Binary Tomography Reconstruction Method with Perimeter Preserving Regularization

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Abstract. We present an energy-minimization and deterministic type method for binary tomography reconstruction problem. The energy or objective function, beside projection data fidelity term, includes prior knowledge about the solution in the form of regularization terms. Next to the already used smooth regularization we propose a perimeter preserving regularization too. Using the convex-concave regularization framework, the binary reconstruction problem is reformulated to a non-integer optimization problem. The objective function is minimized by a Spectral Projected Gradient based optimization approach. Experimental results show that the proposed approach provides better reconstructions, especially in a case of small number of projections.

1 Introduction

Tomography is imaging by sections. It deals with recovering images from a number of projections. Since it is able to explore inside of object without touching it at all, tomography has a various application areas, for example in medicine, archaeology, geophysics and astrophysics. From the mathematical point of view, the object corresponds to a function and the problem posed is to reconstruct this function from its integrals or sums over subsets of its domain. In general, the tomographic reconstruction problem may be continuous or discrete. In *Discrete Tomography* (DT) the range of the function is a finite set. More details about DT and its applications you can find in [6, 5]. In addition to other, it has a wide range of application in medical imaging, for example within Computer Tomography (CT), Positron Emission Tomography (PET) and Electron Tomography (ET). A special case of DT, which is called *Binary Tomography* (BT), deals with the problem of the reconstruction of a binary image.

In this paper we consider the BT reconstruction problem. It usually leads to solving a large-scale and ill-posed optimization problem. To make the BT problem well-posed, usually an appropriate regularization is used. It is often used the *smooth regularization* whose application is based on the a priori knowledge about the solution, that it is composed of compact regions of zeros and ones. This regularization is widely used, for example in DC based algorithm introduced by Schüle et al. [12, 13, 15] or in Simulated Annealing approach presented in [16].

Assuming that the perimeter of the original (true) object is known, we propose a perimeter preserving regularization. Its effect is to preserve the given object's perimeter during the optimization process. The corresponding optimization problem is reformulated to a convex-constrained and *convex-concave* regularized type problem [12]. The optimization of this problem is based on the *Spectral Projected Gradient* (SPG) method introduced by Birgin, Martínez and Raydan (2000) in [3]. The main motivation for application of SPG lies in the fact that SPG is very efficient for solving large-scale and convex-constrained problems.

The paper is organized as follows. In Section 2 we describe the regularized reconstruction problem. In Section 3 we propose a new method, based on the perimeter preserving regularization. Section 4 contains experimental results and finally, Section 5 is for conclusion remarks.

2 Regularized Reconstruction problem

A main problem in connection with BT refers to the image reconstruction. We consider a BT image reconstruction problem where the imaging process is represented by the following linear system of equations

$$Ax = b, \quad A \in R^{m\times n},\ \ x \in \{0,1\}^n,\ b \in R^m. \tag{1}$$

The matrix A is a so called projection matrix, whose each row corresponds to one projection ray, the corresponding components of vector b contain the detected projection values, while binary-vector x represents the unknown image to be reconstructed. The row entries a_i of A represent the length of the intersection of pixels of the discretized volume and the corresponding projection ray, see Figure 1. Components of the vector x are binary variables indicating the membership of the corresponding pixel to the object: for $x_i = 1$ pixel belongs to the object, while for $x_i = 0$ it does not. In a general case the system (1) is under-determined ($m < n$) and has no unique solution. Therefore, the minimization of the squared projection error

$$\min_{x\in\{0,1\}^n} \|Ax - b\|^2$$

can not lead to a satisfactory result. To avoid this problem an appropriate regularization (or eventually more than one), based on a prior information about the true solution, is needed.

The regularized reconstruction problem can be expressed in the following form

$$\min_{x\in\{0,1\}^n} \|Ax - b\|^2 + \alpha \cdot \Phi(x), \tag{2}$$

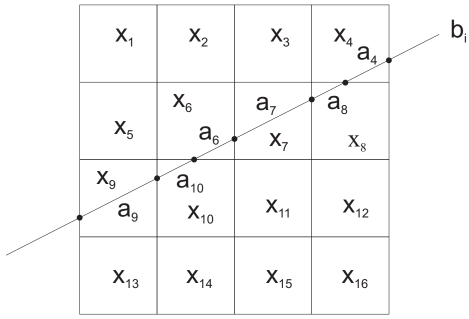


Fig. 1. The discretization model. The corresponding reconstruction problem is represented in a form of a linear system of equations, see (1).

where $\Phi(x)$ is the regularization term and $\alpha > 0$ is its parameter. An often used regularization is a so called smooth regularization and it is defined by,

$$\sum_i \sum_{j \in \mathcal{N}(i)} (x_i - x_j)^2, \tag{3}$$

where $\mathcal{N}(i)$ represents a set of indices of image neighbour pixels right and below from x_i. Using this regularization term in reconstruction algorithms we can enforce the spatial coherency of the solution, that is enforce a solution with possibly compact regions of zeros and ones. Therefore, its application must based on the assumption (a priori knowledge) about compactness of the object of reconstruction. Function (3) is quadratic and convex which make this regularization easy manageable in optimization algorithms.

3 Proposed Method

In order to improve the quality of the reconstruction, beside already used smooth regularization we consider a possibility for inclusion an additional regularization. We consider a situation when the perimeter of the original (true) object is a priori known. This information can be include into the reconstruction process. Following this idea, we propose the *perimeter preserving regularization*, defined by

$$\left(\sum_i \sum_{j \in \mathcal{N}(i)} \chi_\varepsilon(x_i - x_j) - P \right)^2 , \tag{4}$$

where P is the given (true) perimeter, $\mathcal{N}(i)$, like in (3), is a set of neighbour pixels. Function χ_ε is a smooth approximation of the absolute value function $|\cdot|$. It is defined by $\chi_\varepsilon(x) = \sqrt{4\varepsilon^2 + x^2} - 2\varepsilon$, where ε is a small positive number. First term in the brackets estimates the object's perimeter in the image x, [14]. Therefore, the proposed regularization term is a squared distance between true perimeter and object's perimeter in x. Its effect in the optimization process is to penalize the possibly reconstructions where this distance is greater than zero. We note that according to the smoothness of the function χ_ε, the term (4) is differentiable and convex.

The regularized reconstruction problem (2) is expanded with the perimeter preserving term (4) and the proposed reconstruction problem has the following form

$$\min_{x\in\{0,1\}^n} \Gamma_{\alpha,\beta}(x) := \frac{1}{2}\left[\|Ax-b\|^2 + \alpha\cdot\Phi(x) + \beta\cdot\Psi(x)\right], \tag{5}$$

where $\alpha > 0$ and $\beta > 0$ are regularization parameters and

$$\Phi(x) = \sum_i \sum_{j\in\mathcal{N}(i)} (x_i - x_j)^2, \;\; \Psi(x) = \left(\sum_i \sum_{j\in\mathcal{N}(i)} \chi_\varepsilon(x_i - x_j) - P\right)^2.$$

First term in (5) measures the accordance of a solution with a projection data, the role of the second term is to enforce the coherency of the solution while a last term preserves the perimeter of the original object.

We transform the binary optimization problem (5) to the convex-constrained problem defined by

$$\min_{x\in[0,1]^n} \Gamma_{\alpha,\beta}(x) + \mu\cdot x^T(e-x), \quad \mu > 0 \tag{6}$$

where $e = [1,1,1,..,1]^n$. In (6) we relax the feasible set of the optimization to the convex set, $[0,1]^n$ and add a concave regularization term $x^T(e-x)$ with aim to enforce binary solution. Parameter μ regulates the influence of this term. Due to the convex function $\Gamma_{\alpha,\beta}$ and the concave binary enforcing regularization, the problem (6) belongs to the class of convex-concave regularized methods [12]. Soundness of the problem (6) is ensured by the following theorem which establishes an equivalence between the problems (5) and (6).

Theorem 1. *[8, 7] Let E be Lipschitzian on an open set $A \supset [0,1]^n$ and twice continuously differentiable on $[0,1]^n$. Then there exist a $\mu_* \in R$ such that for all $\mu > \mu_*$:*

(i) the integer (binary) programming problem

$$\min_{x\in\{0,1\}^n} E(x)$$

is equivalent with the concave minimization problem

$$\min_{x\in[0,1]^n} E(x) + \tfrac{1}{2}\mu\langle x, e-x\rangle,$$

(ii) the function $E(x) + \frac{1}{2}\mu\langle x, e-x\rangle$ is concave on $[0,1]^n$.

Our strategy is to solve a sequence of optimization problems (6), with gradually increasing μ, which will lead to the binary solution. More precisely, we suggest the following optimization algorithm.

SPG Algorithm for binary tomography.

Parameters: $\epsilon_{in} > 0$; $\epsilon_{out} > 0$; $\delta > 1$; μ_0; $maxit$.
$x^0 = [0.5, 0.5, \dots, 0.5]^T$; $\mu = \mu_0$; $k = 0$;
do
  do
    x^{k+1} from x^k by SPG iterative step; $k = k + 1$;
  until $\| x^{k+1} - x^k \|_\infty > \epsilon_{in}$ and $k < maxit$
  $\mu = \mu + \delta$;
until $\max_i \{\min\{x_i^k, 1 - x_i^k\}\} > \epsilon_{out}$.

The initial configuration is the image with all pixel values equally to 0.5. In each iteration in the outer loop we solve an optimization problem (6) for a fixed binary factor $\mu > 0$ by using the SPG method, described below. By iteratively increasing the value of μ in the outer loop the binary solutions are enforced. The termination criterion for the outer loop, ϵ_{out}, regulates the tolerance for the finally accepted (almost) binary solution.

SPG is a deterministic optimization algorithm, introduced by Birgin, Martínez and Raydan (2000) in [3], for solving a convex-constrained optimization problem

$$\min_{x \in \Omega} f(x),$$

where the feasible region Ω is a closed convex set in $\mathbb{R}^n$. This method combines a *Projected Gradient method* [2] with Grippo type *non-monotone line search* [9] and the *spectral step-length* selection approach [1]. The requirements of the application of the SPG algorithm are: *i*) the objective function, f is defined and has continuous partial derivatives on an open set that contains Ω; *ii*) the projection P_Ω of an arbitrary point $x \in R^n$ onto a set Ω is defined. Global convergence of this method is proved in [3]. The parameters of the algorithm are as follows. Integer $m \geq 1$ is a number of memorized previous objective function values used in line search procedure in each iteration. Parameters $0 < \alpha_{min} < \alpha_{max}$ and $0 < \sigma_1 < \sigma_2 < 1$ have safeguarding function: they keep the spectral step-length, α_k and the trial step-length, λ_{tsp} inside the given limits. Parameter $\gamma \in (0, 1)$ controls the non-monotone objective function decrease condition [9]. Further details about parameters you can find in [4]. Starting from an arbitrary initial iteration $x^0 \in \Omega$, the below computation is iterated until convergence.

SPG iterative step [4]**.**

Given x^k and α_k, the values x^{k+1} and α_{k+1} are computed as follows:

$d^k = P_\Omega(x^k - \alpha_k \nabla f(x^k)) - x^k$;
$f_{max} = \max\{f(x^{k-j}) \mid 0 \leq j \leq \min\{k, m-1\}\}$;
$x^{k+1} = x^k + d^k$; $\delta = \langle \nabla f(x^k), d^k \rangle$; $\lambda_k = 1$;
while $f(x^{k+1}) > (f_{max} + \gamma \lambda_k \delta)$,
  $\lambda_{stl} = -\frac{1}{2}\lambda_k^2 / (f(x^{k+1}) - f(x^k) - \lambda_k \delta)$;
  if $(\lambda_{stl} \geq \sigma_1 \wedge \lambda_{stl} \leq \sigma_2 \lambda)$ **then** $\lambda_k = \lambda_{stl}$;

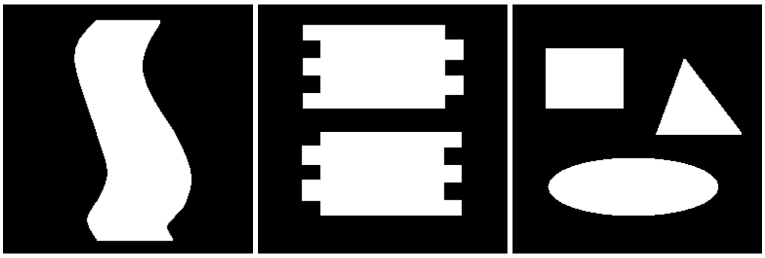
else $\lambda_k = \lambda_k/2$;
end if
$x^{k+1} = x^k + \lambda_k d^k$;
end while;
$s^k = x^{k+1} - x^k;\ y^k = \nabla f(x^{k+1}) - \nabla f(x^k);\ \beta_k = \langle s^k, y^k \rangle$;
if $\beta_k \leq 0$ **then** $\alpha_{k+1} = \alpha_{max}$;
else $\alpha_{k+1} = \min\{\alpha_{max}, \max\{\alpha_{min}, \langle s^k, s^k \rangle / \beta_k\}\}$;
end if

Requirements for application of the SPG algorithm for solving the problem (6) for a fixed μ are satisfied. Indeed, it is obvious that the objective function is differentiable and the projection onto a feasible set $\Omega = [0,1]^n$ is given by

$$[P_r(x)]_i = \begin{cases} 0, & x_i \leq 0 \\ 1, & x_i \geq 1 \\ x_i, & \text{elsewhere} \end{cases}, \qquad \text{where } i = 1, \ldots, n\,.$$

where $x \in \mathbb{R}^n$. P_r is a projection with respect to the Euclidean distance and its calculation is inexpensive.

As we discuss above, $\Gamma_{\alpha,\beta}$ is a convex function. However, by increasing the μ factor during the optimization process the influence of the concave regularization term becomes larger which leads to the non-convex objective function. Therefore, we cannot guaranty that this approach always end up in a global minimum.



PH1 PH2 PH3

Fig. 2. Phantom images used in our experiments. All images have the same resolution 256×256.

4 Experimental Results

We performed experiments on the binary test images (phantoms) presented in Figure 2. Reconstruction problems are composed by taking parallel projections from different directions. We take 256 parallel projections for each direction. Regarding to direction we distinguish reconstructions with 2 and 3 projections. The projection directions are 0^0 and 90^0 for 2 and 0^0, 45^0 and 90^0 for 3 projections.

The parameter settings for SPG based algorithm are following: $M = 5$, $\gamma = 10^{-4}$, $\sigma_1 = 0.1$, $\sigma_2 = 0.9$, $\theta_{min} = 10^{-3}$, $\theta_{max} = 10^{-3}$, $\delta = 0.3$, $E_{in} = 0.01$, $E_{out} = 0.001$, $maxit = 100$. These settings are empirically determined based on our experimental work within this research, but also on our earlier experience regarding the application of the SPG algorithm in image reconstruction problems [10, 11].

The quality of reconstruction (solution) is measured by the following two error measure functions

$$E(x^r) = \|x^r - x^*\|_1,$$

$$PE(x^r) = |\mathrm{per}(x^r) - \mathrm{per}(x^*)|,$$

where x^r is the reconstructed image. Function E gives the number of failed pixels in compare with the original image x^*. PE is the distance between the perimeter of the reconstructed and original object.

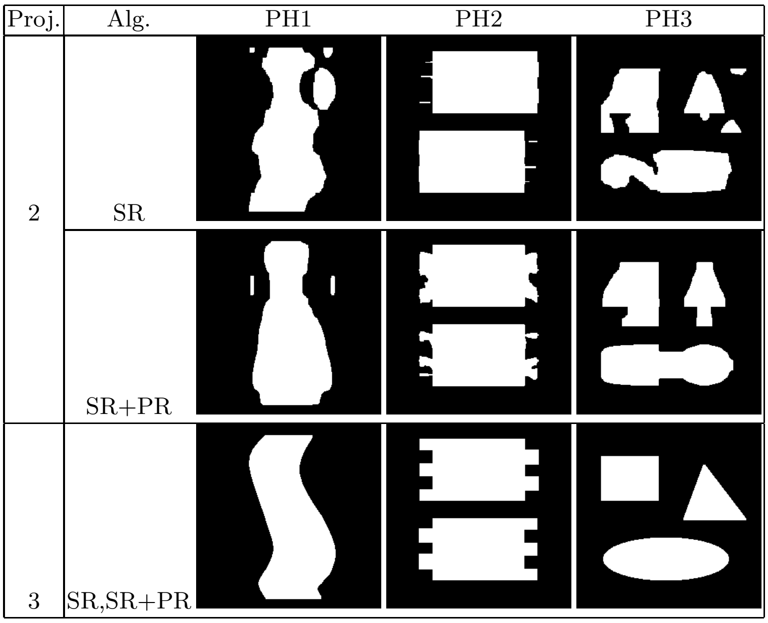


Fig. 3. Reconstructions of the phantom images presented in Figure 2. They are obtained from 2 and 3 projections.

Table (1) shows the obtained error values, E and PE of the reconstructions by the proposed method. For reconstructions obtained from 2 projections the proposed perimeter saving regularized method (SR+PR) provides significantly better results regarding the both criteria, expressed by E and PE. Visual look of the reconstructed images are presented in Figure 3. In a case of 3 projections the obtained reconstructions are exactly the same for both regularization, SR

Proj.		2		3	
Reg.		SR ($\alpha = 5, \beta = 0$)	SR+PR ($\alpha = 5, \beta = 0.5$)	SR ($\alpha = 5, \beta = 0$)	SR+PR ($\alpha = 5, \beta = 0.01$)
PH1	E	7851	4730	5	5
	PE	262	54	2	2
PH2	E	3225	2932	0	0
	PE	126	70	0	0
PH3	E	7147	6902	6	6
	PE	334	86	6	6

Table 1. The measured error values of the reconstructed images. By SR and PR we indicate the using of smooth and perimeter preserving regularization, respectively.

and SR+PR. The solutions in this case are very close to the original images for phantoms PH1 and PH3 or exactly the same for phantom PH2.

5 Concluding Remarks

We have considered a regularized reconstruction method for binary tomography. Beside already used smooth regularization we propose utilizing of the perimeter preserving regularization too. The new method is reformulated to a form of a convex-constrained and convex-concave regularized optimization approach. The using optimization is based on the Spectral Projected Gradient algorithm. Experimental results on three phantom images show that the new approach can improve the quality of the reconstruction obtained from 2 projection directions. This result can be useful in real applications, when the number of projections are limited.

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