A comparison of Several Models of Weighted Tree Automata

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Terms (= trees)

Ranked alphabet : \((\Sigma, rank)\) with \(rank : \Sigma \to \mathbb{N}\)

\[\Sigma^{(m)} = \{\sigma \in \Sigma \mid rank(\sigma) = m\}\]

The set of terms (trees) over \(\Sigma\) and a set \(A\) is the smallest set \(T_{\Sigma}(A)\) satisfying:

(i) \(\Sigma^{(0)} \cup A \subseteq T_{\Sigma}(A)\),

(ii) if \(k \geq 1, \sigma \in \Sigma^{(k)}, t_1, \ldots, t_m \in T_{\Sigma}(A)\), then \(\sigma(t_1, \ldots, t_m) \in T_{\Sigma}(A)\).

\[T_{\Sigma} = T_{\Sigma}(\emptyset) \quad \text{We have } T_{\Sigma} \neq \emptyset \text{ iff } \Sigma^{(0)} \neq \emptyset.\]

Tree language : \(L \subseteq T_{\Sigma}\) (or: \(L : T_{\Sigma} \to \{0, 1\}\) )
Trees (≡ terms)

Example: $\Sigma = \{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\}$
Nondeterministic Tree Automata

The classical definition.

A (finite-state bottom-up) tree automaton is a tuple $M = (Q, \Sigma, F, \delta)$, where

- $Q$ is a finite set (states),
- $\Sigma$ is a ranked alphabet (input ranked alphabet),
- $F \subseteq Q$ is a set (final states), and
- $\delta$ is a family $(\delta_\sigma | \sigma \in \Sigma)$ of mappings $\delta_\sigma : Q^m \rightarrow \mathcal{P}(Q)$ for $\sigma \in \Sigma^{(m)}$.

$M$ is deterministic if $|\delta_\sigma(q_1, \ldots, q_m)|$ has at most one element for all $m \geq 0$, $\sigma \in \Sigma^{(m)}$, and $q_1, \ldots, q_m \in Q$.

The family $\delta$ extends to a mapping $\delta_M : T_\Sigma \rightarrow \mathcal{P}(Q)$. The tree language recognized by $M$ is

$$L_M = \{ s \in T_\Sigma | \delta_M(s) \cap F \neq \emptyset \}.$$
Nondeterministic Tree Automata

Examples of recognizable tree languages:

- the set of derivation trees of a cf grammar
- set of trees which contain the pattern $\sigma(\bullet, \alpha)$
- many other examples

**Theorem.** Bottom-up tree automata and deterministic bottom-up tree automata have the same recognizing power.

**Proof.** The standard powerset construction.
Nondeterministic Tree Automata

Example.

$\Sigma = \{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\}$, show that $L = \{s \in T_\Sigma \mid \sigma(\bullet, \alpha) \text{ occurs in } s\}$ is recognizable.

Let $M = (Q, \Sigma, F, \delta)$, where

- $Q = \{\bot, q_\alpha, q_{ok}\}$,
- $F = \{q_{ok}\}$,
- $\delta_\alpha = \{\bot, q_\alpha\}$,
- $\delta_\sigma(\bot, q_\alpha) = \delta_\sigma(-, q_{ok}) = \delta_\sigma(q_{ok}, -) = \{q_{ok}\}$, otherwise $\delta_\sigma(-, -) = \{\bot\}$,
- $\delta_\gamma(q_{ok}) = \{q_{ok}\}$, otherwise $\delta_\gamma(-) = \{\bot\}$.

Then $L_M = L$. 
Nondeterministic Tree Automata

Example.
Semirings

Semiring : \((K, \oplus, \odot, 0, 1)\)

- \((K, \oplus, 0)\) is a commutative monoid,
- \((K, \odot, 1)\) is a monoid,

and for every \(a, b, c \in K\):

\[
\begin{align*}
(a \oplus b) \odot c &= (a \odot c) \oplus (b \odot c) \\
a \odot (b \oplus c) &= (a \odot b) \oplus (a \odot c) \\
a \odot 0 &= 0 \odot a = 0.
\end{align*}
\]

Examples :

- Boolean semiring : \(\mathbb{B} = (\{0, 1\}, \lor, \land, 0, 1)\)
- semiring of natural numbers : \(\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)\)
- semiring of formal languages :
  \(\text{Lang}_\Delta = (\mathcal{P}(\Delta^*), \cup, \cdot, \emptyset, \{\varepsilon\})\)
  (over \(\Delta\))
- tropical semiring : \(\text{Trop} = (\mathbb{N} \cup \{\infty\}, \text{min}, +, \infty, 0)\)
- arctic semiring : \(\text{Arct} = (\mathbb{N} \cup \{-\infty\}, \text{max}, +, -\infty, 0)\)
Nondeterministic Tree Automata

An algebraic definition

A system $M = (Q, \Sigma, F, \delta)$ (over $\mathbb{B} = (\{0, 1\}, \lor, \land, 0, 1)$)

$F = (F_q \mid q \in Q)$ with $F_q \in \{0, 1\}$

$\delta = (\delta_m : \Sigma^{(m)} \rightarrow \{0, 1\}^{Q^m \times Q} \mid m \geq 0)$ of mappings.

$$\delta_m(\sigma) = \begin{bmatrix} \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \\ q_1 \ldots q_m & \begin{array}{ccc} \ldots & 0/1 & \ldots \end{array} \\ \vdots & \vdots \end{bmatrix} \in \{0, 1\}^{Q^m \times Q}$$

Note, equivalent with $\delta_\sigma : Q^m \rightarrow \mathcal{P}(Q)$.

$M$ is deterministic if, for every $q_1, \ldots, q_m \in Q$, there is at most one $q$ with $\delta_m(\sigma)_{q_1 \ldots q_m, q} \neq 0$. 
Nondeterministic Tree Automata

An algebraic definition

We associate the $\Sigma$-algebra $A_M = (\{0, 1\}^Q, \Sigma_\delta)$, where

$$\Sigma_\delta = (\delta_m(\sigma) \mid m \geq 0, \sigma \in \Sigma^{(m)})$$

and

$$\delta_m(\sigma)(v_1, \ldots, v_m)_q = \bigvee_{q_1, \ldots, q_m \in Q} (v_1)_{q_1} \land \ldots \land (v_m)_{q_m} \land \delta_m(\sigma)_{q_1 \ldots q_m, q}.$$ 

Let $h_\delta : T_\Sigma \to \{0, 1\}^Q$ be the unique $\Sigma$-homomorphism from $T_\Sigma$ to $A_M$.

The tree language recognized by $M$ is $L_M : T_\Sigma \to \{0, 1\}$ defined, for every $s \in T_\Sigma$, by

$$L_M(s) = \bigvee_{q \in Q} h_\delta(s)_q \land F_q.$$
Tree series

(Tree language: $L : T_\Sigma \rightarrow \{0, 1\}$)

*Tree series*: $\varphi : T_\Sigma \rightarrow K$, where $(K, \oplus, \odot, 0, 1)$ is a semiring

Examples of tree series:

- $\text{height} : T_\Sigma \rightarrow \mathbb{N}$, in $\text{Arct} = (\mathbb{N} \cup \{-\infty\}, \text{max}, +, -\infty, 0)$
- $\text{size}_\sigma : T_\Sigma \rightarrow \mathbb{N}$, in $\underline{\mathbb{N}} = (\mathbb{N}, +, \cdot, 0, 1)$
- $\text{size} : T_\Sigma \rightarrow \mathbb{N}$, in $\underline{\mathbb{N}} = (\mathbb{N}, +, \cdot, 0, 1)$
- $\#_{\sigma(\cdot, \alpha)} : T_\Sigma \rightarrow \mathbb{N}$, in $\underline{\mathbb{N}} = (\mathbb{N}, +, \cdot, 0, 1)$
- $\text{shortest}_\alpha : T_\Sigma \rightarrow \mathbb{N}$, in $\text{Trop} = (\mathbb{N} \cup \{-\infty\}, \text{min}, +, -\infty, 0)$
- $\text{yield} : T_\Sigma \rightarrow \mathcal{P}(\Sigma^*)$, in $\text{Lang}_\Sigma = (\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$
- $\text{pos} : T_\Sigma \rightarrow \mathcal{P}(\mathbb{N}^*)$, in $\text{Lang}_\mathbb{N}$
- $\text{pos}_{\sigma(\cdot, \alpha)} : T_\Sigma \rightarrow \mathcal{P}(\mathbb{N}^*)$, in $\text{Lang}_\mathbb{N}$
Tree series

The set of tree series over $K$ and $\Sigma$ is denoted by $K\langle T_\Sigma \rangle$.

For $s \in T_\Sigma$, we write $(\varphi, s)$ for $\varphi(s)$.

The *support* of $\varphi$ is $\text{supp}(\varphi) = \{ s \in T_\Sigma \mid (\varphi, s) \neq 0\}$.

The tree series $\varphi$ is *polynomial* if $\text{supp}(\varphi)$ is finite.

The set of polynomial tree series over $K$ and $\Sigma$ is denoted by $K\langle T_\Sigma \rangle$. 
Generalizations

- recognizability by multilinear mappings over some finite dimensional $K$-vector space, where $K$ is a field, cf. [BR82],

- recognizability by $K$-$\Sigma$-tree automata, where $K$ is a commutative semiring, cf. [Boz99],

- recognizability by weighted tree automata over $K$, where $K$ is a semiring, cf. [AB87],

- recognizability by finite tree automata over $K$ with fixpoint semantics, where $K$ is a commutative and continuous semiring, cf. [Kui98, ÉK03],

- recognizability by polynomially-weighted tree automata, where $K$ is a semiring, cf. [Sei92, Sei94], and

- recognizability by weighted tree automata over M-monoids, cf. [Mal05] and [FMV06].
**$K$-semimodule:**

$(K, \oplus, \circ, 0, 1)$ a commutative semiring, $(V, +, 0)$ a commutative monoid, and $\cdot : K \times V \rightarrow V$ a function:

\[
(k \circ k') \cdot v = k \cdot (k' \cdot v) \tag{1}
\]

\[
k \cdot (v + v') = (k \cdot v) + (k \cdot v') \tag{2}
\]

\[
(k \oplus k') \cdot v = (k \cdot v) + (k' \cdot v) \tag{3}
\]

\[
1 \cdot v = v \tag{4}
\]

\[
k \cdot 0 = 0 \cdot v = 0 \tag{5}
\]

**$K$-vector space:** $K$ is a field and $V$ is a group

A mapping $\omega : V^m \rightarrow V$ is *multilinear* if:

\[
\omega(v_1, \ldots, v_{i-1}, kv + k'v', v_{i+1}, \ldots, v_m) = k\omega(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_m) + k'\omega(v_1, \ldots, v_{i-1}, v', v_{i+1}, \ldots, v_m)
\]
Multilinear mappings over finite-dimensional vector spaces

A multilinear representation [BR82] of $T_\Sigma$ is $(V, \mu, \gamma)$ where

- $(V, +, 0)$ is a finite-dimensional $K$-vector space ($K$ is a field),
- transitions: $\mu = (\mu_m \mid m \geq 0)$ is a family with $\mu_m : \Sigma^{(m)} \to \mathcal{L}(V^m, V)$, the set of multilinear mappings from $V^m$ to $V$
- final behaviour: $\gamma : V \to K$ is a linear form.

The $\Sigma$-algebra associated with $(V, \mu, \gamma)$ is $\mathcal{A}_V = (V, \overline{\Sigma}_\mu)$, where

$\overline{\Sigma}_\mu = (\mu_m(\sigma) \mid m \geq 0, \sigma \in \Sigma^{(m)})$.

$h_\mu : T_\Sigma \to \mathcal{A}_V$ is the unique $\Sigma$-homomorphism.

The tree series recognized by $(V, \mu, \gamma)$ is $\varphi \in K \langle \langle T_\Sigma \rangle \rangle$ such that

$(\varphi, s) = \gamma(h_\mu(s))$ for every $s \in T_\Sigma$.  

Tree series recognizable by multilinear mappings

1) (Example 4.1 of [BR82]) The tree series $\text{size}_\delta$ is recognizable by multilinear mappings over the $\mathbb{Q}$-vector space $V = (\mathbb{Q}^2, +, 0_2)$ with $0_2 = (0, 0)$.

Let $(V, \mu, \gamma)$ defined as follows:

For every $m \geq 0$, $\sigma \in \Sigma^{(m)}$, $e_{i_1}, \ldots, e_{i_l} \in \{e_1 = (1,0), e_2 = (0,1)\}$, we define

$$\mu_m(\sigma)(e_{i_1}, \ldots, e_{i_m}) =$$

$$\begin{cases} 
  e_1 + e_2 & \text{if } \sigma = \delta \text{ and } i_1 = \ldots = i_l = 1 \\
  e_1 & \text{if } \sigma \neq \delta \text{ and } i_1 = \ldots = i_m = 1 \\
  e_2 & \text{if there is exactly one } 1 \leq j \leq m \text{ with } i_j = 2 \\
  0_2 & \text{otherwise.}
\end{cases}$$

$\gamma(e_1) = 0$, $\gamma(e_2) = 1$

For every $s \in T_\Sigma$, we have $h_\mu(s) = e_1 + \text{size}_\delta(s)e_2$. 
Tree series recognizable by multilinear mappings

2) (Example 9.2 of [BR82]) The tree series height is not recognizable by multilinear mappings over any \( \mathbb{Q} \)-vector space.

We denote the class of tree series recognizable by multilinear mappings over some \( K \)-vector space by \( \text{ML}(K) \).

**Theorem.** Every tree language which is recognizable by a deterministic tree automaton \( M = (Q, \Sigma, F, \delta) \) is also recognizable by multilinear mappings over the \( \mathbb{Z}_2 \)-vector space \( \mathbb{Z}_2^Q \).

**Proof.** Let \( Q = \{1, \ldots, n\} \), we define \( (\mathbb{Z}_2^Q, \mu, \gamma) \) with

\[
\mu_m(\sigma)(e_{i_1}, \ldots, e_{i_m}) = e_l \text{ iff } l = \delta_{\sigma}(i_1, \ldots, i_m),
\]

\[
\gamma(e_i) = 1 \text{ iff } i \in F.
\]
**K-Σ-tree automata [Boz99]**

Preparation:

\((K, \oplus, \odot, 0, 1)\) a commutative semiring, \(Q = \{q_1, \ldots, q_\kappa\}\) a finite set.

\((K^Q, +, 0_Q)\) is a \(K\)-semimodule via \(\cdot : K \times K^Q \to K^Q\), with

\[(k \cdot v)_q = k \odot v_q\]

For \(m \geq 0\) and \(\nu : Q^m \to K^Q\), a **multilinear extension of \(\nu\)** is a mapping

\(\overline{\nu} : K^Q \times \ldots \times K^Q \to K^Q\)

such that

- \(\overline{\nu}\) is multilinear

- \(\overline{\nu}(1_{p_1}, \ldots, 1_{p_m}) = \nu(p_1, \ldots, p_m)\).

It is unique and

\[
\overline{\nu}(v_1, \ldots, v_m)_q = \bigoplus_{p_1, \ldots, p_m \in Q} \left( \bigodot_{1 \leq i \leq m} (v_i)_{p_i} \right) \odot \nu(p_1, \ldots, p_m)_q.
\]
**$K$-$\Sigma$-tree automata [Boz99]**

A system $M = (Q, \mu, f)$, where

- $Q$ a finite set,

- $\mu = (\mu_m(\sigma) \mid m \geq 0, \sigma \in \Sigma^{(m)})$ is a family of transition functions $\mu_m(\sigma) : Q^m \to K^Q$, and

- $f : Q \to K$ is a final weight function.

For $m \geq 0$ and $\sigma \in \Sigma^{(m)}$, let $\overline{\mu_m(\sigma)} : (K^Q)^m \to K^Q$ be the multilinear extension of $\mu_m(\sigma)$.

The $\Sigma$-algebra associated with $M$ is $\mathcal{A}_M = (K^Q, \overline{\Sigma}_\mu)$ where $\overline{\Sigma}_\mu = (\overline{\mu_m(\sigma)} \mid m \geq 0, \sigma \in \Sigma^{(m)})$.

The unique $\Sigma$-homomorphism from $T_\Sigma$ to $\mathcal{A}_M$ is $h_\mu : T_\Sigma \to K^Q$.

*The tree series recognized by* $M$ *is* $\varphi_M \in K \ll T_\Sigma \gg$ *such that, for* $s \in T_\Sigma$,

$$\varphi_M, s = \bigoplus_{q \in Q} h_\mu(s)_q \odot f(q).$$
**$K$-$\Sigma$-tree automata [Boz99]**

Example: the tree series **height** is recognizable by an **Arct-$\Sigma$-tree automaton** $M = (Q, \mu, f)$ where $\text{Arct} = (\mathbb{N} \cup \{-\infty\}, \text{max}, +, -\infty, 0)$ and

- $Q = \{p_1, p_2\}$,
- $f(p_1) = 0$ and $f(p_2) = -\infty$,
- $\mu$ is defined in the following way:
  - $\mu_0(\alpha)(p_1) = 0$,
  - $\mu_0(\alpha)(p_2) = 0$,
  - $\mu_2(\sigma)(p_1, p_2)p_1 = 1$,
  - $\mu_2(\sigma)(p_2, p_1)p_1 = 1$,
  - $\mu_2(\sigma)(p_2, p_2)p_2 = 0$,
  - $\mu_2(\sigma)(p, q)r = -\infty$ for every other $p, q, r \in Q$. 

$K$-$\Sigma$-tree automata [Boz99]

We denote the class of tree series recognizable by a $K$-$\Sigma$-tree automaton for some $\Sigma$ by $\text{TA}(K)$.

**Theorem.** For every field $K$, we have $\text{ML}(K) = \text{TA}(K)$.

**Proof.** Let $(V, +, 0)$ be a vector space over the field $(K, \oplus, \odot, 0, 1)$ of dimension $\kappa < \infty$; also let $(V, \mu, \gamma)$ be a multilinear representation of $T_\Sigma$. Moreover, let $M = (Q, \nu, f)$ be a $K$-$\Sigma$-ta over $K$. We say that $(V, \mu, \gamma)$ and $M$ are related if

- $Q$ is a basis of $V$,
- for every $m \geq 0$, $\sigma \in \Sigma^{(m)}$, and $p, p_1, \ldots, p_m \in Q$, the equation $\nu_m(\sigma)(p_1, \ldots, p_m)_p = \mu_m(\sigma)(p_1, \ldots, p_m)_p$ holds, and
- for every $p \in Q$, the equation $f(p) = \gamma(p)$ holds.
A system $M = (Q, \Sigma, K, F, \delta)$ (over the semiring $(K, \oplus, \odot, 0, 1)$)

$F = (F_q \mid q \in Q)$ with $F_q \in K$

$\delta = (\delta_m : \Sigma^{(m)} \to K^{Q^m \times Q} \mid m \geq 0)$ of mappings.

$$
\delta_m(\sigma) = \begin{bmatrix}
\vdots \\
q_1 \cdots q_m \\
\vdots
\end{bmatrix}
\begin{bmatrix}
\vdots \\
\vdots
\end{bmatrix}
\begin{bmatrix}
\vdots \\
\vdots \\
\vdots
\end{bmatrix}
\in K^{Q^m \times Q}
$$
Wta over semirings [AB87]

We define $\overline{\delta_m(\sigma)} : (K^Q)^m \to K^Q$, by

$$\overline{\delta_m(\sigma)}(v_1, \ldots, v_m)_q = \bigoplus_{q_1, \ldots, q_m \in Q} (v_1)_{q_1} \circ \ldots \circ (v_m)_{q_m} \circ \delta_m(\sigma)_{q_1 \ldots q_m,q}.$$

We associate $A_M = (K^Q, \overline{\Sigma}_\delta)$, where $\overline{\Sigma}_\delta = (\overline{\delta_m(\sigma)} \mid m \geq 0, \sigma \in \Sigma^{(m)})$.

Let $h_\delta : T_\Sigma \to K^Q$ be the unique $\Sigma$-homomorphism from $T_\Sigma$ to $A_M$.

The tree language recognized by $M$ is the tree series $\varphi_M : T_\Sigma \to K$
defined for every $s \in T_\Sigma$ by

$$(\varphi_M, s) = \bigoplus_{q \in Q} h_\delta(s)_q \odot F_q.$$
Wta over semirings [AB87]

We denote the class of tree series recognizable by weighted tree automata over the semiring $K$ by $WTA(K)$.

**Theorem.** For every commutative semiring $K$, we have

$$TA(K) = WTA(K).$$

**Corollary.** For every field $K$, we have

$$ML(K) = TA(K) = WTA(K).$$
Wta over semirings [AB87]

Determinization [BV03].

A system $M = (Q, \Sigma, K, F, \delta)$ (over the semiring $(K, \oplus, \odot, 0, 1)$)

$F = (F_q \mid q \in Q)$ with $F_q \in K$

$\delta = (\delta_m : \Sigma^{(m)} \rightarrow K^{Q^m \times Q} \mid m \geq 0)$ of mappings.

\[ \delta_m(\sigma) = q_1 \cdots q_m \begin{bmatrix} \cdots & q & \cdots \\ \vdots & \ddots & \vdots \\ \vdots & & \ddots \\ \vdots & & & \ddots \end{bmatrix} \in K^{Q^m \times Q} \]

$M$ is deterministic if, for every $q_1, \ldots, q_m \in Q$, there is at most one $q$ with $\delta_m(\sigma)_{q_1 \ldots q_m, q} \neq 0$. 
Wta over semirings [AB87]

Determinization.

In general wta over a semiring $K$ and deterministic wta over $K$ do not have the same recognizing power.

B. Borchardt and H. Vogler [BV03]:

- There is a wta over $	ext{Trop}$ which is not equivalent with any deterministic wta over $	ext{Trop}$.

- They give a partial determinization algorithm, which converges in certain cases.
Finite tree automata over semirings with fixpoint semantics [Kui98, ÉK03]

The semiring \((K, \oplus, \otimes, 0, 1)\) must be commutative,

- naturally ordered: \(k \sqsubseteq k'\) iff \((\exists l \in K)k \oplus l = k'\) is a partial order,

- complete: infinite sum exists, and

- continuous: naturally ordered, complete and, for every \(\omega\)-chain
  \(k_1 \sqsubseteq k_2 \sqsubseteq \ldots\) in \(K\) and \(k \in K\),

  \[(\forall n \geq 1) \bigoplus_{i=1}^{n} k_i \sqsubseteq k\] implies that \(\bigoplus_{i=1}^{\infty} k_i \sqsubseteq k\).

Then \(K, K\langle\llceil T_{\Sigma}\rrceil\), and \(K\langle\llceil T_{\Sigma}\rrceil\rangle^n\) become a complete poset with respect to the (extension) of \(\sqsubseteq\).
Finite tree automata over semirings with fixpoint semantics [Kui98, ÉK03]

A **finite tree automaton** (over $K$ and $\Sigma$) is a tuple $M = (Q, \mathcal{M}, S, P)$ where

- $Q$ is a finite set (of states),
- $\mathcal{M} = (\mathcal{M}_m \mid m \geq 0)$ is a family of *transition matrices* $\mathcal{M}_m$ such that $\mathcal{M}_m \in (K\langle T_\Sigma(Y_m) \rangle)^{Q \times Q^m}$ and for almost every $m$ it holds that every entry of $\mathcal{M}_m$ is 0,
- $S \in (K\langle T_\Sigma(Y_1) \rangle)^{1 \times Q}$ is the *initial state vector*, and
- $P \in (K\langle T_\Sigma \rangle)^{Q \times 1}$ is the *final state vector*. 
Finite tree automata over semirings with fixpoint semantics [Kui98, ÉK03]

Such a system induces a continuous mapping

\[ \Phi : K\langle\langle T_\Sigma\rangle\rangle^{Q \times 1} \to K\langle\langle T_\Sigma\rangle\rangle^{Q \times 1}, \]

whose least fixpoint is \( \text{fix} \, \Phi \).

The tree series recognized by \( M \) is

\[ \varphi_M = \bigoplus_{q \in Q} (S_q \leftarrow_{OI} (\text{fix} \Phi)_q), \]

and we denote the class of tree series recognizable by finite tree automata over the semiring \( K \) with fixpoint semantics by \( \text{FTA}(K) \).

**Theorem.** For every commutative and continuous semiring \( K \), we have

\[ \text{WTA}(K) = \text{FTA}(K). \]
Polynomially weighted tree automata over semirings [Sei92]

A system $M = (Q, \Sigma, K, F, \delta)$ (over the semiring $(K, \oplus, \odot, 0, 1)$)

$F = (F_q \mid q \in Q)$ with $F_q \in P_1(K)$

$\delta = (\delta_m : \Sigma^{(m)} \rightarrow P_m(K)^{Q^m \times Q} \mid m \geq 0)$ of mappings.

\[
\delta_m(\sigma) = q_1 \ldots q_m \begin{bmatrix} \vdots & \cdots & q & \cdots & \vdots \\ \vdots & \cdots & f & \cdots & \vdots \end{bmatrix} \in P_m(K)^{Q^m \times Q}
\]
Polynomially weighted tree automata over semirings [Sei92]

We define $\delta_m(\sigma) : (K^Q)^m \to K^Q$, by

$$\delta_m(\sigma)(v_1, \ldots, v_m)_q = \bigoplus_{q_1, \ldots, q_m \in Q} \delta_m(\sigma)(v_1)_{q_1} \cdots (v_m)_{q_m}.$$ 

We associate $A_M = (K^Q, \Sigma_\delta)$, where $\Sigma_\delta = (\overline{\delta_m(\sigma)} | m \geq 0, \sigma \in \Sigma^{(m)})$.

Let $h_\delta : T_\Sigma \to K^Q$ be the unique $\Sigma$-homomorphism from $T_\Sigma$ to $A_M$.

The tree language recognized by $M$ is the tree series $\varphi_M : T_\Sigma \to K$ defined for every $s \in T_\Sigma$ by

$$(\varphi_M, s) = \bigoplus_{q \in Q} F_q(h_\delta(s)_q).$$
Polynomially weighted tree automata over semirings [Sei92]

We denote the class of tree series recognizable by polynomially weighted tree automata over the semiring $K$ by $\text{PWTA}(K)$.

**Theorem.** For every semiring $K$, we have

$$\text{WTA}(K) \subseteq \text{PWTA}(K).$$

**Theorem.** $\text{PWTA}(\mathbb{N}) - \text{WTA}(\mathbb{N}) \neq \emptyset$. 
A *multioperator monoid* (for short: M-monoid) $(K, \oplus, 0, \Omega)$ consists of

- a commutative monoid $(K, \oplus, 0)$ and
- an $\Omega$-algebra $(K, \Omega)$.

A multioperator monoid is *distributive* if

\[
\omega_K(k_1, \ldots, k_{i-1}, \bigoplus_{j=1}^{n} a_j, k_{i+1}, \ldots, k_m) = \bigoplus_{j=1}^{n} \omega_K(k_1, \ldots, k_{i-1}, a_j, k_{i+1}, \ldots, k_m)
\]

holds for every $m \geq 0$, $\omega \in \Omega^{(m)}$, $k_1, \ldots, k_m \in K$, $1 \leq i \leq m$, and $a_1, \ldots, a_n \in K$. (This implies $\omega_K(\ldots, 0, \ldots, ) = 0$).
Wta over M-monoids [Mal05, FMV06]

A system $M = (Q, \Sigma, A, F, \delta)$ (over the M-monoid $A = (K, \oplus, \odot, \Omega)$)

$F = (F_q \mid q \in Q)$ with $F_q \in \Omega^{(1)}$

$\delta = (\delta_m : \Sigma^{(m)} \to (\Omega^{(m)})^{Q^m \times Q} \mid m \geq 0)$ of mappings.

$$
\delta_m(\sigma) = q_1 \ldots q_m \begin{bmatrix}
\vdots & \vdots \\
\vdots & \vdots \\
\vdots & \vdots
\end{bmatrix}
\in (\Omega^{(m)})^{Q^m \times Q}
$$
We define $\overline{\delta_m(\sigma)} : (K^Q)^m \to K^Q$, by

$$\overline{\delta_m(\sigma)}(v_1, \ldots, v_m)_q = \bigoplus_{q_1, \ldots, q_m \in Q} \delta_m(\sigma)_{q_1 \ldots q_m, q}((v_1)_{q_1}, \ldots, (v_m)_{q_m}).$$

We associate $A_M = (K^Q, \Sigma_{\delta})$, where $\Sigma_{\delta} = (\overline{\delta_m(\sigma)} | m \geq 0, \sigma \in \Sigma^{(m)})$.

Let $h_{\delta} : T_\Sigma \to K^Q$ be the unique $\Sigma$-homomorphism from $T_\Sigma$ to $A_M$.

The tree language recognized by $M$ is the tree series $\varphi_M : T_\Sigma \to K$

defined for every $s \in T_\Sigma$ by

$$(\varphi_M, s) = \bigoplus_{q \in Q} F_q(h_{\delta}(s)_q).$$
Wta over M-monoids [Mal05, FMV06]

We denote the class of tree series recognizable by weighted tree automata over the M-monoid $\mathcal{A}$ by $\text{MWTA}(\mathcal{A})$.

**Theorem.** For every semiring $K$, we have $\text{PWTA}(K) = \text{MWTA}(\text{Pol}(K))$.

**Theorem.** $\text{MWTA}(\mathbb{N}_{exp}) \setminus \text{PWTA}(\mathbb{N}) \neq \emptyset$.

**Theorem.** (cf. Corollary 1 of [Mal05]) Let $K$ be a distributive M-monoid and $\varphi$ be a tree series which is recognizable by a deterministic wta over $K$. Then there is a semiring $K'$ such that $K \subseteq K'$ and $\varphi$ is recognizable by a wta over $K'$. 
Wta over M-monoids [Mal05, FMV06]

A new result:

**Theorem.** ([FMV06]). Let $K$ be a distributive M-monoid and $\varphi$ be a tree series over $K$. Then $\varphi$ is rational iff $\varphi$ is recognizable.
Literatur


