{u}=\mathbf{v}$ too. A similar remark applies for the set of the theorems of other formal systems (viz. Ψ_E, Φ_1, Φ_2, $\ldots$, Φ_5) as well as for the set of the terms of some of them (viz. Ψ_E, Φ_1, Φ_2 and Φ_3) too.

[8] We use the expression "predicate" instead of "relation" for the latter is used in this paper in a particular sense.

[9] $\mathfrak{A}_S$ is called *the associative system generated by the system* S *of relations.* It can be characterized (irrespective of isomorphisms) by the following properties. (i) $\mathfrak{A}_S$ is an associative system each element of which can be written as a product each factor of which is one of some elements $a_1, a_2, \ldots, a_l$ of $\mathfrak{A}_S$; (ii) a relation in $a_1, a_2, \ldots, a_l$ holds in $\mathfrak{A}_S$ if and only if it is a consequence of the system S.

[10] A third formulation of the word problem relative to a system S of relations in $a_1, a_2, \ldots, a_l$, perhaps the most interesting one for the algebraist, is to ask for an algo-

2. My proof for the Markov—Post theorem is based on a theorem due to KLEENE[11] according to which a general recursive function $R(x, y)$ of two variables can be given for which there is no algorithm by means of which, given any non-negative integer k, we could decide if the equation $R(k, y) = 0$ in y has a non-negative integer solution. Here a general recursive function[12] is an arithmetical function (i. e., a function with non-negative integers as arguments and values) for which, together with the successor function $F(x) = x+1$ and some additional arithmetical functions (called, together with F and R, the functions *"needed to the definition of R"*) a system E of equations (called a *"defining system of equations for R"*) can be given with the following properties. Each equation of E has to be a formula in the formal system Ψ_E defined below. For each function G needed to the definition of R, and for every sequence $k_1, k_2, \ldots, k_s$ of non-negative integers the number s of which coincides with the number of arguments of G, the equation $G(F^{k_1}(0), F^{k_2}(0), \ldots, F^{k_s}(0)) = F^k(0)$ has to be a theorem of the formal system Ψ_E for one and only one non-negative integer k which we denote by $k = G(k_1, k_2, \ldots, k_s)$. Here $F^k(0)$ is an abbreviation for $F(F(\ldots F(0)\ldots))$ with k symbols F and the formal system Ψ_E is defined as follows. *Symbols* of Ψ_E are 0; an enumerable infinite set of *"variables"* $x, y, \ldots$; a finite set of symbols $F, \ldots, R$, called *"functors"*, for the functions needed to the definition of R to each of which a positive integer (for F, the integer 1) is attached as the *"number of its arguments"*;[13] parentheses (and) ; comma , ; equality symbol = . *Terms* of Ψ_E are (i) 0 and the variables; (ii) for any functor G, the sequence of symbols $G(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_s)$ where s is the number of arguments of the functor G and $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_s$ are terms of Ψ_E; (iii) nothing else. *Formulae* of Ψ_E are the sequences of symbols of the form $\mathbf{t} = \mathbf{u}$ where $\mathbf{t}$ and $\mathbf{u}$ are terms of Ψ_E. *Axioms* of Ψ_E are the equations of E.

rithm by means of which, given any relation in $a_1, a_2, \ldots, a_l$, we could decide if it holds in the associative system $\mathfrak{A}_S$ generated by S.

[11] S. C. KLEENE, General recursive functions of natural numbers, *Math. Annalen,* **112** (1936), pp. 727—742, especially theorem XV, p. 741. We do not make use of the fact that $R(x, y)$ is actually a *primitive recursive* function. The same result follows, with a 2-*recursive* function R, from an unpublished proof of R. PÉTER for CHURCH's theorem on the existence of decision problems which are not solvable by any algorithm, by means of GÖDEL's theorem on the existence of truth problems unsolvable in a given postulate system, and, with an *elementary* function R, from an unpublished proof of mine for CHURCH's above theorem as a particular case of GÖDEL's above theorem (see also L. KALMÁR, On unsolvable mathematical problems, *Proceedings of the tenth International Congress of Philosophy* (Amsterdam, August 11—18, 1948), pp. 756—758).

[12] See KLEENE, loc. cit.[11], Definition 2*b*, p. 731.

[13] We suppose that we never use the same functor for functions with a different number of variables (as $F(x)$ and $F(x, y)$, e. g.).

Theorems of Ψ_E are (i) the axioms of Ψ_E; (ii) the result of substitution of 0 or of $F(x)$ for a variable[14] x throughout a theorem $\mathbf{T}$ of Ψ_E; (iii) the result of replacement of a particular occurrence of $\mathbf{t}$ in $\mathbf{T}$ by $\mathbf{u}$, where $\mathbf{T}$ and $\mathbf{t} = \mathbf{u}$ are theorems of Ψ_E; (iv) nothing else.[15]

Instances for general recursive functions are given by $Q(x) = x!$ for which[16]

$$S(x, 0) = x,$$
$$S(x, F(y)) = F(S(x, y)),$$
$$P(x, 0) = 0,$$
$$P(x, F(y)) = S(x, P(x, y)),$$
$$Q(0) = F(0),$$
$$Q(F(x)) = P(Q(x), F(x)),$$

further, by the Ackermann function[17] $A(x, y, z)$ for which

$$A(x, y, 0) = S(x, y),$$
$$A(x, 0, F(0)) = 0,$$
$$A(x, 0, F^2(0)) = F(0),$$
$$A(x, 0, F^3(z)) = x,$$
$$A(x, F(y), F(z)) = A(x, A(x, y, F(z)), z)$$

is a defining system of equations. (Here, of course, $F^3(z)$ stands for $F(F(F(z)))$.)

3. In the sequel, let E denote the particular defining system of equations for KLEENE's general recursive function $R(x, y)$ referred to; Φ_1 denote the

[14] "For *a variable x*" (and not "for *the variable x*") means that x can be replaced by any other variable too. In a similar sense we use the clause "for a functor G". — Note that by substitution of $F(x)$ for x k times in succession and then, by substitution of 0 for x, we can substitute $F^k(0)$ for x. — Also remark that by substitution of a term for a variable throughout a term or a formula of Ψ_E or more generally, by replacement of one or more occurrences of some terms Ψ_E by other terms of Ψ_E in a term or a formula of Ψ_E, we get a term or a formula, respectively, of Ψ_E again. A similar remark holds for the formal systems Φ_2, and Φ_3 (and, trivially, for Φ_4 and Φ_5) too.

[15] As easily shown (by means of the proof method stated in footnote [7]), a formula $\mathbf{T}$ of Ψ_E is a theorem of Ψ_E if and only if there is a finite sequence $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_l$ of formulae of Ψ_E such that $\mathbf{T}_l$ is $\mathbf{T}$, and, for each $i = 1, 2, \ldots, l$, $\mathbf{T}_i$ is either an axiom of Ψ_E, or the result of substitution of 0 or of $F(x)$ for a variable x in one of $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{i-1}$, or the result of replacement of a particular occurrence of $\mathbf{t}$ in one of $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{i-1}$ by $\mathbf{u}$, where $\mathbf{t} = \mathbf{u}$ is one $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{i-1}$. Such a sequence $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{i-1}$ is called a *proof of* $\mathbf{T}$ *in the formal system* Ψ_E. Of course, the same holds for the formal systems $\Phi_1, \Phi_2, \ldots, \Phi_5$ too.

[16] On reasons which will become clear later on, we insist on consequent functional notation; e. g., we write $F(x), S(x, y), P(x, y), Q(x)$ rather than $x+1, x+y, xy, x!$.

[17] See W. ACKERMANN, Zum Hilbertschen Aufbau der reellen Zahlen, *Math. Annalen,* **99** (1928), pp. 118—133. $A(x, y, z)$ is an instance for a general recursive function which is not primitive recursive, whereas $Q(x)$ is a primitive recursive function.

corresponding formal system Ψ_{E}. I shall transform the question "has the equation $R(k,y)=0$ in y a non-negative integer soluton?" into the question "is the relation $\mathfrak{r}_k$ a consequence of the system $\mathfrak{S}$ of relations?", $\mathfrak{r}_0, \mathfrak{r}_1, \ldots$ being a particular sequence of relations in the letters of some alphabet and $\mathfrak{S}$ a particular system of relations in the same letters. Then, *the word problem for associative systems, relative to the system* $\mathfrak{S}$ *of relations, cannot be solvable by any algorithm,* for in the opposite case, there would be an algorithm by means of which, given any non-negative integer k, we could decide if $\mathfrak{r}_k$ is a consequence of $\mathfrak{S}$, i. e. if the equation $R(k,y)=0$ has a non-negative integer solution y, in contradiction to KLEENE's theorem referred to.

Our first step towards this transformation is to transform the above question to an equation depending on k. To this effect, let us introduce the arithmetical function [18]

$$(2)\qquad U(x,y)=\begin{cases}0, \text{ if there is an integer } m\geqq y \text{ for which } R(x,m)=0,\\ 1 \text{ otherwise.}\end{cases}$$

Then, the equation $R(k,y)=0$ has a non-negative integer solution if and only if we have $U(k,0)=0$.

Now U is no general recursive function.[19] However, with the aid of the auxiliary (general recursive) function

$$(3)\qquad V(x,y)=\begin{cases}0 & \text{if } y=0,\\ x & \text{if } y\neq 0\end{cases}$$

it can be partially [20] defined by the system E' of equations arising from E by subjoining the equations [21]

$$(4)\qquad U(x,y)=V(U(x,F(y)),R(x,y)),$$
$$(5)\qquad V(U(x,F(y)),0)=0,$$
$$(6)\qquad V(U(x,F(y)),F(z))=U(x,F(y)).$$

[18] We suppose, U (and V) is a functor different from those for the functions $F,\ldots,R$ needed to the definition of R.

[19] Indeed, in the opposite case, the computation algorithm of $U(k,0)$ by means of a defining system of equations for U would furnish an algorithm for deciding if we have $U(k,0)=0$, i. e. ifthere is a non-negative integer solution of the equation $R(k,y)=0$ in y.

[20] In a sense similar to the notion of a partial recursive function; see S. C. KLEENE, On notation for ordinal numbers, *The Journal of Symbolic Logic*, 3 (1938), pp. 150—155, especially pp. 151—152. However, the function $U(x,y)$ as defined by (2), is not partial recursive (for it is defined for all values of its arguments and it is not general recursive); E' is rather a defining system of equations for the partial recursive function

$$U(x,y)=\begin{cases}0 \text{ if there is an integer } m\geqq y \text{ for which } R(x,m)=0,\\ \text{undefined otherwise.}\end{cases}$$

[21] The idea of introducing these equations is due, in another form, to KLEENE; see S. C. KLEENE, A theory of positive integers in formal logic, loc. cit. [5], pp. 230—231.

We define a formal system Φ_2, analogous to Φ_1 (except a slight modification) but based on the system of equations E' instead of E as follows. *Symbols* of Φ_2 are 0, the variables $x, y, \ldots$, the functors $F,\ldots,R$ as well as U and V; parentheses (and) ; comma , ; equality symbol $=$. *Terms* of Φ_2 are (i) 0 and the variables; (ii) for any functor G, the sequence of symbols $G(\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}_s)$ where s is the number of arguments of G and $\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}_s$ are terms of Φ_2; (iii) nothing else. *Formulae* of Φ_2 are the sequences of symbols of the form $\mathbf{t}=\mathbf{u}$ where $\mathbf{t}$ and $\mathbf{u}$ are terms of Φ_2. *Axioms* of Φ_2 are (i) the equations of E'; (ii) the formulae of the form $\mathbf{t}=\mathbf{t}$ where $\mathbf{t}$ is a term of Φ_2. (The purpose of the modification (ii) will be clear in section **6**, lemma 9). *Theorems* of Φ_2 are (i) the axioms of Φ_2; (ii) the result of substitution of 0 or of $F(x)$ for a variable x throughout a theorem of Φ_2; (iii) the result of replacement of a particular occurrence of $\mathbf{t}$ in $\mathbf{T}$ by $\mathbf{u}$, where $\mathbf{T}$ and $\mathbf{t}=\mathbf{u}$ are theorems of Φ_2: (iv) nothing else. Then we have

LEMMA 1. *Let k denote a non-negative integer. The formula*

$$(7)\qquad U(F^k(0),0)=0$$

of Φ_2 is a theorem of Φ_2 if and only if the equation $R(k,y)=0$ has a non-negative integer solution in y. Hence, there is no algorithm by means of which, given a non-negative integer k, we could decide if (7) is a theorem of Φ_2.

Indeed, suppose first, the equation $R(k,y)=0$ has a solution in non-negative integers y; let l denote its least solution. Then, we have on the one hand $R(k,i)\neq 0$, i. e. $R(k,i)=r_i+1$ for some non-negative integers r_i ($i=0,1,\ldots,l-1$), on the other hand $R(k,l)=0$. Hence, the formulae $R(F^k(0),F^i(0))=F^{r_i+1}(0)$, i. e.

$$(8)\qquad R(F^k(0),F^i(0))=F(F^{r_i}(0))\qquad (i=0,1,\ldots,l-1)$$

and

$$(9)\qquad R(F^k(0),F^l(0))=0$$

are theorems of Φ_1 and thus of Φ_2 too. On the other hand, by (4) (see footnote [14], second sentence),

$$U(F^k(0),F^i(0))=V(U(F^k(0),F^{i+1}(0)),R(F^k(0),F^i(0))\qquad (i=0,1,\ldots,l),$$

hence, by (8) and (9),

$$(10)\quad U(F^k(0),F^i(0))=V(U(F^k(0),F^{i+1}(0)),F(F^{r_i}(0)))\qquad (i=0,1,\ldots,l-1),$$

and

$$(11)\qquad U(F^k(0),F^l(0))=V(U(F^k(0),F^{l+1}(0)),0)$$

are theorems of Φ_2. The same holds, by (6) and (5), for

$$(12)\quad V(U(F^k(0),F^{i+1}(0)),F(F^{r_i}(0)))=U(F^k(0),F^{i+1}(0))\qquad (i=0,1,\ldots,l-1)$$

and

(13) $$V(U(F^k(0), F^{l+1}(0)), 0) = 0$$

too. By (10) and (12) on the one hand, (11) and (13) on the other hand, we see that

$$U(F^k(0), 0) = U(F^k(0), F(0)),$$
$$U(F^k(0), F(0)) = U(F^k(0), F^2(0)),$$
$$\cdots \qquad \cdots$$
$$U(F^k(0), F^{l-1}(0)) = U(F^k(0), F^l(0)),$$
$$U(F^k(0), F^l(0)) = 0$$

are theorems of Φ_2; thus $U(F^k(0), 0) = 0$, i. e. (7) too.

Suppose now, (7) is a theorem of Φ_2. By $F(x) = x + 1$ and the equations of E, the arithmetical functions needed to the definition of R have been defined so as to render each equation of E verifiable, i. e. true for each replacement of its variables by non-negative integers. Defining the additional arithmetical functions U and V by (2) and (3), the same holds for the equations (4), (5) and (6) too. This is obvious for (5) and (6), for we have $F(m) = m + 1 \neq 0$ for each non-negative integer m. As to (4), we have for any non-negative integers k and l, in case $R(k, l) = 0$

$$U(k, l) = 0 = V(U(k, F(l)), 0) = V(U(k, F(l), R(k, l)),$$

whereas in case $R(k, l) \neq 0$ we have

$$U(k, l) = U(k, l+1) = U(k, F(l)) = V(U(k, F(l)), R(k, l)),$$

for either we have $R(k, m) = 0$ for some integer $m \geqq l+1$, thus $U(k, l) = U(k, l+1) = 0$, or $R(k, m) \neq 0$ for every integer $m \geqq l$ (for we have $R(k, l) \neq 0$), thus $U(k, l) = U(k, l+1) = 1$. Also, obviously, the equations of the form $\mathbf{t} = \mathbf{t}$ are verifiable.

Obviously, substitution of 0 or of $F(x)$ for a variable x in a verifiable formula of Φ_2 yields a verifiable formula again; also, if $\mathbf{T}$ and $\mathbf{t} = \mathbf{u}$ are verifiable formulae of Φ_2, then so is the formula obtained by replacement of a particular occurrence of $\mathbf{t}$ in $\mathbf{T}$ by $\mathbf{u}$. Hence, each theorem of Φ_2 is verifiable, thus, by hypothesis, (7) too; i. e. there is an integer $n \geqq 0$ such that $R(k, n) = 0$ as stated.

4. The second part of the proof of lemma 1 (from "suppose now" on), besides being somewhat sketchy, does not fulfil the requirements of proof theory for it does not provide a calculation method of a non-negative integer n such that $R(k, n) = 0$ by means of a given proof of the formula $U(F^k(0), 0) = 0$ in the formal system Φ_2. Indeed, the notion of verifiability as defined above is not a constructive one, owing to the non-constructive definition (2) of $U(x, y)$. For those readers who are interested in constructiveness distinctions, I give a proof-theoretical alternative for the second part of the proof. The other readers may continue reading with section **5.**

Let us call the terms $F^n(0)$ of Φ_2 $(n = 0, 1, \ldots)$ *numerals*. A term or a formula obtained from a term $\mathbf{t}$ or a formula $\mathbf{T}$ of Φ_2 by substituting a numeral for each of its variables is called a *numerical instance* of $\mathbf{t}$ or of $\mathbf{T}$, respectively. A term or a formula containing no variables is called a numerical term or formula;[22] it is the only numerical instance of itself. E. g., (7) is a numerical formula. A finite sequence $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_l$ of (numerical) formulae such that, for $i = 1, 2, \ldots, l$, $\mathbf{T}_i$ is either a numerical instance of an axiom of Φ_2, or the result of replacement of a particular occurrence of $\mathbf{t}$ in one of $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{i-1}$ by $\mathbf{u}$, where $\mathbf{t} = \mathbf{u}$ is one of $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{i-1}$, is called a *numerical proof of* $\mathbf{T}_l$. Our arguments are based on

LEMMA 2. *If* $\mathbf{T}$ *is a theorem of* Φ_2, *and* $\overline{\mathbf{T}}$ *a numerical instance of* $\mathbf{T}$, *then there is a numerical proof of* $\overline{\mathbf{T}}$.

Indeed,[23] this holds if $\mathbf{T}$ is an axiom of Φ_2, for then, $\overline{\mathbf{T}}$ alone forms a numerical proof of $\overline{\mathbf{T}}$. Suppose, lemma 2 holds for a theorem $\mathbf{T}$ of Φ_2 and let $\mathbf{T}'$ and $\mathbf{T}''$ be the results of substitution of 0 and of $F(x)$, respectively, for a variable x throughout $\mathbf{T}$. Then, lemma 2 holds for $\mathbf{T}'$ and $\mathbf{T}''$ too. Indeed, any numerical instance of $\mathbf{T}'$ or $\mathbf{T}''$ is a numerical instance of $\mathbf{T}$ too. For if $x, y, \ldots, v$ are the variables of $\mathbf{T}$, then substitution of the numerals $F^n(0), \ldots, F^q(0)$ for the variables $y, \ldots, v$, respectively, in $\mathbf{T}'$ amounts to substitution of $0, F^n(0), \ldots, F^q(0)$ for the variables $x, y, \ldots, v$, respectively, in $\mathbf{T}$, whereas substitution of the numerals $F^m(0), F^n(0), \ldots, F^q(0)$ for the variables $x, y, \ldots, v$, respectively, in $\mathbf{T}''$ amounts to substitution of $F^{m+1}(0), F^n(0), \ldots, F^q(0)$ for the variables $x, y, \ldots, v$, respectively, in $\mathbf{T}$. Again, suppose, lemma 2 holds for some theorems $\mathbf{T}$ and $\mathbf{T}'$ of Φ_2, and let $\mathbf{T}''$ be the result of replacement of a particular occurrence of $\mathbf{t}$ in $\mathbf{T}$ by $\mathbf{u}$, where $\mathbf{T}'$ is $\mathbf{t} = \mathbf{u}$. Then lemma 2 holds for $\mathbf{T}''$ too. Indeed, let $\overline{\mathbf{T}''}$ be any numerical instance of $\mathbf{T}''$. Substitute in $\mathbf{T}$ as well as in $\mathbf{T}'$ (i. e., in $\mathbf{t}$ and $\mathbf{u}$) the same numerals for the variables figuring in $\mathbf{T}''$, (in so far as they figure in $\mathbf{T}$ or in $\mathbf{T}'$) as when forming $\overline{\mathbf{T}''}$ out of $\mathbf{T}''$, whereas for the other variables figuring in $\mathbf{T}$ and $\mathbf{T}'$, substitute 0, say. (There are such variables if and only if there is a single occurrence of $\mathbf{t}$ in $\mathbf{T}$ and some variables figure in $\mathbf{t}$ but neither

[22] Obviously, the notion of a numerical term and of a numerical formula could be defined as follows. Numerical terms are (i) 0; (ii) for any functor G, the sequence of symbols $G(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_s)$, where s is the number of arguments of G and $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_s$ are numerical terms; (iii) nothing else. Numerical formulae are the sequences of symbols of the form $\mathbf{t} = \mathbf{u}$ where $\mathbf{t}$ and $\mathbf{u}$ are numerical terms. Hence, a function can be defined over the class of the numerical terms by defining it for 0 and, supposing its value known for $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_s$, by defining it for $G(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_s)$, where G is a functor having s as its number of arguments.

[23] The proof is a variant of HILBERT's method of removing back the substitutions (Rückverlegung der Einsetzungen), however, without dissolution into proof-files (Auflösung in Beweisfäden); see D. HILBERT and P. BERNAYS, *Grundlagen der Mathematik*, I (Berlin, 1934), pp. 221—228.

in $\mathbf{u}$, nor in $\mathbf{T}$ outside the occurrence of $\mathbf{t}$.) Thus, we get a numerical instance $\bar{\mathbf{T}}$ of the formula $\mathbf{T}$ as well as numerical instances $\bar{\mathbf{t}}$ and $\bar{\mathbf{u}}$ of the terms $\mathbf{t}$ and $\mathbf{u}$, respectively; and obviously, $\bar{\mathbf{T}}''$ is the result of replacement of some occurrence of $\bar{\mathbf{t}}$ in $\bar{\mathbf{T}}$ by $\bar{\mathbf{u}}$. By hypothesis, there are numerical proofs of $\bar{\mathbf{T}}$ as well as of the numerical instance $\bar{\mathbf{t}} = \bar{\mathbf{u}}$ of $\mathbf{T}'$. By juxtaposing them and subjoining the formula $\bar{\mathbf{T}}''$, we get a numerical proof of $\bar{\mathbf{T}}''$. Hence, lemma 2 holds for every theorem $\mathbf{T}$ of Φ_2.

Given a non-negative integer k, we attach to each numerical term a non-negative integer, called its value, by means of the following definition. The value of 0 is 0 (zero). If G is one of the functors $F, \ldots, R$ for the functions needed to the definition of R, and s is the number of its arguments, then if $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_s$ are numerical terms whose values are $k_1, k_2, \ldots, k_s$, respectively, we define the value of the numerical term $G(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_s)$ as the only non-negative integer l for which $G(F^{k_1}(0), F^{k_2}(0), \ldots, F^{k_s}(0)) = F^l(0)$ is a theorem of Φ_1. If $\mathbf{t}$ and $\mathbf{u}$ are numerical terms the values of which are m and n, respectively, then we define the value of the numerical term $U(\mathbf{t}, \mathbf{u})$ to be 1 or 0 according as $m = k$ or $m \neq k$ and the value of the numerical term $V(\mathbf{t}, \mathbf{u})$ to be 0 or m according as $n = 0$ or $n \neq 0$. Obviously, the value of a numerical term $\mathbf{t}$ does not change if an occurrence of a term in $\mathbf{t}$ is replaced by another numerical term with the same value. We call a numerical formula $\mathbf{t} = \mathbf{u}$ *true* or *false* according as the numerical terms $\mathbf{t}$ and $\mathbf{u}$ have the same value or not.

Suppose now, the numerical formula (7) is a theorem of Φ_2. Then, by lemma 2, there is a numerical proof of (7). Let $\mathbf{T}$ be the first formula of this numerical proof which is false (there is such a formula for (7) is false, the value of its left-hand side being 1 and that of its right-hand side 0). Then, $\mathbf{T}$ is a numerical instance of an axiom of Φ_2. For in the opposite case, there would be numerical formulae $\mathbf{T}'$ and $\mathbf{t} = \mathbf{u}$, belonging to, and prior to $\mathbf{T}$ in, the numerical proof in question, thus true, such that $\mathbf{T}$ is the result of replacement of a particular instance of $\mathbf{t}$ in $\mathbf{T}'$ by $\mathbf{u}$. However, this is impossible, for, $\mathbf{t} = \mathbf{u}$ being true, $\mathbf{t}$ and $\mathbf{u}$ have the same value; thus, the left and the right hand side of $\mathbf{T}$ have the same value as those of $\mathbf{T}'$; but, $\mathbf{T}'$ being true, its left and right hand side have the same value and thus, the same holds for $\mathbf{T}$ too; i. e. $\mathbf{T}$ would be true.

Now, the numerical instances of the equations of E are true in consequence of the unicity requirement in the definition of general recursive functions. Also, the numerical instances of the axioms (5) and (6) are true by the definition of the value of a numerical term of the form $V(\mathbf{t}, \mathbf{u})$. Plainly, also the numerical instances of an equation of the form $\mathbf{t} = \mathbf{t}$ are true. Hence, $\mathbf{T}$ must be a numerical instance of the axiom (4), i. e. a formula of the form $U(F^m(0), F^n(0)) = V(U(F^m(0), F^{n+1}(0)), R(F^m(0), F^n(0)))$. Here, we must have $m = k$, for in the opposite case, $U(F^m(0), F^n(0))$ and $U(F^m(0), F^{n+1}(0))$

would have the value 0, hence the same would hold for $V(U(F^m(0), F^{n+1}(0)), R(F^m(0), F^n(0)))$ independently of the value of $R(F^m(0), F^n(0))$ and thus, $\mathbf{T}$ would be true. Hence, $\mathbf{T}$ is of the form $U(F^k(0), F^n(0)) = V(U(F^k(0), F^{n+1}(0)), R(F^k(0), F^n(0)))$. Here, $U(F^k(0), F^n(0))$ and $U(F^k(0), F^{n+1}(0))$ have the value 1; hence, $R(F^k(0), F^n(0))$ has the value 0, for in the opposite case, $V(U(F^k(0), F^{n+1}(0)), R(F^k(0), F^n(0)))$ would have the value 1 and $\mathbf{T}$ would be true again. I. e., we have $R(k, n) = 0$, as stated.

5. Our second step towards the transformation of the question "has the equation $R(k, y) = 0$ in y a non-negative integer solution?" into a question of being a relation a consequence of a system of relations is the omission of all parentheses and commas from the formulae of Φ_2; i. e. the use of ŁUKASIEWICZ's notation system.[24] For this purpose, we define a counterpart Φ_3 of the formal system Φ_2 as follows. *Symbols* of Φ_3 are 0, the variables $x, y, \ldots$, and the functors $F, \ldots, R, U, V$, as well as the equality symbol $=$. *Terms* of Φ_3 are (i) 0 and the variables $x, y, \ldots$; (ii) for any functor G, the sequence of symbols $G\mathbf{t}_1\mathbf{t}_2 \ldots \mathbf{t}_s$ where $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_s$ are terms of Φ_3 whose number s coincides with the number of arguments of G; (iii) nothing else. Obviously, omitting the parentheses and commas from a term $\mathbf{t}$ of Φ_2, we obtain a term $\bar{\mathbf{t}}$ of Φ_3 which we call the *simplification*[25] of $\mathbf{t}$; on the other part, each term of Φ_3 is the simplification of at least one (and, as we shall show in the sequel, only one) term of Φ_2. *Formulae* of Φ_3 are the sequences of symbols of the form $\mathbf{t} = \mathbf{u}$ where $\mathbf{t}$ and $\mathbf{u}$ are terms of Φ_3. If $\bar{\mathbf{t}}, \bar{\mathbf{u}}$ are the simplifications of the terms $\mathbf{t}, \mathbf{u}$ of Φ_2, respectively, then we call the formula $\bar{\mathbf{t}} = \bar{\mathbf{u}}$ of Φ_3 the simplification of the formula $\mathbf{t} = \mathbf{u}$ of Φ_2. Again, each formula

[24] See J. ŁUKASIEWICZ and A. TARSKI, Untersuchungen über den Aussagenkalkül, *Sprawozdania z posiedzeń Towarzystwa Naukowego Warszawskiego*, Wydz. III, 23 (1930), pp. 30—50. For a reader, who is acquainted with ŁUKASIEWICZ's notation system as well as with its properties, I could begin the proof in writing down the system E' of equations in this notation system and then, continue with section **6**. Alternatively, I could menage without ŁUKASIEWICZ's notation system; then, in the formal system Φ_4, I should allow to multiply both sides of an equation by *any* symbol used, also (or , or), on the left or on the right. However, such a variant of the proof would not be simpler than that of the text, and it would be very strange for the algebraist; indeed, it would amount to consider also the parentheses (and) and the comma , as elements of an associative system, whereas our procedure amounts to consider, besides 0 (which is in no sense a "zero element"), the "indeterminates" $x, y, \ldots, w$ as well as the "operators" $F, \ldots, R, U, V$ as elements of an associative system and to consider the function value $G(x, y, \ldots, v)$ as the product of $G, x, y, \ldots$, and v (e. g., $F(x)$ as the product of the operator F and the indeterminate x), which is much more familiar to the algebraist.

[25] Alternatively, we could define the simplication of a term of Φ_2 thus: (i) the simplification of 0 or of a variable x is 0 or x itself, respectively; (ii) for any functor G and for any terms $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_s$ of Φ_3 the number of which is the same as the number of arguments of G, the simplification of $G(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_s)$ is $G\bar{\mathbf{t}}_1\bar{\mathbf{t}}_2 \ldots \bar{\mathbf{t}}_s$, where, for $i = 1, 2, \ldots, s$, $\bar{\mathbf{t}}_i$ is the simplification of $\mathbf{t}_i$.

of Φ_3 is the simplification of a formula of Φ_2. *Axioms* of Φ_3 are the simplifications of the equations belonging to E, as well as the formulae of the form $\mathbf{t}=\mathbf{t}$ where $\mathbf{t}$ is a term of Φ_3. *Theorems* of Φ_3 are (i) the axioms of Φ_3; (ii) the result of substitution of 0 or of Fx for a variable x throughout a theorem $\mathbf{T}$ of Φ_3; (iii) the result of replacement of a particular occurrence of $\mathbf{t}$ in $\mathbf{T}$ by $\mathbf{u}$, where $\mathbf{T}$ and $\mathbf{t}=\mathbf{u}$ are theorems of Φ_3; (iv) nothing else.

The "rules of inference" (ii) and (iii) corresponding exactly to those of the formal system Φ_2, it is obvious that the simplification of a theorem of Φ_2 is a theorem of Φ_3. In particular, by lemma 1, $UF^k00=0$ is certainly a theorem of Φ_3 if the equation $R(k,y)=0$ has a non-negative integer solution in y. (Here F^k0 is an abbreviation for the term $FF\cdots F0$ of Φ_3 with k symbols F; also, we shall use UF^k0^2 as an abbreviation of UF^k00.) However, the converse is not obvious at all; as a matter of fact, it is a consequence of the point of ŁUKASIEWICZ's notation system, viz. its unequivocality.[26] To prove it, the notion of the *valency* of a symbol (except $=$) of Φ_3, as defined below, will be useful. The valency of 0 or of one of the variables $x, y, \ldots$ is, by definition, the number -1; the valency of a functor G is, by definition, $s-1$, where s is the number of arguments of G. Also, we attach to a finite sequence of symbols, except $=$, of Φ_3, the sum of valencies of its members as its valency. Throughout this section, we call such a sequence a *word*; for any word $\alpha_1\alpha_2\cdots\alpha_l$ ($\alpha_1, \alpha_2, \ldots, \alpha_l$ symbols of Φ_3 except $=$), we call the words[27] e (the empty word), $\alpha_1, \alpha_1\alpha_2, \ldots, \alpha_1\alpha_2\ldots\alpha_{l-1}$ the *proper sections of* $\alpha_1\alpha_2\ldots\alpha_l$. Then we have

LEMMA 3. *Each term of Φ_3 has the valency*[28] -1 *whereas its proper sections have non-negative valencies.*

Indeed, this holds obviously for 0 and the variables $x, y, \ldots$ Supposing, the statement of the lemma holds for the terms $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_s$ of Φ_3 and G is one of the functors $F, \ldots, R, U, V$, having s as the number of its arguments, it holds for $G\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_s$ too. Indeed, the valency of this term is $s-1\overbrace{-1-1-\ldots-1}^{s\text{ times}}=s-1-s=-1$, whereas each of its proper sections (other than e) has the form $G\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_{i-1}\mathbf{u}$ where $i=1,2,\ldots,$ or s, and $\mathbf{u}$ is a proper section of $\mathbf{t}_i$; hence, its valency is $s-1\overbrace{-1-1-\ldots-1}^{i-1\text{ times}}+v=s-1-(i-1)+v=s-i+v\geqq v\geqq 0$, v denoting the valency of $\mathbf{u}$. Hence, lemma 3 holds for each term of Φ_3.

[26] According to my knowledge ŁUKASIEWICZ did not publish any proof of the unequivocality of his notation system. The note of K. MENGER, Eine elementare Bemerkung über die Struktur logischer Formeln, *Ergebnisse eines math. Kolloquiums*, **3** (1930—31), pp. 22—23, furnishes a proof of the unequivocality in question for a particular case. The general case is treated in H. B. CURRY, *Leçons de logique algébrique* (Paris—Louvain, 1952), pp. 143—145 as well as in P. C. ROSENBLOOM, *Elements of mathematical logic* (New York, 1950), pp. 153—157 and 205. (The work of CURRY has appeared after the manuscript of this paper has been completed; as to the work of ROSENBLOOM, I got aware of its existence by a quotation of CURRY, loc. cit, p. 142.)

[27] We do not make any distinction between a symbol and the word formed of that symbol alone.

[28] In contrast to chemistry, here "saturated compounds" have the valency -1 and not 0, for they can serve at the same time as radicals out of which further compounds can be made (by means of a functor). For the same reason, a functor which needs s terms in order to get saturated, has the valency $s-1$ and not s as it would have in chemistry.

LEMMA 4. *Two different terms $\mathbf{t}$ and $\mathbf{u}$ of Φ_2 cannot have the same term $\bar{\mathbf{t}}$ of Φ_3 as their simplifications.*

This is obvious for terms $\bar{\mathbf{t}}$ of Φ_3 containing but one symbol. Assuming it to hold for terms $\bar{\mathbf{t}}$ of Φ_3 containing less than r symbols ($r=2,3,\ldots$), suppose the term $\bar{\mathbf{t}}$ of Φ_3 containing r symbols is the simplification of the terms $\mathbf{t}$ and $\mathbf{u}$ of Φ_2. Then, the first symbol of $\mathbf{t}$ and $\mathbf{u}$ must coincide with that of $\bar{\mathbf{t}}$, viz. a functor G. Thus, $\mathbf{t}$ and $\mathbf{u}$ have the forms $G(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_s)$ and $G(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s)$, respectively, where s is the number of arguments of G and $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_s, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s$ are terms of Φ_2, each containing less than r symbols. Hence, denoting, for $i=1,2,\ldots,s$, by $\bar{\mathbf{t}}_i$ and $\bar{\mathbf{u}}_i$ the simplifications of $\mathbf{t}_i$ and $\mathbf{u}_i$, respectively, $\bar{\mathbf{t}}$ is, on the one hand $G\bar{\mathbf{t}}_1\bar{\mathbf{t}}_2\ldots\bar{\mathbf{t}}_s$, on the other hand $G\bar{\mathbf{u}}_1\bar{\mathbf{u}}_2\ldots\bar{\mathbf{u}}_s$. Thus, for $i=1,2,\ldots,s$, $\bar{\mathbf{t}}_i$ is identical with $\bar{\mathbf{u}}_i$. For in the opposite case, let $\bar{\mathbf{t}}_i$ be the first of $\bar{\mathbf{t}}_1, \bar{\mathbf{t}}_2, \ldots, \bar{\mathbf{t}}_s$ which is different from the corresponding $\bar{\mathbf{u}}_i$. Then, the words $G\bar{\mathbf{t}}_1\bar{\mathbf{t}}_2\ldots\bar{\mathbf{t}}_{i-1}$ and $G\bar{\mathbf{u}}_1\bar{\mathbf{u}}_2\ldots\bar{\mathbf{u}}_{i-1}$ are identical, whereas $G\bar{\mathbf{t}}_1\bar{\mathbf{t}}_2\ldots\bar{\mathbf{t}}_{i-1}\bar{\mathbf{t}}_i$ and $G\bar{\mathbf{u}}_1\bar{\mathbf{u}}_2\ldots\bar{\mathbf{u}}_{i-1}\bar{\mathbf{u}}_i$, both sections of $\bar{\mathbf{t}}$ (i. e. either proper sections of $\bar{\mathbf{t}}$ or identical with $\bar{\mathbf{t}}$), are different. Hence, one of $\bar{\mathbf{t}}_i$ and $\bar{\mathbf{u}}_i$ would be a proper section of the other, which is impossible, for both $\bar{\mathbf{t}}_i$ and $\bar{\mathbf{u}}_i$ have, according to lemma 3, the valency -1, whereas their proper sections have non-negative valencies. By the induction hypothesis, the identity of $\bar{\mathbf{t}}_i$ with $\bar{\mathbf{u}}_i$ ($i=1,2,\ldots,s$) implies that of $\mathbf{t}_i$ with $\mathbf{u}_i$; hence, also $\mathbf{t}$ and $\mathbf{u}$ cannot be different.

As a corollary of lemma 4, we see that two different formulae of Φ_2 cannot have the same formula of Φ_3 as their simplifications.

Now, we can prove the converse of lemma 3, i. e.

LEMMA 5. *Each word whose valency is -1 while ist proper sections have non-negative valencies, is a term of Φ_3.*

Indeed, this holds for words containing a single letter (0, or one of the variables $x, y, \ldots$). Supposing the statement of the lemma to hold for words containing less than r letters ($r=2,3,\ldots$), let $\mathbf{w}$ be a word containing r letters and satisfying the conditions of the lemma. Then the first letter of $\mathbf{w}$, having a non-negative valency, is a functor G. Denote by s the number of arguments of G, and, for $i=0,1,2,\ldots,s$, by $\mathbf{w}_i$ the first (i. e., the shortest) non-empty section of $\mathbf{w}$ (proper section or $\mathbf{w}$ itself) whose valency is less than or equal to $s-1-i$. (There is such a section for $\mathbf{w}$ has the valency $-1=s-1-s\leq s-1-i$.) Obviously, $\mathbf{w}_0$ is G, and $\mathbf{w}_s$ is $\mathbf{w}$. For $i=0,1,\ldots,s-1$, $\mathbf{w}_i$ is a section of $\mathbf{w}_{i+1}$, for the valency of $\mathbf{w}_{i+1}$ is less than or equal to $s-1-(i+1)<s-1-i$, thus $\mathbf{w}_{i+1}$ cannot be shorter than $\mathbf{w}_i$. We prove that, for $i=0,1,\ldots,s$, the valency of $\mathbf{w}_i$ is exactly $s-1-i$ (hence, for $i=0,1,\ldots,s-1$, $\mathbf{w}_i$ is a *proper* section of $\mathbf{w}_{i+1}$). Obviously, this holds for $i=0$. For $i=1,2,\ldots,s$, $\mathbf{w}_i$ contains more than one letter, for the first letter G of $\mathbf{w}_i$ has the valency $s-1>s-1-i$. If $\mathbf{w}_i$ had not the valency $s-1-i$, its valency would be $s-2-i$ or less; hence, omitting its last letter, we should obtain a shorter word the valency of which would be $s-1-i$ or less (for the valency of a letter is -1 or more), which is impossible.

For $i=1,2,\ldots,s$, let $\mathbf{t}_i$ be the word the subjoining of which to $\mathbf{w}_{i-1}$ yields $\mathbf{w}_i$. Then, $\mathbf{t}_i$ has the valency -1 whereas its proper sections have non-negative valencies; for in the opposite case, subjoining a proper section having a negative valency to $\mathbf{w}_{i-1}$, we should get a word shorter than $\mathbf{w}_i$ which has a valency $s-1-i$ or less. By the hypothesis, $\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}$ are terms of Φ_3, hence the same holds for $\mathbf{w}$, which is $G\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_s$.

Now, we can prove the converse of the remark made after the definition of the theorems of Φ_3, i. e.

LEMMA 6. *Each theorem* $\overline{\mathbf{T}}$ *of* Φ_3 *is the simplification of a theorem* $\mathbf{T}$ *of* Φ_2.

Indeed, this holds for the axioms of Φ_3. Suppose it to hold for a theorem $\overline{\mathbf{T}}$ of Φ_3, then it holds for the result of substitution of 0 or of Fx for a variable x in $\overline{\mathbf{T}}$ too; for it is obviously the simplification of the result of substitution of 0 or of $F(x)$, respectively, for x in the theorem $\mathbf{T}$ of Φ_2 the simplification of which is $\overline{\mathbf{T}}$. Suppose now, the lemma holds for the theorems $\overline{\mathbf{T}}$ as well as $\overline{\mathbf{t}}=\overline{\mathbf{u}}$ of Φ_2; i. e. they are the simplifications of some theorems $\mathbf{T}$ and $\mathbf{t}=\mathbf{u}$ of Φ_3. Let $\overline{\mathbf{U}}$ be the result of replacement of a particular occurrence of $\overline{\mathbf{t}}$ in $\overline{\mathbf{T}}$ by $\overline{\mathbf{u}}$. Replace that occurrence of $\overline{\mathbf{t}}$ in $\overline{\mathbf{T}}$ by a variable v not occurring in $\overline{\mathbf{T}}$. Thus, we get an equation $\overline{\mathbf{V}}$ between two words which we prove to be a formula of Φ_3. Indeed, one of the sides of $\overline{\mathbf{V}}$ is identical with the corresponding side of $\overline{\mathbf{T}}$ whereas the other side differs by replacement of an occurrence of $\overline{\mathbf{t}}$ by v. Now, v has the valency -1 just as $\overline{\mathbf{t}}$; hence, the side in question of $\overline{\mathbf{V}}$ has the same valency as the corresponding side of $\overline{\mathbf{T}}$ and its proper sections have the same valencies as some proper sections of that. Thus, by lemmas 3 and 5, both sides of $\overline{\mathbf{V}}$ are terms of Φ_3, that is, $\overline{\mathbf{V}}$ is a formula of Φ_3, hence, the simplification of a formula $\mathbf{V}$ of Φ_2. Now, substituting $\mathbf{t}$ for v in $\mathbf{V}$, we get a formula of Φ_2 which has obviously $\overline{\mathbf{T}}$ as its simplification. By the above corollary of lemma 4, this formula is identical with the theorem $\mathbf{T}$ of Φ_2. Substituting, on the other hand, $\mathbf{u}$ for v in $\mathbf{V}$, we get another formula $\mathbf{U}$ of Φ_2 which has $\overline{\mathbf{U}}$ as its simplification. Now, $\mathbf{U}$ is plainly the result of replacement of an occurrence of $\mathbf{t}$ in $\mathbf{T}$ by $\mathbf{u}$, hence, a theorem of Φ_2; consequently, lemma 6 holds for $\overline{\mathbf{U}}$ too.

As a corollary, we see that $UF^k0^2=0$ is a theorem of Φ_3 if and only if the formula $U(F^k(0),0)=0$ of Φ_2 having it as simplification is a theorem of Φ_2, that is, if we have $R(k,n)=0$ for some non-negative integer n. Hence, there is no algorithm by means of which, given any non-negative integer k, we could decide if $UF^k0^2=0$ is a theorem of Φ_3.

6. Remark that no theorem of Φ_3, except those of the form $\mathbf{t}=\mathbf{t}$, contains any variable which does not figure in at least one equation of E′. Hence, the class of theorems of Φ_3 does not change essentially if we modify system Φ_3 by allowing no variables other than those figuring in the equations of E′ (besides 0, $=$, and the functors $F,\ldots,R,U,V$) as symbols.[29]

By such a modification, the terms of Φ_3 become particular words formed of the letters of a *finite* alphabet, subject to the valency conditions of lemma 3 or 5. Our next step consists in removing those conditions, and, at the same time, in making the rules of inference more similar to those of Φ_S. For this purpose, we define a formal system Φ_4 as follows. *Symbols* of Φ_4 are 0, the variables $x,y,\ldots,w$ figuring in at least one equation of E′, the functors $F,\ldots,R,U,V$, and the equality sign $=$. *Terms* of Φ_4 are arbitrary finite (possibly empty) sequences of symbols of Φ_4 except $=$, i. e. words formed of the letters of the alphabet $\{0,x,y,\ldots,w,F,\ldots,R,U,V\}$. *Formulae* of Φ_4 are the sequences of symbols of the form $\mathbf{t}=\mathbf{u}$ where $\mathbf{t}$ and $\mathbf{u}$ are terms of Φ_4. *Axioms* of Φ_4 are the simplifications of the equations belonging to E′, as well as the formula $e=e$ (e denoting the empty word). *Theorems* of Φ_4 are (i) the axioms of Φ_4; (ii) the result of substitution of 0 or of Fx for a variable x throughout a theorem $\mathbf{T}$ of Φ_4; (iii) $\alpha\mathbf{t}=\alpha\mathbf{u}$ and $\mathbf{t}\alpha=\mathbf{u}\alpha$ if $\mathbf{t}=\mathbf{u}$ is a theorem of Φ_4 and α a symbol of Φ_4 except $=$; (iv) $\mathbf{u}=\mathbf{v}$ if $\mathbf{t}=\mathbf{u}$ and $\mathbf{t}=\mathbf{v}$ are theorems of Φ_4; (v) nothing else.

As an easy consequence of (i), (iii) and (iv), we have the following

LEMMA 7. *For any term* $\mathbf{t}$ *of* Φ_4, $\mathbf{t}=\mathbf{t}$ *is a theorem of* Φ_4. *If* $\mathbf{t}=\mathbf{u}$ *is a theorem of* Φ_4 *then the same holds for* $\mathbf{u}=\mathbf{t}$, *as well as, for any term* $\mathbf{v}$ *of* Φ_4, *for* $\mathbf{vt}=\mathbf{vu}$ *and* $\mathbf{tv}=\mathbf{uv}$. *If* $\mathbf{t}=\mathbf{u}$ *and* $\mathbf{T}$ *are theorems of* Φ_4, *then the same holds for the result of replacement of a particular occurrence of* $\mathbf{t}$ *in* $\mathbf{T}$ *by* $\mathbf{u}$.

Indeed, if $\mathbf{t}=\mathbf{u}$ is a theorem and $\mathbf{v}$ a term of Φ_4, then a repeated application of (iii) gives $\mathbf{vt}=\mathbf{vu}$ and $\mathbf{tv}=\mathbf{uv}$ as theorems of Φ_4. In particular, $\mathbf{t}=\mathbf{t}$ is a theorem of Φ_4 for each term $\mathbf{t}$ of Φ_4, for $e=e$ is a theorem of Φ_4. Further, we get for any theorem $\mathbf{t}=\mathbf{u}$ of Φ_4, by application of (iv) to the theorems $\mathbf{t}=\mathbf{u}$ and $\mathbf{t}=\mathbf{t}$ of Φ_4 (with $\mathbf{t}$ for $\mathbf{v}$), $\mathbf{u}=\mathbf{t}$ as a theorem

[29] A similar remark applies for Φ_2 and (with E instead of E′) for Φ_1 too.

of Φ_4. Finally, suppose that $\mathbf{t}=\mathbf{u}$ and $\mathbf{T}$ are theorems of Φ_4 and that there is an occurrence of $\mathbf{t}$ in $\mathbf{T}$. Then $\mathbf{T}$ has one of the forms $\mathbf{t}_1\mathbf{t}\mathbf{t}_2=\mathbf{t}_3$ and $\mathbf{t}_3=\mathbf{t}_1\mathbf{t}\mathbf{t}_2$ with terms $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ of Φ_4. By what has been proved already, $\mathbf{t}_1\mathbf{t}\mathbf{t}_2=\mathbf{t}_3$ is in any case a theorem of Φ_4 and the same holds for $\mathbf{t}_1\mathbf{t}\mathbf{t}_2=\mathbf{t}_1\mathbf{u}\mathbf{t}_2$ too. Hence, by (iv), $\mathbf{t}_1\mathbf{u}\mathbf{t}_2=\mathbf{t}_3$ and $\mathbf{t}_3=\mathbf{t}_1\mathbf{u}\mathbf{t}_2$ are theorems of Φ_4; and, choosing $\mathbf{t}_1, \mathbf{t}_2$ and $\mathbf{t}_3$ appropriately, one of them is the result of replacement of the particular occurrence in question of $\mathbf{t}$ in $\mathbf{T}$ by $\mathbf{u}$.

As a corollary of lemma 7, we see that each theorem of Φ_3 is a theorem of Φ_4. Indeed, the axioms of Φ_3 are theorems of Φ_4 (for they are either axioms of Φ_4 or of the form $\mathbf{t}=\mathbf{t}$ with a term $\mathbf{t}$ of Φ_4); and the rules of inference of Φ_3 hold in Φ_4 too.

Now, we shall show that, in spite of allowing "meaningless" words (i. e. those which are no terms of Φ_3) as terms of Φ_4 and of admitting the new rules of inference, the class of the theorems did not change "essentially". For this purpose, we shall analyse the structure of the terms of Φ_4 as to their "meaningful" components.

A sequence of consecutive symbols of a term of Φ_4 is called a *sub-term* of it. A particular occurrence of a functor G in a term $\mathbf{t}$ of Φ_4 is called *saturated* (or G is called saturated at that occurrence), if there is a sub-term $\mathbf{u}$ of $\mathbf{t}$ beginning with that occurrence of G which is a term of Φ_3. In this case, $\mathbf{u}$ is uniquely determined, for of two different sub-terms of $\mathbf{t}$ beginning with the same occurrence of G, one is a proper section of the other, and, by lemma 3, a proper section of a term of Φ_3 cannot be a term of Φ_3. If a particular occurrence of G in $\mathbf{t}$ is not saturated, it is called *unsaturated* (or G is called unsaturated at that occurrence).

Now, we can describe the structure of the terms of Φ_4 by

LEMMA 8. *Each term* $\mathbf{t}$ *of* Φ_4 *can be decomposed in one and only one way into a product* (i. e., juxtaposition) *of "components" each of which is either a term of* Φ_3 *or an unsaturated occurrence of a functor.* (We call this decomposition the "standard decomposition" of $\mathbf{t}$.)

Indeed, this holds for a term of Φ_4 containing a single symbol (for it is either a term of Φ_3, viz. if it is 0 or one of the variables, or a functor in which case it is unsaturated). Suppose, the statement of the lemma holds for terms of Φ_4 containing less than r symbols ($r=2, 3, \ldots$) and let $\mathbf{t}$ be a term of Φ_4 containing r symbols. If the first symbol α of $\mathbf{t}$ is 0 or a variable, or an unsaturated functor, then $\mathbf{t}$ is $\alpha\mathbf{u}$ where $\mathbf{u}$ is a term of Φ_4 containing $r-1$ symbols; and $\mathbf{t}$ can be decomposed, as required, into α and the components of $\mathbf{u}$. This is the only such decomposition of $\mathbf{t}$ for there is no term of Φ_3 beginning with α and the standard decomposition of $\mathbf{u}$ is unique. If the first symbol of $\mathbf{t}$ is a saturated functor G, i. e. $\mathbf{t}$ is $\mathbf{u}\mathbf{v}$ where $\mathbf{u}$ is a term of Φ_3 beginning with G and $\mathbf{v}$ a term of Φ_4, containing less than r symbols, then

$\mathbf{t}$ can be decomposed as required into $\mathbf{u}$ and the components of $\mathbf{v}$. This is the only such decomposition, for by lemma 3, neither a proper section of $\mathbf{u}$ nor a term of Φ_4 having $\mathbf{u}$ as its proper section can be a term of Φ_3, and the standard decomposition of $\mathbf{v}$ is unique.

By means of lemma 8, we can prove the following necessary (and as easily seen, but not used in the sequel, sufficient) condition of being a theorem of Φ_4.

LEMMA 9. *If* $\mathbf{t}=\mathbf{u}$ *is a theorem of* Φ_4, *then* $\mathbf{t}$ *and* $\mathbf{u}$ *have the same number* r *of components. If* $\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_r$ *and* $\mathbf{u}_1\mathbf{u}_2\ldots\mathbf{u}_r$ *are the standard decompositions of* $\mathbf{t}$ *and* $\mathbf{u}$, *respectively, then, for* $i=1, 2, \ldots, r$, $\mathbf{t}_i$ *and* $\mathbf{u}_i$ *are either both occurrences of the same unsaturated functor, or both are terms of* Φ_3 *and* $\mathbf{t}_i=\mathbf{u}_i$ *is a theorem of* Φ_3.

Indeed, this holds for the axioms of Φ_4, for, in case of the axiom $e=e$, e has no components, and, in case of the other axioms, each side of them has a single component and they are axioms of Φ_3. Supposing, the statement of the lemma holds for a theorem $\mathbf{t}=\mathbf{u}$ of Φ_4, let $\mathbf{t}'$ and $\mathbf{u}'$ denote the terms formed of $\mathbf{t}$ and $\mathbf{u}$, respectively, by substituting throughout 0 or Fx (the same in both cases) for a variable x. Let $\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_r$ and $\mathbf{u}_1\mathbf{u}_2\ldots\mathbf{u}_r$ be the standard decompositions of $\mathbf{t}$ and $\mathbf{u}$, respectively; and, for $i=1, 2, \ldots, r$, let $\mathbf{t}_i'$ and $\mathbf{u}_i'$ denote the terms of Φ_4 formed of $\mathbf{t}_i$ and $\mathbf{u}_i$, respectively, by substituting throughout 0 or Fx, respectively, for x. Then, each of $\mathbf{t}_1', \mathbf{t}_2', \ldots, \mathbf{t}_r', \mathbf{u}_1', \mathbf{u}_2', \ldots, \mathbf{u}_r'$ is either an unsaturated functor, or a term of Φ_3. Indeed, a functor does not change and a term of Φ_3 does not cease being a term of Φ_3 by substitution of 0 or Fx for x. Also, an unsaturated functor cannot become saturated by this substitution. For, if an occurrence of the functor G, viz. one of $\mathbf{t}_1', \mathbf{t}_2', \ldots, \mathbf{t}_r'$ or of $\mathbf{u}_1', \mathbf{u}_2', \ldots, \mathbf{u}_r'$, would be saturated in $\mathbf{t}'$ or $\mathbf{u}'$ respectively, then there would be a sub-term of $\mathbf{t}'$ or $\mathbf{u}'$, respectively, beginning with the said occurrence of G which is a term of Φ_3. However, we could regain a sub-term of $\mathbf{t}$ or $\mathbf{u}$, respectively, beginning with the corresponding occurrence of G, thus, with one of $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_r$, or of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$, respectively, by means of replacing some of the 0's or Fx's by x according as 0 or Fx has been substituted for x. Now, as remarked above (see footnote [14]), we get by this replacement a term of Φ_3 again which is impossible, for the said occurrence of G in $\mathbf{t}$ or $\mathbf{u}$, respectively, is an unsaturated one. Hence $\mathbf{t}_1'\mathbf{t}_2'\ldots\mathbf{t}_r'$ and $\mathbf{u}_1'\mathbf{u}_2'\ldots\mathbf{u}_r'$ are the standard decompositions of $\mathbf{t}'$ and $\mathbf{u}'$, respectively. Now, if $\mathbf{t}_i$ and $\mathbf{u}_i$ are occurrences of some unsaturated functor, then the same holds for $\mathbf{t}_i'$ and $\mathbf{u}_i'$ too; if $\mathbf{t}_i$ and $\mathbf{u}_i$ are terms of Φ_3 such that $\mathbf{t}_i=\mathbf{u}_i$ is a theorem of Φ_3, then the same holds for $\mathbf{t}_i'=\mathbf{u}_i'$ too. Hence, the statement of the lemma holds for the theorem $\mathbf{t}'=\mathbf{u}'$ of Φ_4 too.

We have next to prove that if the statement of the lemma holds for the theorem $\mathbf{t}=\mathbf{u}$ of Φ_4, then, for an arbitrary symbol α of Φ_4 except $=$, it holds for the theorems $\alpha\mathbf{t}=\alpha\mathbf{u}$ as well as $\mathbf{t}\alpha=\mathbf{u}\alpha$ of Φ_4 too. Suppose again that $\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_r$ and $\mathbf{u}_1\mathbf{u}_2\ldots\mathbf{u}_r$ are the standard decompositions of $\mathbf{t}$ and $\mathbf{u}$, respectively. First we treat $\alpha\mathbf{t}=\alpha\mathbf{u}$. If α is 0 or a variable, or else a functor unsaturated in both $\alpha\mathbf{t}$ and $\alpha\mathbf{u}$, then our statement is trivial for then α is the first component of both $\alpha\mathbf{t}$ and $\alpha\mathbf{u}$ and their further components are $\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}_r$, and $\mathbf{u}_2,\mathbf{u}_2,\ldots,\mathbf{u}_r$, respectively. Suppose, α is a functor G and let s be the number of arguments of G. In $\alpha\mathbf{t}$, that is $G\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_r$, the first G is saturated if and only if $r\geqq s$ and $\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}_s$ are terms of Φ_3. Indeed, in this case, the sub-term $G\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_s$ of $\alpha\mathbf{t}$ is a term of Φ_3; conversely, if the first G is saturated in $G\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_r$, i.e., if there is a sub-term of $G\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_r$ beginning with the first G, thus of the form $G\mathbf{t}'_1\mathbf{t}'_s\ldots\mathbf{t}'_s$ with some terms $\mathbf{t}'_1,\mathbf{t}'_2,\ldots,\mathbf{t}'_s$ of Φ_2, then $G\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_r$ is $G\mathbf{t}'_1\mathbf{t}'_2\ldots\mathbf{t}'_s\mathbf{v}$ with some term $\mathbf{v}$ of Φ_4, thus $\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_r$ is $\mathbf{t}'_1\mathbf{t}'_2\ldots\mathbf{t}'_s\mathbf{v}$, hence we have $r\geqq s$ and $\mathbf{t}_1$ is $\mathbf{t}'_1$, $\mathbf{t}_2$ is $\mathbf{t}'_2,\ldots,\mathbf{t}_s$ is $\mathbf{t}'_s$ for the (unique) standard decomposition of $\mathbf{t}'_1\mathbf{t}'_2\ldots\mathbf{t}'_s\mathbf{v}$ is $\mathbf{t}'_1\mathbf{t}'_2\ldots\mathbf{t}'_s$ followed by the standard decomposition of $\mathbf{v}$. Taking into account that, for $i=1,2,\ldots,r$, $\mathbf{t}_i$ and $\mathbf{u}_i$ are either both terms of Φ_3 or neither, we see that the first G is either saturated in both $\alpha\mathbf{t}$ and $\alpha\mathbf{u}$, or unsaturated in both. The second case being settled already, let us suppose that the first G is saturated in both $\alpha\mathbf{t}$ and $\alpha\mathbf{u}$, i. e. we have $r\geqq s$ and $\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}_s,\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_s$ are terms of Φ_3. Then, the components of $\mathbf{t}$ and $\mathbf{u}$ are $G\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_s$, $\mathbf{t}_{s+1},\mathbf{t}_{s+2},\ldots,\mathbf{t}_r$, and $G\mathbf{u}_1\mathbf{u}_2\ldots\mathbf{u}_s,\mathbf{u}_{s+1},\mathbf{u}_{s+2},\ldots,\mathbf{u}_r$, respectively. We have only to show that $G\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_s=G\mathbf{u}_1\mathbf{u}_2\ldots\mathbf{u}_s$ is a theorem of Φ_3, for if $i=s+1,s+2,\ldots,r$, then either $\mathbf{t}_i$ and $\mathbf{u}_i$ are the same unsaturated functor or $\mathbf{t}_i=\mathbf{u}_i$ is a theorem of Φ_3. Now, $G\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_s=G\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_s$ and $\mathbf{t}_1=\mathbf{u}_1$, $\mathbf{t}_2=\mathbf{u}_2,\ldots,\mathbf{t}_s=\mathbf{u}_s$ being theorems of Φ_3, the same holds for $G\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_s=$ $=G\mathbf{u}_1\mathbf{u}_2\ldots\mathbf{u}_s$ too, for it can be obtained from $G\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_s=G\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_s$ by replacement of the occurrences of $\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}_s$ on the right-hand side by $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_s$, respectively.

Now, let us examine $\mathbf{t}\alpha=\mathbf{u}\alpha$ where $\mathbf{t}=\mathbf{u}$ is a theorem of Φ_3 and α an arbitrary symbol of Φ_4 except $=$. If α is a functor, then it is obviously unsaturated, thus, the statement of the lemma is trivial again, the components of $\mathbf{t}\alpha$ and $\mathbf{u}\alpha$ being those of $\mathbf{t}$ and $\mathbf{u}$, respectively, and α. The same holds if α is 0 or a variable, provided no functor which is unsaturated in $\mathbf{t}$ or $\mathbf{u}$ becomes saturated by subjoining α. Suppose, some components of $\mathbf{t}$, say, which are unsaturated functors, are saturated in $\mathbf{t}\alpha$ and let $\mathbf{t}_i$ be the first of those components (from the left to the right). Then $\mathbf{t}_i$ is a functor G and $G\mathbf{t}_{i+1}\mathbf{t}_{i+2}\ldots\mathbf{t}_r\alpha$ is a term of Φ_3 (for this is the only sub-term of $\mathbf{t}\alpha$ beginning with $\mathbf{t}_i$ which is not a sub-term of $\mathbf{t}$). Replacing the term $\mathbf{t}_j$ of Φ_3 by the term $\mathbf{u}_j$ of Φ_3 $(j=i+1,i+2,\ldots,r)$, we get another term $G\mathbf{u}_{i+1}\mathbf{u}_{i+2}\ldots\mathbf{u}_r\alpha$ of Φ_3 (see footnote [11]), which is a sub-term of $\mathbf{u}\alpha$ beginning with $\mathbf{t}_i$, i. e. $\mathbf{u}_i$; hence, $\mathbf{u}_i$ is saturated in $\mathbf{u}\alpha$. Also, $\mathbf{u}_i$ is the first component of $\mathbf{u}$ which is an unsaturated functor in $\mathbf{u}$ but saturated in $\mathbf{u}\alpha$, for a similar argument

shows that if $\mathbf{u}_{i'}$ were another such component of $\mathbf{u}$ $(i'<i)$, then $\mathbf{t}_{i'}$ would be a component of $\mathbf{t}$ which is an unsaturated functor in $\mathbf{t}$ but saturated in $\mathbf{t}\alpha$, contrary to the fact that $\mathbf{t}_i$ is the first such component of $\mathbf{t}$. Hence, the components of $\mathbf{t}\alpha$ are $\mathbf{t}_1,\mathbf{t}_2,\ldots,\mathbf{t}_{i-1}$, and $G\mathbf{t}_{i+1}\mathbf{t}_{i+2}\ldots\mathbf{t}_r\alpha$, and those of $\mathbf{u}\alpha$ are $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_{i-1}$, and $G\mathbf{u}_{i+1}\mathbf{u}_{i+2}\ldots\mathbf{u}_r\alpha$. Now, $G\mathbf{t}_{i+1}\mathbf{t}_{i+2}\ldots\mathbf{t}_r\alpha=G\mathbf{u}_{i+1}\mathbf{u}_{i+2}\ldots\mathbf{u}_r\alpha$ is a theorem of Φ_3, for it can be obtained from the theorem $G\mathbf{t}_{i+1}\mathbf{t}_{i+2}\ldots\mathbf{t}_r\alpha=$ $=G\mathbf{t}_{i+1}\mathbf{t}_{i+2}\ldots\mathbf{t}_r\alpha$ of Φ_3 by replacement of some of the $\mathbf{t}_j$, $j=i+1,i+2,\ldots,r$, viz. those which are terms of Φ_3, by the corresponding $\mathbf{u}_j$ and, for these j, $\mathbf{t}_j=\mathbf{u}_j$ is a theorem of Φ_3. Hence, the statement of the lemma holds for $\mathbf{t}\alpha=\mathbf{u}\alpha$ too.

We have still to prove that if the statement of the lemma holds for the theorems $\mathbf{t}=\mathbf{u}$ and $\mathbf{t}=\mathbf{v}$ of Φ_4, then it holds for $\mathbf{u}=\mathbf{v}$ too. Let $\mathbf{t}_1\mathbf{t}_2\ldots\mathbf{t}_p$, $\mathbf{u}_1\mathbf{u}_2\ldots\mathbf{u}_q$ and $\mathbf{v}_1\mathbf{v}_2\ldots\mathbf{v}_r$ be the standard decompositions of $\mathbf{t},\mathbf{u}$ and $\mathbf{v}$, respectively. By the hypothesis, we have $p=q$ and $p=r$, thus $q=r$. Also, if, for some $i=1,2,\ldots,r$, $\mathbf{t}_i$ is an unsaturated functor in $\mathbf{t}$, then $\mathbf{u}_i$ and $\mathbf{v}_i$ are unsaturated functors in $\mathbf{u}$ and $\mathbf{v}$, respectively, and $\mathbf{u}_i$ is identical with $\mathbf{v}_i$, for both are identical with $\mathbf{t}_i$. If $\mathbf{t}_i$ is a term of Φ_3, the same holds for $\mathbf{u}_i$ and $\mathbf{v}_i$ too, and $\mathbf{t}_i=\mathbf{u}_i,\mathbf{t}_i=\mathbf{v}_i$ are theorems of Φ_3; hence, the same holds for $\mathbf{u}_i=\mathbf{v}_i$ too, for it can be obtained from $\mathbf{t}_i=\mathbf{v}_i$ by replacement ot the occurrence of $\mathbf{t}_i$ on the left-hand side by $\mathbf{u}_i$. Hence, the statement of the lemma holds for the theorem $\mathbf{u}=\mathbf{v}$ of Φ_4 too, which completes the proof of the lemma.

As a corollary, we see that if $\mathbf{t}=\mathbf{u}$ is a formula of Φ_3 and at the same time a theorem of Φ_4, then it is a theorem of Φ_3. Indeed, in this case, $\mathbf{t}$ and $\mathbf{u}$ are terms of Φ_3, thus, the only components of themselves.

In particular, by lemmas 7 and 9, we see that $UF^k0^2=0$ is a theorem of Φ_4 if and only if it is a theorem of Φ_3, that is, if we have $R(k,n)=0$ for some non-negative integer n. Hence, there is no algorithm by means of which, given any non-negative integer k, we could decide if $UF^k0^2=0$ is a theorem of Φ_4.

7. The formal system Φ_4 is no particular case of Φ_S because of the rule of inference (ii), i. e. of the rule of substitution which is alien from Φ_S. However, we show that it is possible to dispense with this rule by introducing some new symbols and axioms. For this purpose, let us define a formal system Φ_5 as follows. *Symbols* of Φ_5 are those of Φ_4, i. e. 0, the variables $x,y,\ldots,w$, the functors $F,\ldots,R,U,V$, the equality sign $=$; and, in addition, to each of the variables $x,y,\ldots,w$, two new [30] symbols, $x_0,x_+,y_0,y_+,\ldots,w_0,w_+$, called *substitutors*, and a single new symbol a, called *absorptor*. *Terms* of Φ_5 are arbitrary finite (possibly empty) sequences of symbols of Φ_5 except $=$,

[30] I. e., we suppose that $x_0,x_+,y_0,y_+,\ldots,w_0,w_+$ are different from the symbols $0,x,y,\ldots,w,F,\ldots,R,U,V$ as well as from each other.

i. e. words formed of the letters of the alphabet $\{0, x, x_0, x_+, y, y_0, y_+, \ldots, \ldots, w, w_0, w_+, F, \ldots, R, U, V\}$. *Formulae* of Φ_5 are sequences of symbols of the form $\mathbf{t}=\mathbf{u}$ where $\mathbf{t}$ and $\mathbf{u}$ are terms of Φ_5; i. e., relations between such words. *Axioms* of Φ_5 are those of Φ_4 (i. e., the simplifications of the equations belonging to E′ as well as $e=e$, e denoting the empty word); and, in addition, for each variable[31] x the relations

(14) $$x_0\, x = 0 x_0,$$

(15) $$x_+ x = F x x_+,$$

(16) $$x_0\, a = e,$$

(17) $$x_+ a = e,$$

(18) $$x_0\, 0 = 0 x_0,$$

(19) $$x_+ 0 = 0 x_+,$$

and, for each variable[32] y different from x and for each functor G,

(20) $$x_0\, y = y x_0,$$

(21) $$x_+\, y = y x_+,$$

(22) $$x_0\, G = G x_0,$$

(23) $$x_+ G = G x_+.$$

Theorems of Φ_5 are (i) the axioms of Φ_5; (ii) $\alpha\mathbf{t}=\alpha\mathbf{u}$ and $\mathbf{t}\alpha=\mathbf{u}\alpha$ if $\mathbf{t}=\mathbf{u}$ is any theorem of Φ_5 and α any symbol of Φ_5 except $=$; (iii) $\mathbf{u}=\mathbf{v}$ if $\mathbf{t}=\mathbf{u}$ and $\mathbf{t}=\mathbf{v}$ are theorems of Φ_5; (iv) nothing else.

First we prove that the new symbols $a, x_0, x_+, y_0, y_+, \ldots, w_0, w_+$ together with the new axioms (14) to (23) suffice to replace the missing rule of substitution; i. e., we prove

LEMMA 10. *Each theorem of Φ_4 is a theorem of Φ_5.*

To prove this lemma we observe that lemma 7 holds for Φ_5 instead of Φ_4 too, for the rule of substitution, missing in Φ_5, has not been used in its proof.

Now, the statement of lemma 10 holds for the axioms of Φ_4 for they are axioms of Φ_5. Suppose, it holds for a theorem $\mathbf{T}$ of Φ_4; then it holds for the theorems $\mathbf{U}$ and $\mathbf{V}$ of Φ_4 obtained by substitution of 0 and Fx, respectively, for a variable x throughout $\mathbf{T}$. Indeed, suppose $\mathbf{T}$ is $\alpha_1\alpha_2\cdots\alpha_r = \beta_1\beta_2\cdots\beta_s$, where $\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \beta_2, \ldots, \beta_s$ are symbols of Φ_5, except $=$. Then $\mathbf{U}$ is $\alpha'_1\alpha'_2\cdots\alpha'_r=\beta'_1\beta'_2\cdots\beta'_s$ and $\mathbf{V}$ is $\alpha''_1\alpha''_2\cdots\alpha''_r=\beta''_1\beta''_2\cdots\beta''_s$, α'_i denoting 0 or α_i and α''_i denoting Fx or α_i according as α_i is x or not $(i=1,2,\ldots,r)$, and β'_i denoting 0 or β_i, β''_i denoting Fx or β_i according as β_i is x or not $(i=1,2,\ldots,s)$. Now, by (ii),

(24) $$x_0\,\alpha_1\alpha_2\cdots\alpha_r\, a = x_0\,\beta_1\beta_2\cdots\beta_s\, a$$

and

(25) $$x_+\,\alpha_1\alpha_2\cdots\alpha_r\, a = x_+\,\beta_1\beta_2\cdots\beta_s\, a$$

are theorems of Φ_5. By (14), (18), (20), and (22), the same holds for $x_0\alpha_i=\alpha'_i x_0$ $(i=1,2,\ldots,r)$, $x_0\beta_i=\beta'_i x_0$ $(i=1,2,\ldots,s)$, and by (15), (19), (21), and (23), for $x_+\alpha_i=\alpha''_i x_+$ $(i=1,2,\ldots,r)$, $x_+\beta_i=\beta''_i x_+$ $(i=1,2,\ldots,s)$ too. Hence, by lemma 7, last assertion, we can replace $x_0\alpha_1$ by $\alpha'_1 x_0$, then $x_0\alpha_2$ by $\alpha'_2 x_0, \ldots,$ finally $x_0\alpha_r$ by $\alpha'_r x_0$ on the left-hand side of (24), and $x_0\beta_1$ by $\beta'_1 x_0$, then $x_0\beta_2$ by $\beta'_2 x_0, \ldots,$ finally $x_0\beta_s$ by $\beta_s x_0$ on the right-hand side of (24), and, similarly, $x_+\alpha_1$ by $\alpha''_1 x_+$, then $x_+\alpha_2$ by $\alpha''_2 x_+, \ldots,$ finally $x_+\alpha_r$ by $\alpha''_r x_+$ on the left-hand side of (25), and $x_+\beta_1$ by $\beta''_1 x_+$, then $x_+\beta_2$ by $\beta''_2 x_+, \ldots,$ finally $x_+\beta_s$ by $\beta''_s x_+$ on the right-hand side of (25). Thus, we see that

$$\alpha'_1 x'_2 \cdots \alpha'_r x_0\, a = \beta'_1\beta'_2\cdots\beta'_s x_0\, a$$

and

$$\alpha''_1\alpha''_2\cdots\alpha''_r x_+\, a = \beta''_1\beta''_2\cdots\beta''_s x_+\, a$$

are theorems of Φ_5. Here, by (16) and (17), we can replace $x_0 a$ and $x_+ a$ by e, i. e. we can omit them, on both sides; thus, we get $\alpha'_1\alpha'_2\cdots\alpha'_r = \beta'_1\beta'_2\cdots\beta'_s$ and $\alpha''_1\alpha''_2\cdots\alpha''_r=\beta''_1\beta''_2\cdots\beta''_s$, i. e. $\mathbf{U}$ and $\mathbf{V}$ as theorems of Φ_5.

The rest of the proof of lemma 10 is quite trivial for the rules of inference (iii) and (iv) of Φ_4 are those (viz. (ii) and (iii)) of Φ_5 too.

Now, we shall examine the relation of the theorems of Φ_5 to those of Φ_4 in order to show that a theorem of Φ_5 which is a formula of Φ_4 is a theorem of Φ_4.

We call a sequence of consecutive symbols of a term of Φ_5 a *sub-term* of it. We call a term of Φ_5 *reduced* if it does not contain any occurrence of a substitutor preceding an occurrence of a symbol which is no substitutor. To each term $\mathbf{t}$ of Φ_5, we attach a reduced term $\bar{\mathbf{t}}$, called the *reductum* of $\mathbf{t}$, by the following *reduction process*. If $\mathbf{t}$ is reduced, then $\bar{\mathbf{t}}$ is $\mathbf{t}$. If $\mathbf{t}$ is not reduced, let α be the last occurrence of a substitutor in $\mathbf{t}$ preceding an occurrence of a symbol which is no substitutor. If there is an occurrence of the absorptor a preceded by α, then perform as first "*reduction step*" the substitution indicated by α (i. e. of 0 for x if α is x_0, and of Fx for x if α is x_+, say) throughout the sub-term of $\mathbf{t}$ between α and the next occurrence of a after α, and cancel both α and this occurrence of a. If there is no occurrence of a preceded by α, then perform as first reduction step the substitution indicated by α throughout the sub-term of $\mathbf{t}$ between α and the next occurrence of a substitutor after α, or between α and the end of $\mathbf{t}$, if such occurrence does not exist, and transplant α to the place immediately after this sub-term. By means of this reduction step, we obtain another term $\mathbf{u}$

[31] "For *each* variable x" has been used here in an analogous sense as "for a variable x" (see footnote [14]).

[32] I. e., y can be replaced by any variable which is different from the variable by which x has been replaced.

of Φ_5; and the reductum of $\mathbf{t}$ is, by definition, the reductum of $\mathbf{u}$. This definition has a sense, for in $\mathbf{u}$, the number of the occurrences of substitutors, preceding an occurrence of a symbol which is not a substitutor, is one less than in $\mathbf{t}$ so that the iterated application of the said reduction step comes to an end in a finite number of steps.

A reduced term $\mathbf{t}$ has obviously the form $\mathbf{t}_0 a \mathbf{t}_1 a \mathbf{t}_2 \ldots a \mathbf{t}_r \mathbf{v}$ where $\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_r$ are terms of Φ_4 (possibly the empty word e) and $\mathbf{v}$ is a finite (possibly empty) sequence of substitutors. We call $\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_r$ the *intervals* and $\mathbf{v}$ the *suffix* of $\mathbf{t}$. For a non-reduced term $\mathbf{t}$, we call the intervals and the suffix of its reductum also the intervals and the suffix, respectively, of $\mathbf{t}$.

Now, the relation between the theorems of Φ_5 and those of Φ_4 is displayed in the following lemma which gives a necessary (and, as easily seen, but not used in the sequel, sufficient) condition of being a theorem of Φ_5.

LEMMA 11. *If* $\mathbf{t} = \mathbf{u}$ *is a theorem of* Φ_5, *then* $\mathbf{t}$ *and* $\mathbf{u}$ *have the same number* r *of intervals. If* $\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_r$ *and* $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ *are in succession the intervals of* $\mathbf{t}$ *and* $\mathbf{u}$, *respectively, then for* $i = 0, 1, 2, \ldots, r$, $\mathbf{t}_i = \mathbf{u}_i$ *is a theorem of* Φ_4. *Also, the suffixes of* $\mathbf{t}$ *and* $\mathbf{u}$ *are identical.*

Indeed, the statement of the lemma holds for the axioms of Φ_5, for a term of Φ_4 is its only interval and has an empty suffix, and, for any variable x, $x_0 x, x_+ x, x_0 a, x_+ a, x_0 \alpha, x_+ \alpha$ (for any symbol α of Φ_5 except x, a, and $=$) have the reducta $0x_0, Fxx_+, e, e, \alpha x_0, \alpha x_+$, respectively. Suppose the statement of the lemma holds for a theorem $\mathbf{t} = \mathbf{u}$ of Φ_5. Then, for any symbol α of Φ_5 except $=$, it holds for $\alpha\mathbf{t} = \alpha\mathbf{u}$ too. Indeed, if α is no substitutor, then the result of each reduction step, performed on $\alpha\mathbf{t}$, is the same as that of the corresponding reduction step, performed on $\mathbf{t}$, except that α is prefixed; hence, if α is not the absorptor a either, the first interval of $\alpha\mathbf{t}$ is $\alpha\mathbf{t}_0$, the others are, in succession, $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_r$, ($\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_r$ denoting the intervals of $\mathbf{t}$) and the suffix of $\alpha\mathbf{t}$ is the same as that of $\mathbf{t}$. Analogously, the intervals of $\alpha\mathbf{u}$ are, in succession, $\alpha\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$, $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ denoting the intervals of $\mathbf{u}$, and the suffix of $\alpha\mathbf{u}$ is the same as that of $\mathbf{u}$. Now, $\mathbf{t}_1 = \mathbf{u}_1, \mathbf{t}_2 = \mathbf{u}_2, \ldots, \mathbf{t}_r = \mathbf{u}_r$ are, by the hypothesis, theorems of Φ_4; and the same holds for $\alpha\mathbf{t}_0 = \alpha\mathbf{u}_0$ too, for $\mathbf{t}_0 = \mathbf{u}_0$ is, by the hypothesis, a theorem of Φ_4. Also, the suffixes of $\alpha\mathbf{t}$ and $\alpha\mathbf{u}$ are, by the hypothesis, the same; hence, the statement of the lemma holds for the theorem $\alpha\mathbf{t} = \alpha\mathbf{u}$ of Φ_5 too. If α is the absorptor a, then the number of the intervals of $\alpha\mathbf{t}$ and $\alpha\mathbf{u}$ is one more than that of the intervals of $\mathbf{t}$ and $\mathbf{u}$, respectively, the first interval of both $\alpha\mathbf{t}$ and $\alpha\mathbf{u}$ being e, and the others, in succession, the same as those of $\mathbf{t}$ and $\mathbf{u}$, respectively; also, the suffixes of $\alpha\mathbf{t}$ and $\alpha\mathbf{u}$ are the same as those of $\mathbf{t}$ and $\mathbf{u}$, respectively. Hence, the statement of the lemma holds for the theorem $\alpha\mathbf{t} = \alpha\mathbf{u}$ of Φ_5 too. If α is a substitutor, x_0 or x_+, say, then the

result of each but the last reduction step, performed on $\alpha\mathbf{t}$, differs but in a prefixed α from that of the corresponding reduction step, performed on $\mathbf{t}$. As to the last reduction step, its effect is substitution of 0 or Fx, respectively, for x in the first interval of $\mathbf{t}$ and cancellation of the first occurrence of a in the reductum of $\mathbf{t}$, if $\mathbf{t}$ has more than one interval; and substitution of 0 or Fx, respectively, for x in the single interval of $\mathbf{t}$ and prefixing of α to the suffix of $\mathbf{t}$ if there is but one interval of $\mathbf{t}$. Hence, the intervals of $\alpha\mathbf{t}$ are $\mathbf{t}_0'\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \ldots, \mathbf{t}_r$ and similarly, those of $\alpha\mathbf{u}$ are $\mathbf{u}_0'\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_r$ in the first case, $\mathbf{t}_0'$ and $\mathbf{u}_0'$, respectively, in the second case, $\mathbf{t}_0'$ and $\mathbf{u}_0'$ denoting the result of substitution of 0 or of Fx for x, according as α is x_0 or x_+, in $\mathbf{t}_0$ and $\mathbf{u}_0$, respectively. Now, $\mathbf{t}_0 = \mathbf{u}_0, \mathbf{t}_1 = \mathbf{u}_1, \mathbf{t}_2 = \mathbf{u}_2, \ldots, \mathbf{t}_r = \mathbf{u}_r$ being theorems of Φ_4, the same holds for $\mathbf{t}_0' = \mathbf{u}_0'$, and, by lemma 7, for $\mathbf{t}_0'\mathbf{t}_1 = \mathbf{u}_0'\mathbf{u}_1$ too (for $\mathbf{t}_0'\mathbf{t}_1 = \mathbf{t}_0'\mathbf{t}_1$ is a theorem of Φ_4 and by replacement of $\mathbf{t}_0'$ by $\mathbf{u}_0'$ on the right-hand side we get $\mathbf{t}_0'\mathbf{t}_1 = \mathbf{u}_0'\mathbf{t}_1$, then, by replacement of $\mathbf{t}_1$ by $\mathbf{u}_1$ on the right-hand side, $\mathbf{t}_0'\mathbf{t}_1 = \mathbf{u}_0'\mathbf{u}_1$). Also, the suffixes of $\alpha\mathbf{t}$ and $\alpha\mathbf{u}$ are the same in both cases; hence, the statement of lemma 11 holds for the theorem $\alpha\mathbf{t} = \alpha\mathbf{u}$ of Φ_5 also if α is a substitutor.

Suppose again, the statement of the lemma holds for a theorem $\mathbf{t} = \mathbf{u}$ of Φ_5; we have next to prove that, for any symbol α of Φ_5 except $=$, it holds for the theorem $\mathbf{t}\alpha = \mathbf{u}\alpha$ of Φ_5 too. If α is a substitutor, then the result of each reduction step, performed on $\mathbf{t}\alpha$, differs from that of the corresponding reduction step, performed on $\mathbf{t}$, but in a suffixed α to the suffix of $\mathbf{t}$; hence, the intervals of $\mathbf{t}\alpha$ are the [same as those of $\mathbf{t}$ and the suffix of $\mathbf{t}\alpha$ differs from that of $\mathbf{t}$ but in a suffixed α. An analogous assertion holding for $\mathbf{u}\alpha$, the statement of the lemma holds for the theorem $\mathbf{t}\alpha = \mathbf{u}\alpha$ of Φ_5 too. If α is the absorptor a, then its effect in the result of the successive reduction steps is suffixing a to the result of the corresponding reduction steps, performed on $\mathbf{t}$, so far as no substitutor appears on the end of this result; afterwards its effect is cancelling the last substitutor appearing on the end of this result. Hence, the intervals of $\mathbf{t}a$ are the same as those of $\mathbf{t}$ and the suffix of $\mathbf{t}a$ is the same as that of $\mathbf{t}$ except that the last substitutor has been cancelled, provided the suffix of $\mathbf{t}$ is not empty; and, if the suffix of $\mathbf{t}$ is empty, the intervals of $\mathbf{t}a$ are the same as those of $\mathbf{t}$ and, in addition, the empty word e, whereas, in this case, the suffix of $\mathbf{t}a$ is empty as well. Hence, the statement of the lemma holds for the theorem $\mathbf{t}a = \mathbf{u}a$ of Φ_5 too. Finally, if α is 0, a variable, or a functor, then its effect in the result of the successive reduction steps is suffixing α to the result of the corresponding reduction steps, performed on $\mathbf{t}$, so far as no substitutor appears on the end of this result; and afterwards, suffixing the result of the substitutions in α, indicated by the substitutors standing on the end of this result, to the last interval of $\mathbf{t}$. Hence, the intervals of $\mathbf{t}\alpha$ are the same as those of $\mathbf{t}$, except that the last one is suffixed by the result α' of substitutions in α, indicated by the suffix of $\mathbf{t}$ (i. e. by α, if α is 0 or a functor; by F^l0 if α is a variable x and there are l occurrences of the substitutor x_+ preceded by the last occurrence of the substitutor x_0 in the suffix of $\mathbf{t}$; and by F^lx if α is a variable x, there are l occurrences of the substitutor x_+ and no occurrence of the substitutor x_0 in the suffix of $\mathbf{t}$); and the suffix of $\mathbf{t}\alpha$ is the same as that of $\mathbf{t}$. An analogous assertion holds for $\mathbf{u}\alpha$; hence, the statement of the lemma holds for the theorem $\mathbf{t}\alpha = \mathbf{u}\alpha$ of Φ_5 in this case too, for, together with $\mathbf{t}_r = \mathbf{u}_r$, $\mathbf{t}_r\alpha' = \mathbf{u}_r\alpha'$ is a theorem of Φ_4.

We have still to prove that if the statement of the lemma holds for the theorems $\mathbf{t} = \mathbf{u}$ and $\mathbf{t} = \mathbf{v}$ of Φ_5, then it holds for $\mathbf{u} = \mathbf{v}$ too. Now, if $\mathbf{t}$ and $\mathbf{u}$ have the same number of intervals, and the same holds for $\mathbf{t}$ and $\mathbf{v}$ too, then also $\mathbf{u}$ and $\mathbf{v}$ have the same number of intervals. Let $r+1$ be the number of these intervals, and let, in succession, $\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_r$ be the intervals of $\mathbf{t}$, $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_r$ those of $\mathbf{u}$ and $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_r$ those of $\mathbf{v}$. Then, by the hypothesis, $\mathbf{t}_0 = \mathbf{u}_0, \mathbf{t}_1 = \mathbf{u}_1, \ldots, \mathbf{t}_r = \mathbf{u}_r$ as well as $\mathbf{t}_0 = \mathbf{v}_0, \mathbf{t}_1 = \mathbf{v}_1, \ldots, \mathbf{t}_r = \mathbf{v}_r$ are theorems of Φ_4; hence, $\mathbf{u}_0 = \mathbf{v}_0, \mathbf{u}_1 = \mathbf{v}_1, \ldots, \mathbf{u}_r = \mathbf{v}_r$ too. Also, the suffixes of $\mathbf{t}$ and $\mathbf{u}$ are identical, and those of $\mathbf{t}$ and $\mathbf{v}$ too; hence, $\mathbf{u}$ and $\mathbf{v}$ have the same suffixes, so that the statement of the lemma holds for the theorem $\mathbf{u} = \mathbf{v}$ of Φ_5 too, which concludes the proof of the lemma.

As a corollary, we see that if $\mathbf{t} = \mathbf{u}$ is a theorem of Φ_5 and, at the same time, a formula of Φ_4, then it is a theorem of Φ_4, for then, $\mathbf{t}$ and $\mathbf{u}$ are the only intervals of themselves. Hence, a formula of Φ_4 is a theorem of Φ_5 if and only if it is a theorem of Φ_4. In particular $UF^k0^2 = 0$ is a theorem of Φ_5 if and only if it is a theorem of Φ_4, i. e. if we have $R(k, n) = 0$ for some non-negative integer n. Hence, there is no algorithm by means of which, given any non-negative integer k, we could decide if $UF^k0^2 = 0$ is a theorem of Φ_5.

8. Now, the formal system Φ_5 is a particular case of Φ_{S}. Indeed, as immediately seen, it is the formal system Φ_{S} belonging to the particular system S of relations formed of the simplifications of the equations E' by subjoining the relations (14) to (23) (for each variable x figuring in at least one equation of E' as well as for each such variable y different from x, and for each functor G figuring in at least one equation of E'). Hence, *there is no algorithm by means of which, given any relation in* $\{0, x, x_0, x_+, y, y_0, y_+, \ldots, w, w_0, w_+, F, \ldots, R, U, V\}$, *we could decide if it is a consequence of this particular system* S *of relations; i. e., the word problem for associative systems, relative to this particular system* S *of relations, is unsolvable by any algorithm.* Thus, the Markov—Post theorem has been proved.

9. By a slight modification of the proof method, we can prove some further results of MARKOV concerning the impossibility of some algorithms for associative systems. Indeed, for the word problem for associative systems, relative to the system of relations in $\{0, x, x_0, x_+, y, y_0, y_+, \ldots, w, w_0, w_+, F, \ldots, R\}$, formed of the simplifications of the equations E (instead of E') by subjoining the relations (14) to (23) for each pair of different variables x and y as well as for each functor G figuring in at least one equation of E (instead of E'), a solving algorithm can be given by means of methods of the proof theory. On the other hand, the question of the existence of a word $\mathbf{w}$ formed of the letters of the above alphabet, for which $RF^k0\mathbf{w} = 0$ is a consequence of this system of relations, can be proved to be equivalent to the question of the existence of a non-negative integer solution of the equation $R(k, y) = 0$ in y. Hence, *for this system of relations, the word problem can be solved by an algorithm, whereas the (left-hand side) "divisibility problem" cannot be solved by any algorithm.* The existence of a system of relations with essentially[33] the same property has been first proved by MARKOV.[34]

On the other hand, subjoining the relation[35]

$$SxP0y = x$$

to the simplifications of the equations of E, with functors S and P not figuring[36] in the equations of E, and introducing new substitutors indicating, for each pair of variables x and y, substitution of y for x, as well as those indicating, for each functor G, the substitution of $G(x, y, \ldots, v)$ for x (where the number of variables $x, y, \ldots, v$ is the same as that of the arguments of G), we get a system of relations such that in the associative system generated by it every element $\mathbf{t}$ is a right-hand divisor of every element $\mathbf{u}$ (indeed, $S\mathbf{u}P0\mathbf{t} = \mathbf{u}$ is a consequence of that system of relations), whereas RF^k0 is a left-hand divisor of 0 if and only if there is a non-negative integer solution of the equation $R(k, y) = 0$ in y. For this system of relations, the word problem can again be solved by an algorithm; hence, *we have a system of relations, for which both the word problem and the one hand divisibility problem can be solved by an algorithm whereas the other hand divisibility problem not.* The existence of such a system of relations has been first proved also by MARKOV.[37]

To a detailed exposition of the ideas sketched in this section, I shall come back in another publication.

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[33] In MARKOV, right-hand side divisibility has been considered instead of left-hand side one. However, this makes no difference, for from any associative system we get another by reinterpreting ab as ba and, in the latter, right-hand side divisibility means the same as left-hand side divisibility in the former.

[34] See loc. cit. [2], theorem 2.

[35] Its intuitive meaning is $x + 0 . y = x$ (Sxy and Pxy standing for $x + y$ and xy, respectively).

[36] Or figuring "in the sense" of the (two member) addition and (two factor) multiplication functor.

[37] See loc. cit. [4].

НОВОЕ ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ МАРКОВА—POST'A

Л. КАЛЬМАР (Сегед)

(Резюме)

Автор дает новое доказательство теоремы, доказанной одновременного и независимо друг от друга Марковым и Post'ом, согласно которой существует конечно-образованная ассоциативная система, для которой проблема тождества неразрешима (конечным общерекурсивным) алгоритмом. Новое доказательство не опирается на теорию λ-конверсируемости Church'a как доказательства Маркова и Post'a. Вместо этого доказательство основано на теореме Kleene, согласно которой существует двухпеременная функция $R(x, y)$ для которой нет такого альгоритма, с помощью которого, если дано неотрицательное целое число k, можно узнать, разрешимо-ли уравнение $R(k, y) = 0$ по y в неотрицательных целых числах.

Основные идеи доказательства состоят в следующем: Напишем определяющую систему E уравнений функции R в обозначении без скобок Лукашиевича. Вместе с этими уравнениями рассмотрим также уравнения

$$(1) \quad Uxy = VUxFyRxy,$$
$$(2) \quad VUxFy0 = 0,$$
$$(3) \quad VUxFyFz = UxFy,$$

где U и V— два новых функтори и F— знак последования, значит $Fx = x+1$. Легко видеть, что уравнение $R(k, y) = 0$ разрешимо в неотрицателных целых числах тогда и только тогда, если уравнение $UF^k00 = 0$ является следствием вышеуказанной системы E' уравнений с помощью шагов, допустимых по ходу вычисления значений общерекурсивных функций, т. е. с помощью замен какого-либо переменного x на Fx или 0 (повторение которого допускает замену x также на $F0$, $FF0 = F^20$, $FFF0 = F^30$ и т. д.) и с помощью замещения одного члена некоторого уравнения другим. Поставим в соотношение с каждой переменной $x, y, \ldots, w$ два „субститутора" $x_0, x_+, y_0, y_+, \ldots, w_0, w_+$, дальше один общий „абсорптор" a и обширим системы уравнений E' с уравнениями

$$(4) \quad x_0x = 0x_0, \qquad (5) \quad x_+x = Fxx_+,$$
$$(6) \quad x_00 = 0x_0, \qquad (7) \quad x_+0 = 0x_+,$$
$$(8) \quad x_0y = yx_0, \qquad (9) \quad x_+y = yx_+,$$
$$(10) \quad x_0g = gx_0, \qquad (11) \quad x_+g = gx_+,$$
$$(12) \quad x_0a = e, \qquad (13) \quad x_+a = e$$

и

$$(14) \quad e = e,$$

где e— пустое слово, x— любая переменная фигурирующая в E', y— любая такая-же переменная, разнящаяся от x и g— любой функтор фигурирующий в E'. Тогда вместо замена x на Fx или 0 в некотором уравнении можно ее умножить на x_0 соответственно x_+ слева и на a с права и после этого применить уравнения (4), (6), (8), (10), (12) соответственно (5), (7), (9), (11), (13). Из этого следует, что проблема тождества ассоциативной системы, образованной над алфавитом $\{0, x, x_0, x_+, y, y_0, y_+, \ldots, w, w_0, w_+, a, F, \ldots, R, U, V\}$ (где $F, \ldots, R, U, V$— функторы фигурирующие в E') с помощью уравнений E' и уравнений (4)—(14)

неразрешима в алгоритмах. В самом деле, в противном случае существовал бы алгоритм посредством которого для каждого k можно было бы узнать, следует-ли соотношение $UF_k00 = 0$ из вышеупомянутых образующих уравнений.

Метод с легкими модификациями может быть применен для доказательства теоремы Маркова, согласно которой существует конечно-образованная ассоциативная система, проблема тождества которой разрешима, но одна из односторонних проблем делимости неразрешима с помощью алгоритма. По существу тот же метод дает доказательство следующей теоремы Маркова: существует конечно-образованная система с разрешимой с помощью алгоритма проблемой тождества и одной из односторонних проблем делимости но с неразрешимой другой проблемой делимости.