3D reconstruction from 2D images: Discrete tomography

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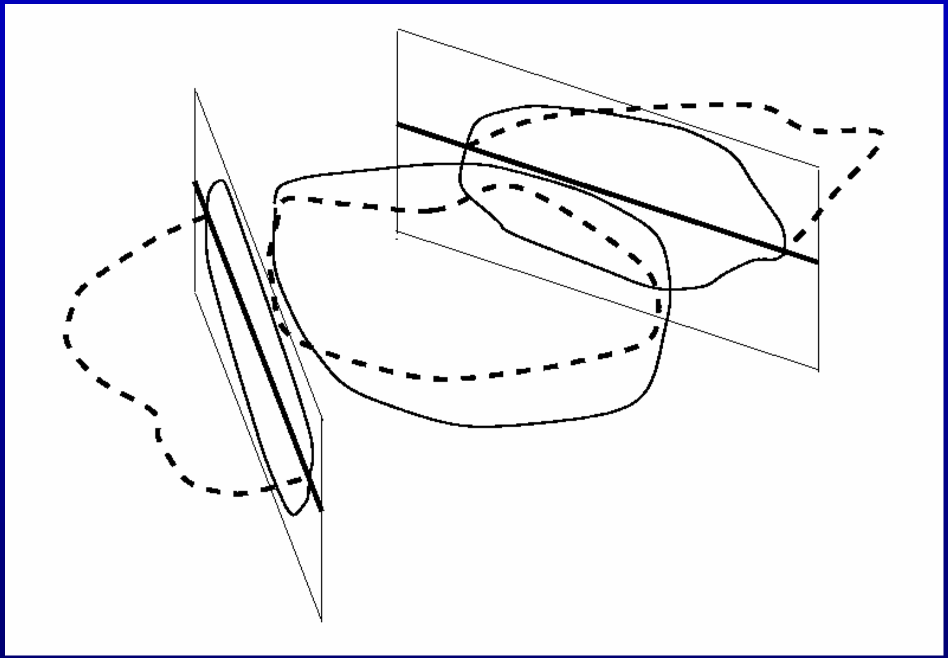
OUTLINE

- What is Discrete Tomography (DT) ?
- Reconstruction of binary matrices / discrete sets
- Optimization
- Simulation and physical experiments

TOMOGRAPHY

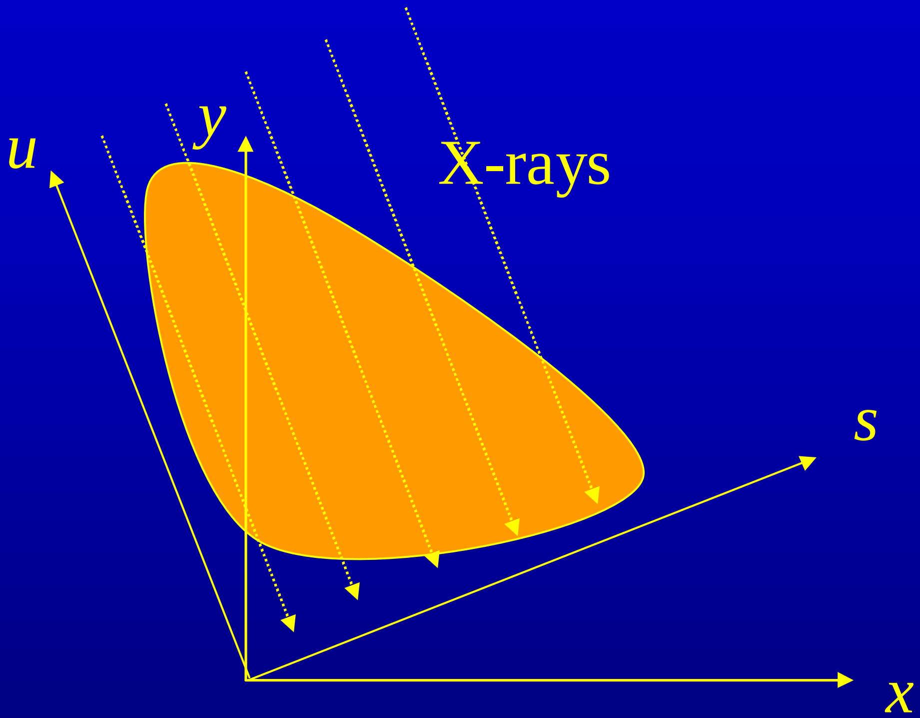
technique for imaging the cross-sections of 3D objects

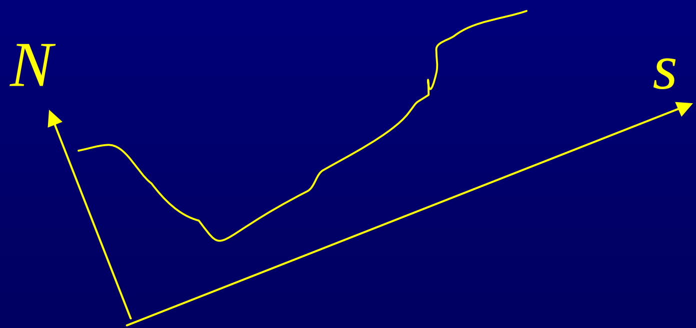
reconstruction tomography: the images are reconstructed from the projections of the objects



for example: computerized tomography (CT)
reconstruction of the cross-sections of the human body from X-ray images

X-ray projections





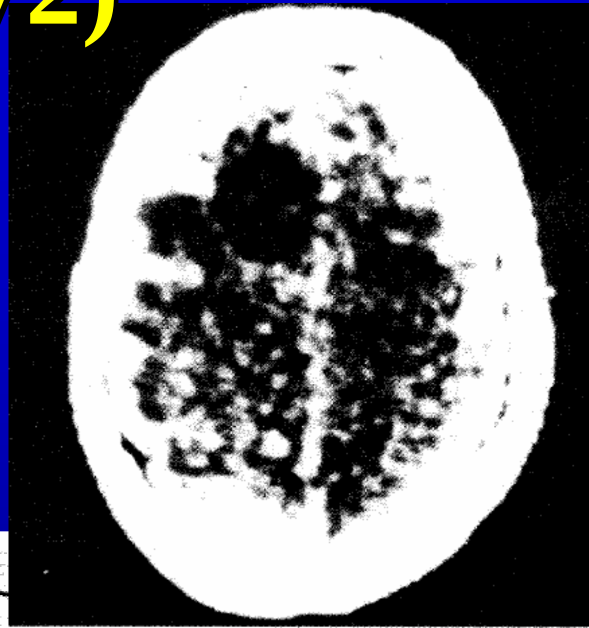
$$N = N_0 \cdot e^{-\int \mu(x,y)\,du}$$

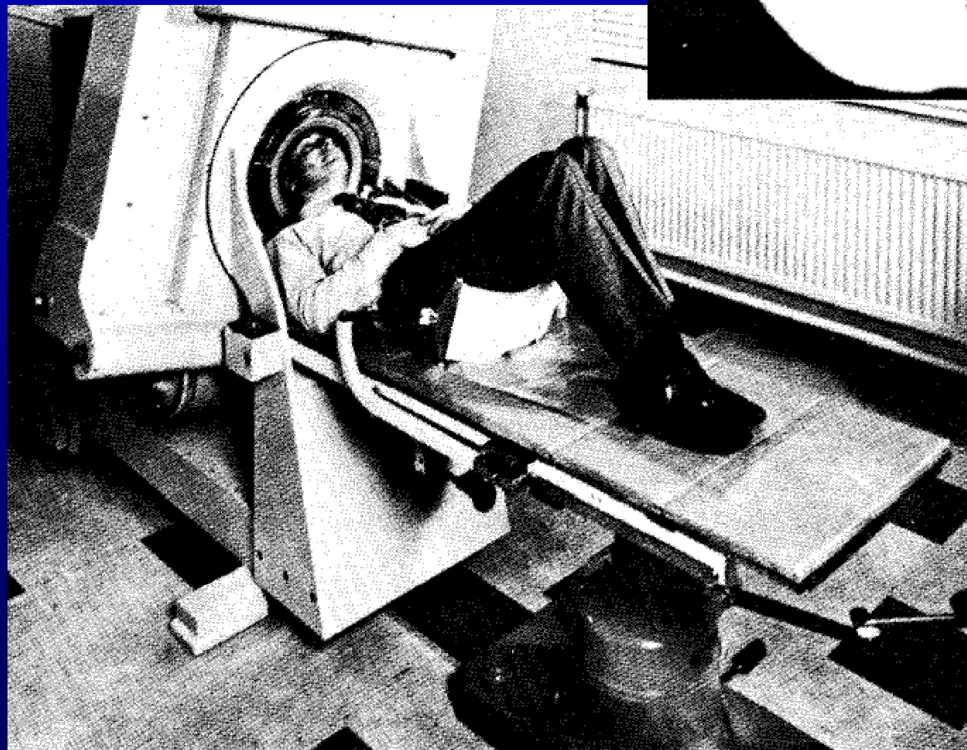
line integral

$$\log \frac{N}{N_0} = \int \mu(x, y)\,du$$

The first CT (1972)

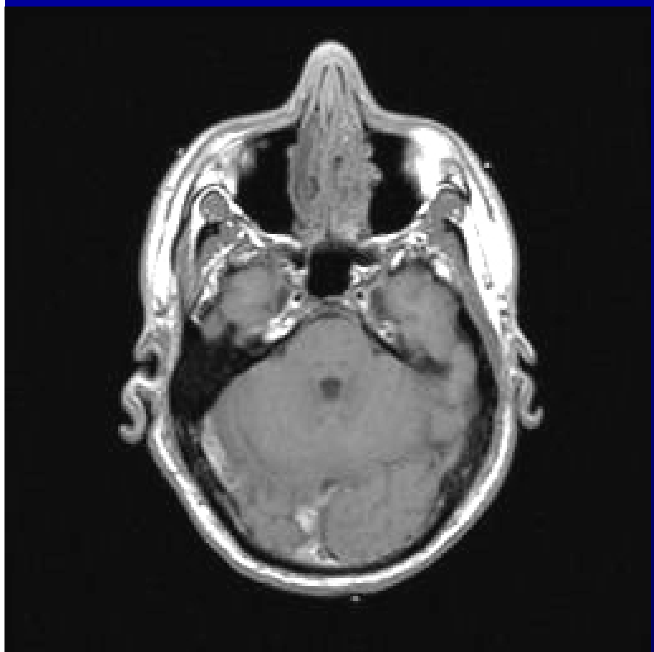
Godfrey N. Hounsfield
Nobel-prize 1979

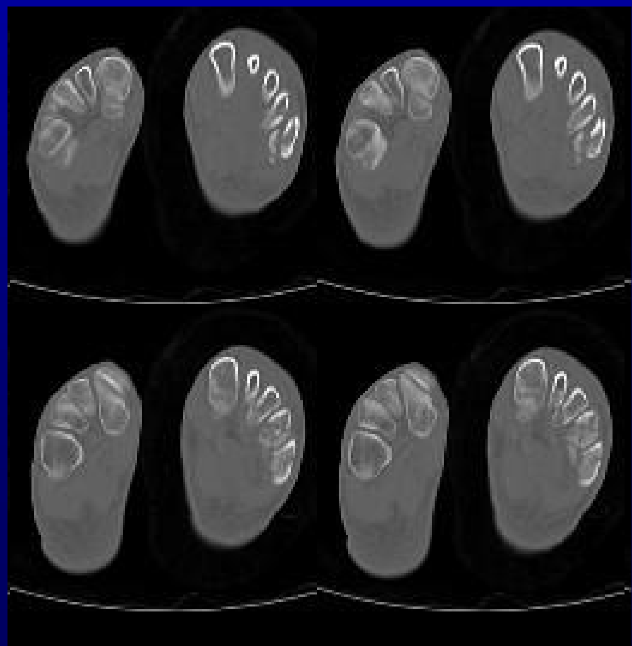


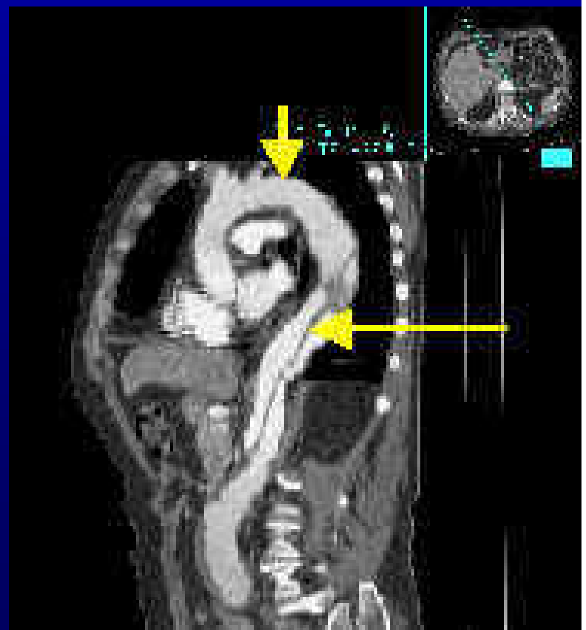


CT

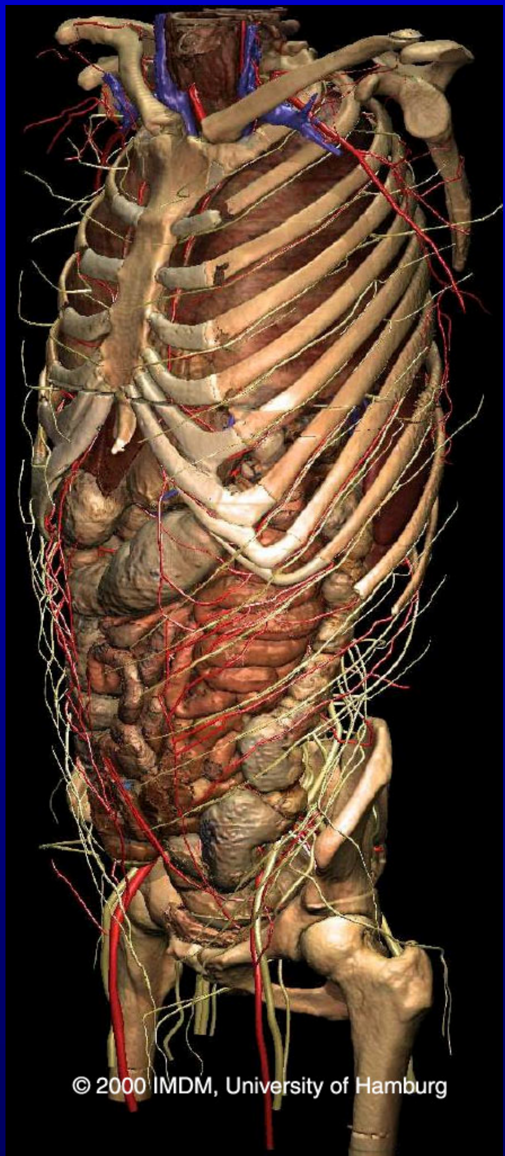
Spiral Xray CT
Spiral scanning causes the
focal spot to follow a spiral
around the patient.

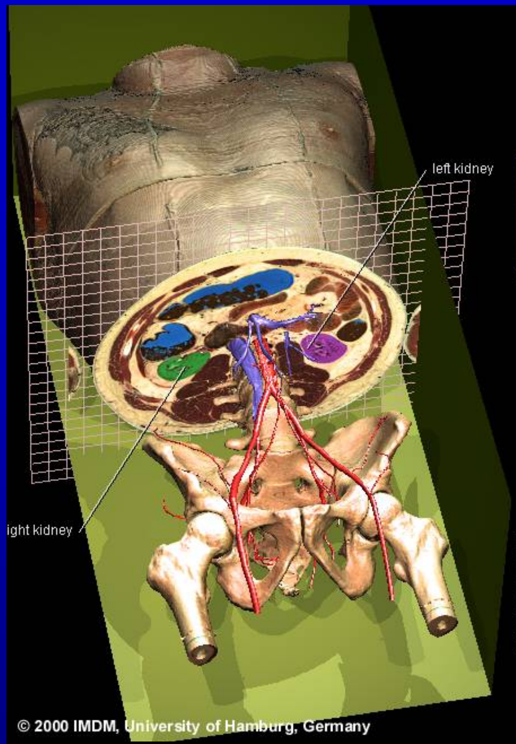


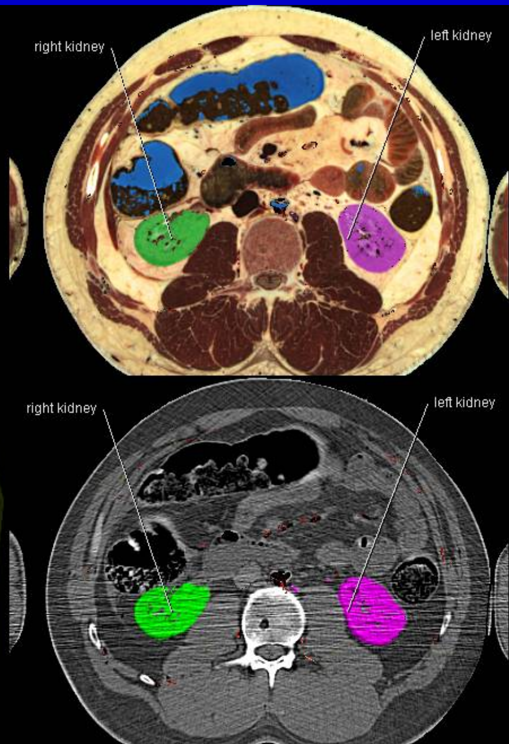




Electronic atlas

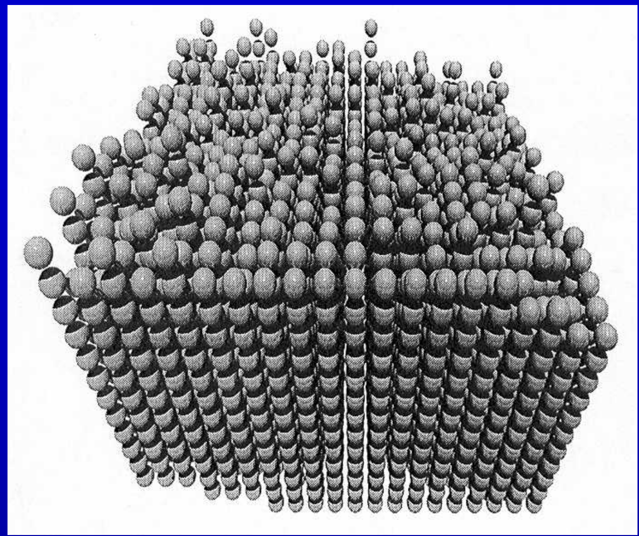


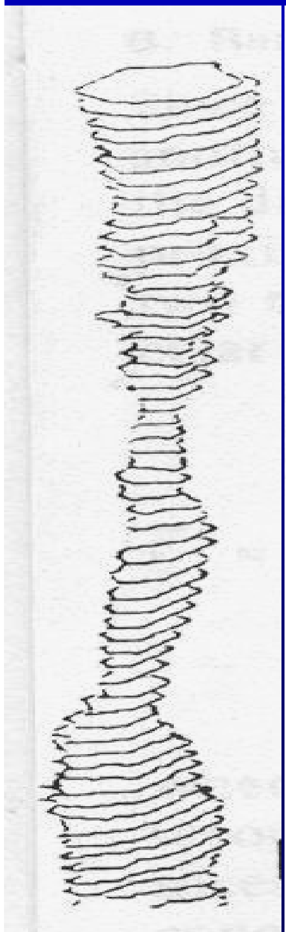


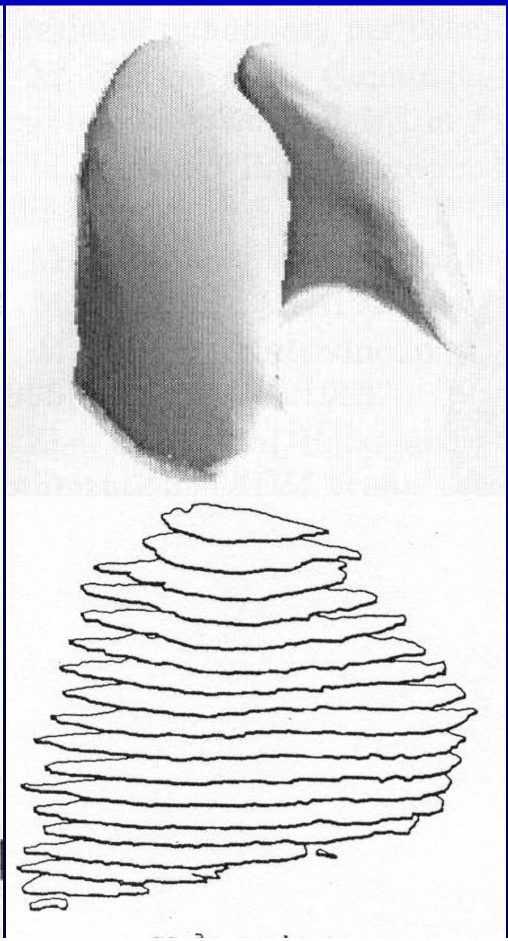


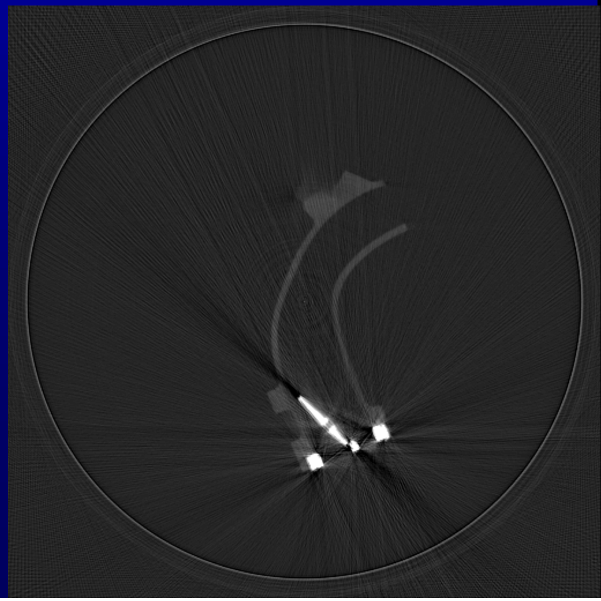
Karl Heinz Höhne, Hamburg 7

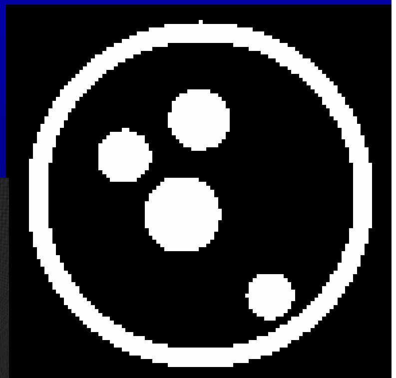
WHAT ABOUT „SIMPLE" OBJECTS?

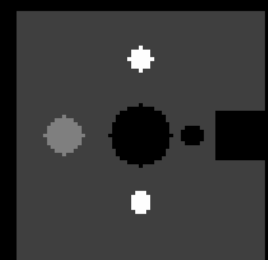




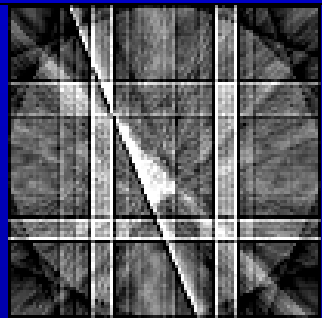
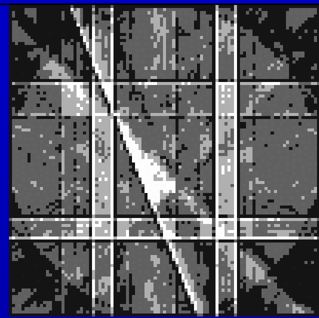
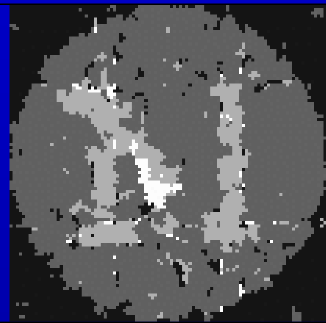
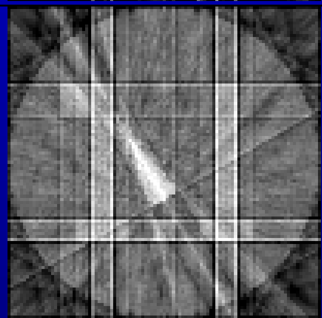
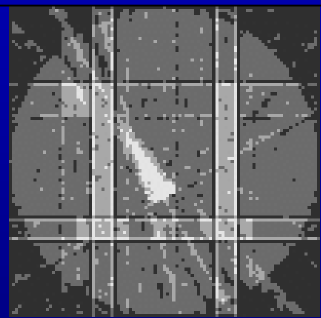
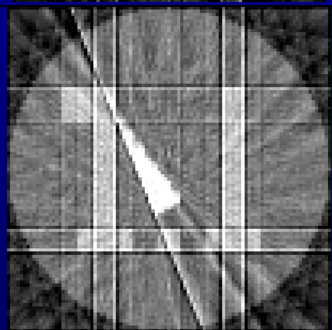
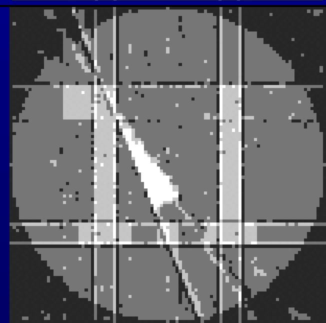








KNOWING THE DISCRETE RANGE

# projs.	Conv. method	Discretized image	DT method
8			
12			
16			

L. Ruskó, A.K., Z. Kiss, L. Rodek, 2003

DISCRETE TOMOGRAPHY (DT)

special tomography when the function *f* to be reconstructed has a known discrete domain *D*,

$$f : R^2 \rightarrow D$$

for example, *D*={0,1} means that *f* has only binary values

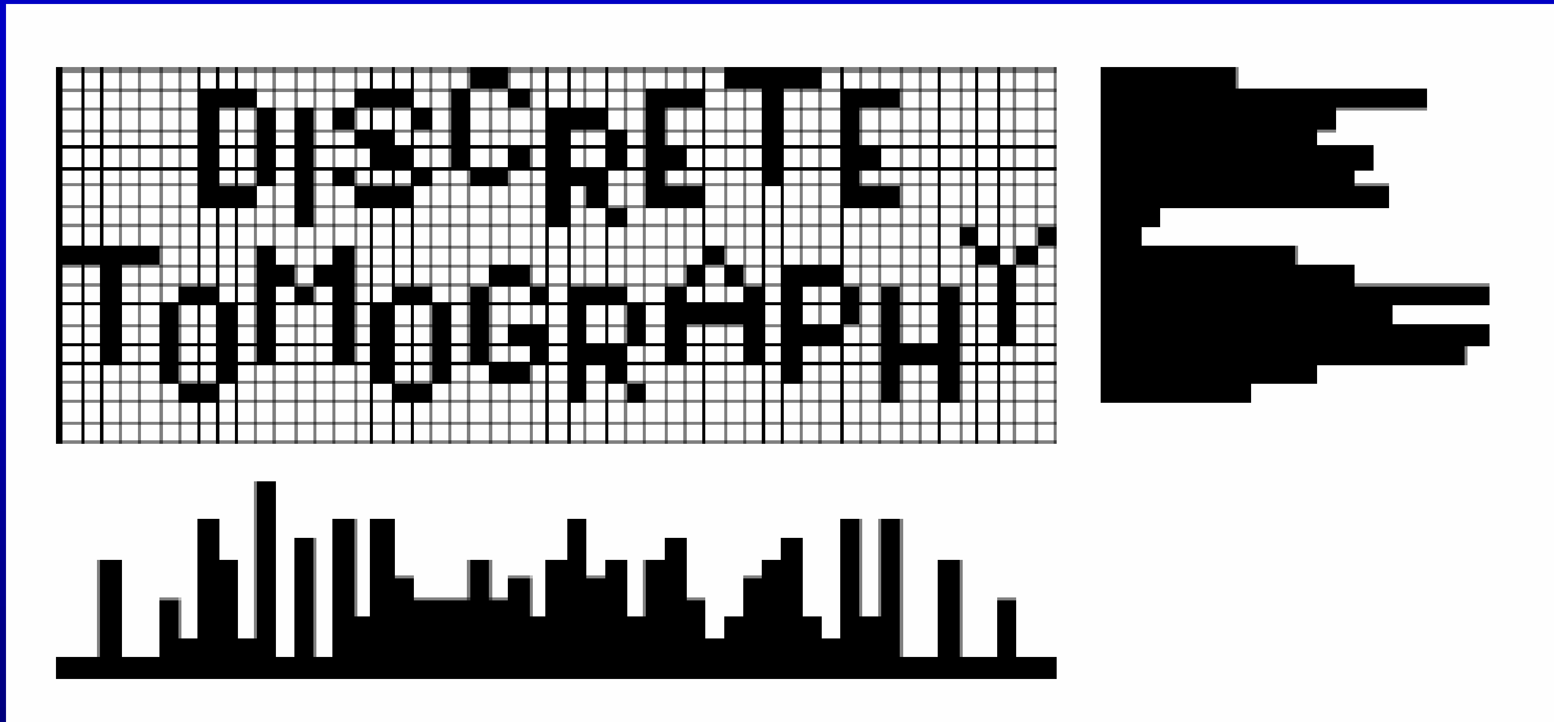
WHY DISCRETE TOMOGRAPHY ?

let us use the fact that the range of the function to be reconstructed is discrete and known

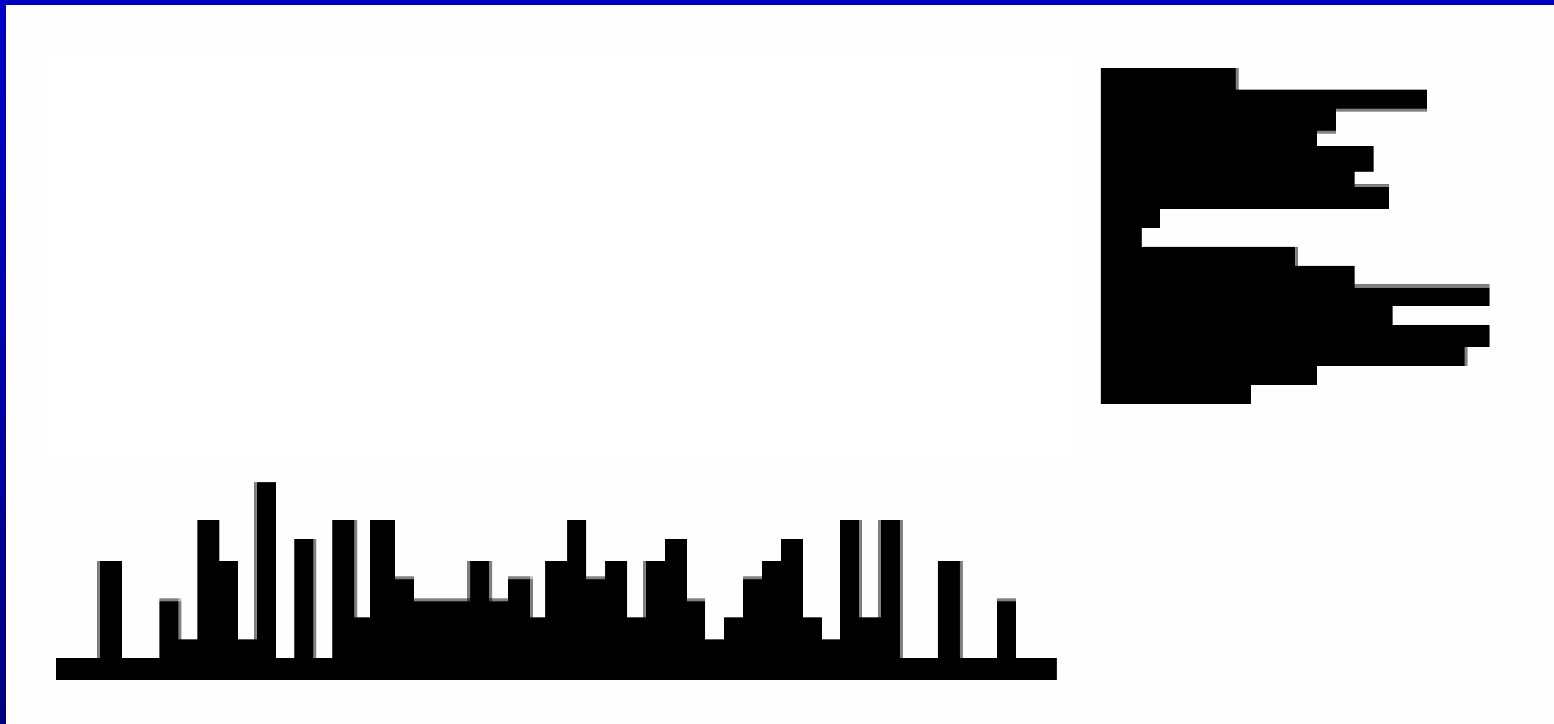
Consequence:

in DT we need a few (e.g., 2-10) projections,

(in CT we need a few hundred projections)


DISCRETE
TOMOGRAPHY

?



?

A CLASSICAL PROBLEM

Reconstruction of binary matrices from their row and column sums

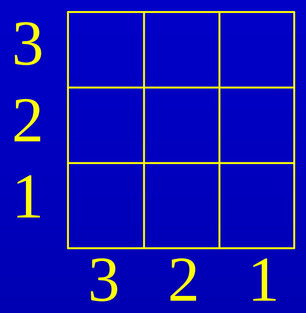
2	0	1	1	0	0	0
4	0	1	1	1	0	1
3	1	1	0	1	0	0
4	1	1	1	0	1	0
1	1	0	0	0	0	0
	3	4	3	2	1	1

2		1	1			
4		1	1	1		1
3	1	1		1		
4	1	1	1		1	
1	1					
	3	4	3	2	1	1

How to reconstruct?

Is a binary matrix uniquely determined by these sums?

EXAMPLES



EXAMPLES

3			
2			
1			
	3	2	1

3	1	1	1
2			
1			
	3	2	1

EXAMPLES

3			
2			
1			
	3	2	1

3	1	1	1
2			
1			
	3	2	1

3	1	1	1
2	1		
1	1		
	3	2	1

EXAMPLES

3			
2			
1			
	3	2	1

3	1	1	1
2			
1			
	3	2	1

3	1	1	1
2	1		
1	1		
	3	2	1

3	1	1	1
2	1	1	
1	1		
	3	2	1

EXAMPLES

3			
2			
1			
	3	2	1

3	1	1	1
2			
1			
	3	2	1

3	1	1	1
2	1		
1	1		
	3	2	1

3	1	1	1
2	1	1	
1	1		
	3	2	1

unique

EXAMPLES

3			
2			
1			
	3	2	1

3	1	1	1
2			
1			
	3	2	1

3	1	1	1
2	1		
1	1		
	3	2	1

3	1	1	1
2	1	1	
1	1		
	3	2	1

unique

3			
3			
1			
	3	3	1

EXAMPLES

3			
2			
1			
	3	2	1

3	1	1	1
2			
1			
	3	2	1

3	1	1	1
2	1		
1	1		
	3	2	1

3	1	1	1
2	1	1	
1	1		
	3	2	1

unique

3			
3			
1			
	3	3	1

3	1	1	1
3			
1			
	3	3	1

EXAMPLES

3			
2			
1			
	3	2	1

3	1	1	1
2			
1			
	3	2	1

3	1	1	1
2	1		
1	1		
	3	2	1

3	1	1	1
2	1	1	
1	1		
	3	2	1

unique

3			
3			
1			
	3	3	1

3	1	1	1
3			
1			
	3	3	1

3	1	1	1
3	1	1	1
1			
	3	3	1

EXAMPLES

3			
2			
1			
	3	2	1

3	1	1	1
2			
1			
	3	2	1

3	1	1	1
2	1		
1	1		
	3	2	1

3	1	1	1
2	1	1	
1	1		
	3	2	1

unique

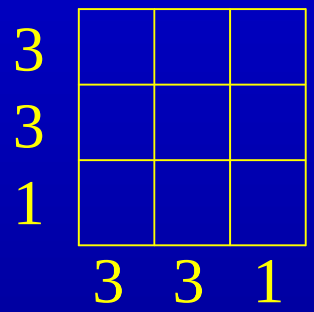
3			
3			
1			
	3	3	1

3	1	1	1
3			
1			
	3	3	1

3	1	1	1
3	1	1	1
1			
	3	3	1

inconsistent

CLASSIFICATION



inconsistent

CLASSIFICATION

3			
3			
1			
	3	3	1

inconsistent

3	1	1	1
2	1	1	
1	1		
	3	2	1

unique

CLASSIFICATION

3			
3			
1			
	3	3	1

inconsistent

3	1	1	1
2	1	1	
1	1		
	3	2	1

unique

1	1	
1		1
	1	1

1		1
1	1	
	1	1

non-unique

SWITCHING COMPONENT

configuration

1	
	1

	1
1	

2		1	1			
4		1	1	1	**1**	
3	1	1		1		
4	1	1	1			**1**
1	1					
	3	4	3	2	1	1

2		1	1			
4		1	1	1		**1**
3	1	1		1		
4	1	1	1		**1**	
1	1					
	3	4	3	2	1	1

It is necessary and sufficient for the non-uniqueness.

A RECONSTRUCTION ALGORITHM

Input: a (compatible) pair of vectors (R,S)

construct S' from S;
let $B=A^*$ and $k=n$;
while (k>1) {
    while ($s'_k > \bullet\ b_{ik}$) {
        let $j_0=\max\{j<k|b_{\mathrm{ij}}=1, b_{\mathrm{i,j+1}}=\ldots=b_{\mathrm{ik}}=0\}$;
        let row i_0 be where such a j_0 was found;
        set $b_{i0j0}=0$ and $b_{i0k}=1$ (i.e., shift the 1 to the right)
        }
    k=k-1;
    }

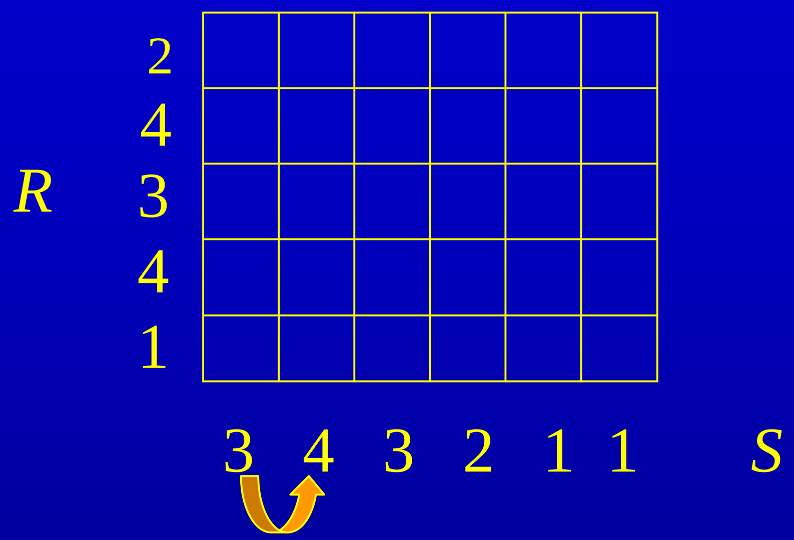
Ryser, 1957

complexity: $O(n\cdot(m+\log n))$

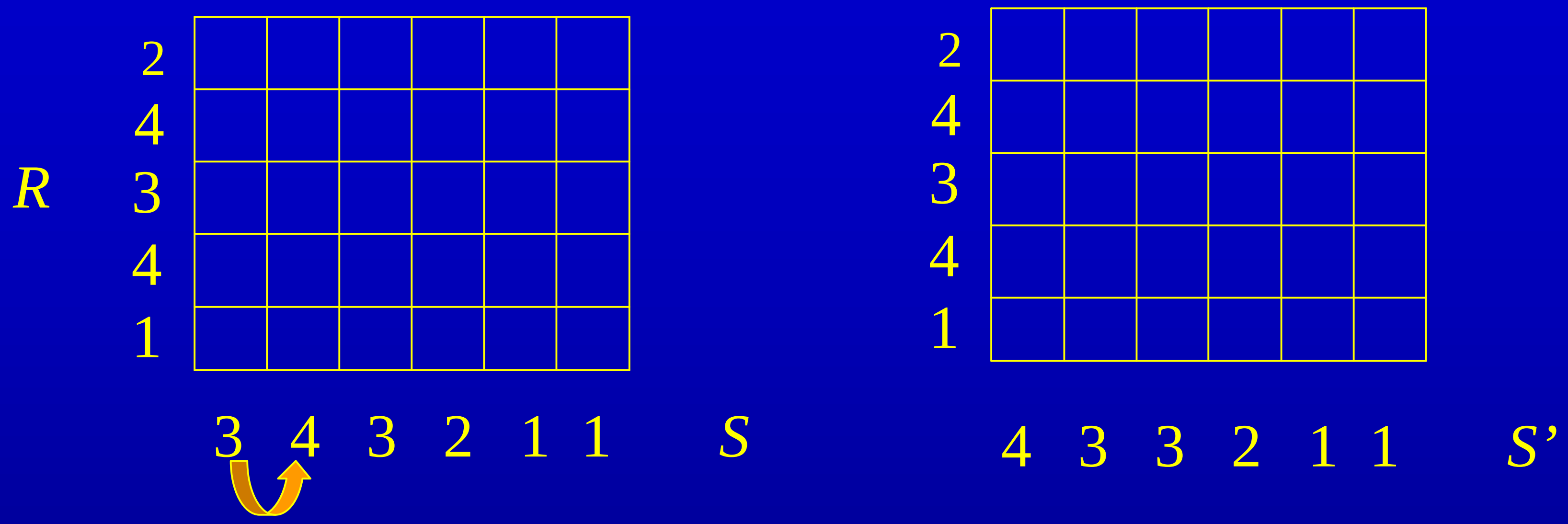
RECONSTRUCTION

	2							
	4							
R	3							
	4							
	1							
		3	4	3	2	1	1	*S*

RECONSTRUCTION



RECONSTRUCTION



RECONSTRUCTION

R						
2						
4						
3						
4						
1						
S	3	4	3	2	1	1

2						
4						
3						
4						
1						
S'	4	3	3	2	1	1

R						
2	1	1				
4	1	1	1	1		
3	1	1	1			
4	1	1	1	1		
1	1					
S(*B*)	5	4	3	2	0	0
S'	4	3	3	2	1	1

=B

RECONSTRUCTION

R						
2						
4						
3						
4						
1						
S	3	4	3	2	1	1

R						
2						
4						
3						
4						
1						
S'	4	3	3	2	1	1

R						
2	1	1				
4	1	1	1	1		
3	1	1	1			
4	1	1	1	1		
1	1					
S(*B*)	5	4	3	2	0	0
S'	4	3	3	2	1	1

=B

RECONSTRUCTION

R

R						
2						
4						
3						
4						
1						
S	3	4	3	2	1	1

R						
2						
4						
3						
4						
1						
S'	4	3	3	2	1	1

R

R						
2	1	1				
4	1	1	1	1		
3	1	1	1			
4	1	1	1	1		
1	1					
S(*B*)	5	4	3	2	0	0
S'	4	3	3	2	1	1

=B

R						
2	1	1				
4	1	1	1			1
3	1	1	1			
4	1	1	1	1		
1	1					
S(*B*)	5	4	3	1	0	1
S'	4	3	3	2	1	1

RECONSTRUCTION

R						
2						
4						
3						
4						
1						
S	3	4	3	2	1	1

R						
2						
4						
3						
4						
1						
S'	4	3	3	2	1	1

=B

R						
2	1	1				
4	1	1	1	1		
3	1	1	1			
4	1	1	1	1		
1	1					
S(*B*)	5	4	3	2	0	0
S'	4	3	3	2	1	1

R						
2	1	1				
4	1	1	1			1
3	1	1	1			
4	1	1	1	1		
1	1					
S(*B*)	5	4	3	1	0	1
S'	4	3	3	2	1	1

RECONSTRUCTION

	2	1	1				
	4	1	1	1			1
R	3	1	1	1			
	4	1	1	1		1	
	1	1					
		5	4	3	0	1	1	*S*(*B*)
		4	3	3	2	1	1	*S'*

RECONSTRUCTION

R							
	2	1	1				
	4	1	1	1			1
	3	1	1	1			
	4	1	1	1		1	
	1	1					
$S(B)$		5	4	3	0	1	1
S'		4	3	3	2	1	1

RECONSTRUCTION

R						
2	1	1				
4	1	1	1 →			1
3	1	1	1 →			
4	1	1	1		1	
1	1					
S(*B*)	5	4	3	0	1	1
S'	4	3	3	2	1	1

2	1	1				
4	1	1		1		1
3	1	1		1		
4	1	1	1		1	
1	1					
S(*B*)	5	4	1	2	1	1
S'	4	3	3	2	1	1

RECONSTRUCTION

R							
2	1	1					
4	1	1	1			1	
3	1	1	1				
4	1	1	1		1		
1	1						
	5	4	3	0	1	1	$S(B)$
	4	3	3	2	1	1	S'

2	1	1					
4	1	1		1		1	
3	1	1		1			
4	1	1	1		1		
1	1						
	5	4	1	2	1	1	$S(B)$
	4	3	3	2	1	1	S'

R							
2	1	1					
4	1	1		1		1	
3	1	1		1			
4	1	1	1		1		
1	1						
	5	4	1	2	1	1	$S(B)$
	4	3	3	2	1	1	S'

RECONSTRUCTION

R						
2	1	1				
4	1	1	1			1
3	1	1	1			
4	1	1	1		1	
1	1					
$S(B)$	5	4	3	0	1	1
S'	4	3	3	2	1	1

2	1	1				
4	1	1		1		1
3	1	1		1		
4	1	1	1		1	
1	1					
$S(B)$	5	4	1	2	1	1
S'	4	3	3	2	1	1

R						
2	1	1				
4	1	1		1		1
3	1	1		1		
4	1	1	1		1	
1	1					
$S(B)$	5	4	1	2	1	1
S'	4	3	3	2	1	1

2	1		1			
4	1		1	1		1
3	1	1		1		
4	1	1	1		1	
1	1					
$S(B)$	5	2	3	2	1	1
S'	4	3	3	2	1	1

RECONSTRUCTION

R						
2	1	1				
4	1	1	1			1
3	1	1	1			
4	1	1	1		1	
1	1					
S(*B*)	5	4	3	0	1	1
S'	4	3	3	2	1	1

2	1	1				
4	1	1		1		1
3	1	1		1		
4	1	1	1		1	
1	1					
S(*B*)	5	4	1	2	1	1
S'	4	3	3	2	1	1

R						
2	1	1				
4	1	1		1		1
3	1	1		1		
4	1	1	1		1	
1	1					
S(*B*)	5	4	1	2	1	1
S'	4	3	3	2	1	1

2	1		1			
4	1		1	1		1
3	1	1		1		
4	1	1	1		1	
1	1					
S(*B*)	5	2	3	2	1	1
S'	4	3	3	2	1	1

RECONSTRUCTION

R							
	2		1	1			
	4	1		1	1		1
R	3	1	1		1		
	4	1	1	1		1	
	1	1					
		4	3	1	2	1	1	$S(B)$
		4	3	3	2	1	1	S'

RECONSTRUCTION

	2		1	1			
	4	1		1	1		1
R	3	1	1		1		
	4	1	1	1		1	
	1	1					
		4	3	1	2	1	1	*S*(*B*)
		4	3	3	2	1	1	*S'*

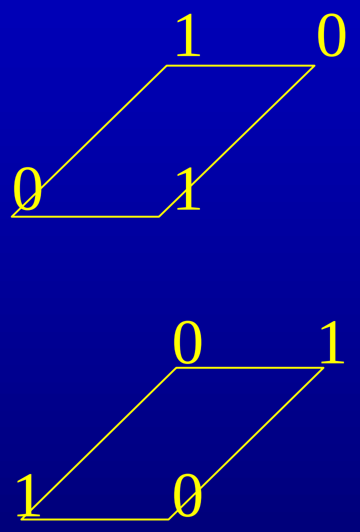
RECONSTRUCTION

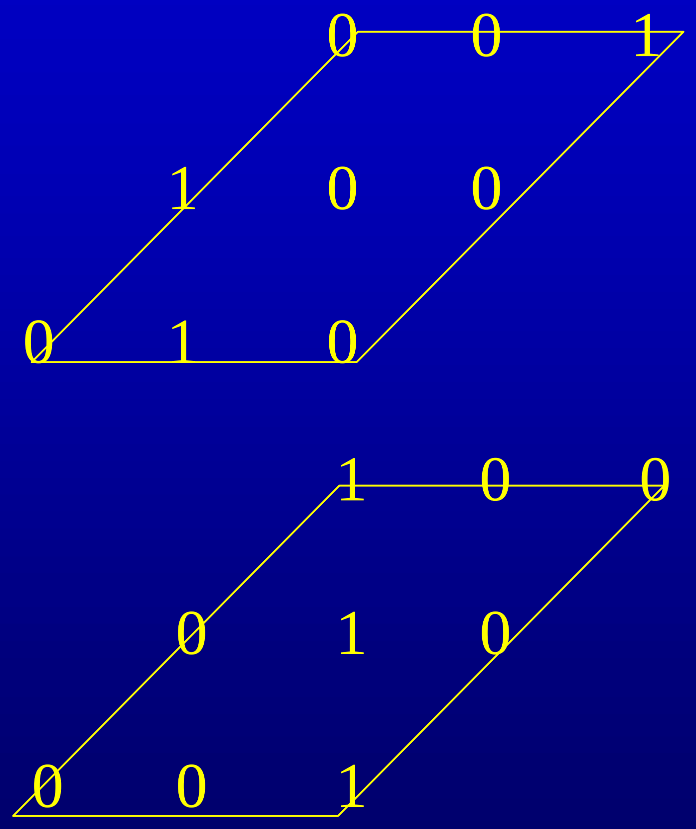
R						
2		1	1			
4	1		1	1		1
3	1	1		1		
4	1	1	1		1	
1	1					
S(*B*)	4	3	1	2	1	1
S'	4	3	3	2	1	1

2	1		1			
4		1	1	1		1
3	1	1		1		
4	1	1	1		1	
1		1				
S	3	4	3	2	1	1

3D RECONSTRUCTION FROM 3 PROJECTIONS

3D switching component





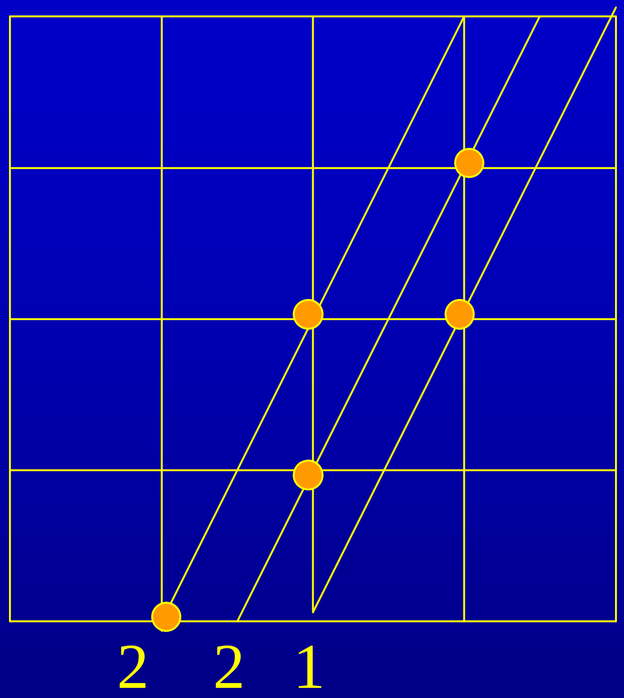
it is not necessary
for the uniqueness

3D

3D uniqueness, existence, and reconstruction problems are NP-hard.

Herman, Kong, 1999
Gardner, Gritzmann, 1998

MORE THAN 2 PROJECTIONS



(also) further projections are taken along lattice directions

In the case of more than 2 projections the uniqueness, existence, and reconstruction problems are NP-hard (in any dimensions).

A PRIORI INFORMATION

in order to reduce the number of possible solutions let us include some ***a priori*** information into the reconstruction of binary matrices

e.g.

hv-convexity

	1	1			
	1	1	1	1	1
1	1				
1	1	1			
1					

h-convex

	1				
	1		1	1	1
1	1	1			
1	1	1			
1					

v-convex

	1				
	1	1	1	1	1
1	1	1			
1	1	1			
1					

hv-convex

A PRIORI INFORMATION

e.g.
4-connectedness (NP-hard – Del Lungo, 1996)

	1	1			
	1	1	1		1
1	1			1	
1	1	1			
1					

not 4-connected
but 8-connected

	1				
	1		1	1	1
1	1	1	1		
1	1	1			
1					

4-connected

A PRIORI INFORMATION

e.g.

hv-convex, 4-connected

$O(mn \cdot \min\{m^2, n^2\})$ - Chrobak, Dürr, 1999

	1				
	1	1	1	1	1
1	1	1			
1	1	1			
1					

A PRIORI INFORMATION

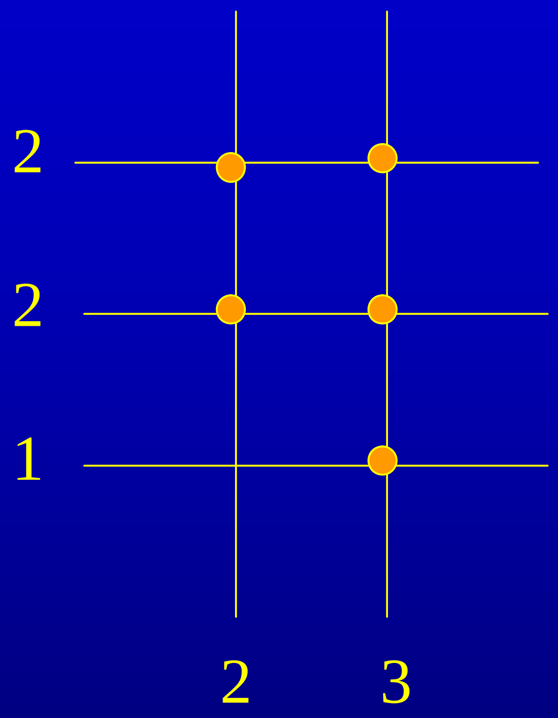
e.g.

hv-convex, 8-connected

$$O(mn\cdot\min\{m^2,n^2\})$$

	1				
	1	1	1	1	1
1					
1					
1					

SOLUTION
AS A LINEAR EQUATION SYSTEM?



$$\underbrace{\begin{aligned}
x_1 + x_2 \qquad\qquad\qquad\qquad &= 2 \\
x_3 + x_4 \qquad\qquad &= 2 \\
x_5 + x_6 &= 1 \\
x_1 \quad + x_3 \quad + x_5 \quad &= 2 \\
x_2 \quad + x_4 \quad + x_6 &= 2
\end{aligned}}_{Px} \quad \left.\vphantom{\begin{aligned}1\\1\\1\\1\\1\end{aligned}}\right\} b$$

$$P = \begin{pmatrix} 1 & 1 & & & & \\ & & 1 & 1 & & \\ & & & & 1 & 1 \\ 1 & & 1 & & 1 & \\ & 1 & & 1 & & 1 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

SOLUTION AS A LINEAR EQUATION SYSTEM ?

$$Px = b \qquad x \in \{0,1\}^{m \times n}$$

problems:

- binary! *x*,
- big system,
- underdeterined (#equation < #unknown),
- inconsistent (if there is noise)

OPTIMIZATION

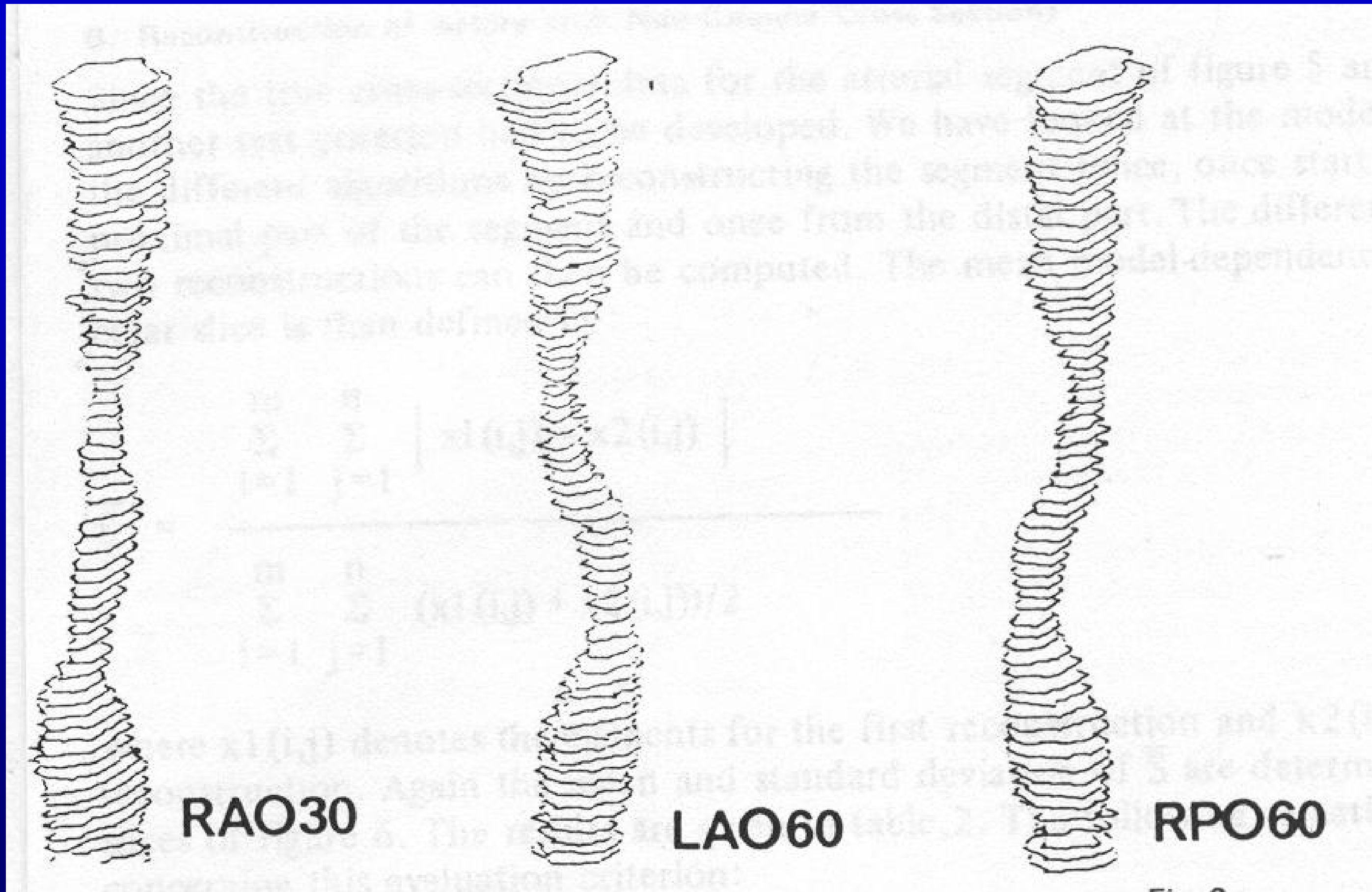
$$x \in \{0,1\}^{m \times n}$$
$$\Phi(x) = \|Px - b\|^2 \to \min$$

more generally:

$$x \in \{0,1\}^{m \times n}$$
$$\Phi(x) = \|Px - b\|^2 + g(x) \to \min$$

optimization method: e.g., simulated annealing

ANGIOGRAPHY

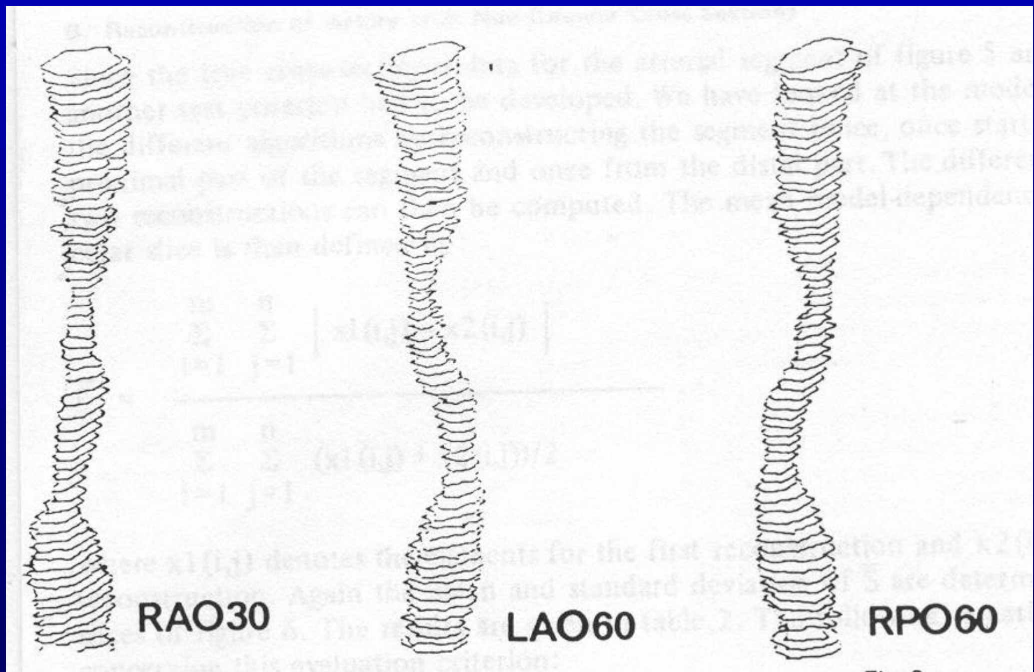


coronary arterial segments from two projections
Reiber, 1982

ANGIOGRAPHY

$$x \in \{0,1\}^{m \times n}$$

$$\Phi(x) = \|Px - b\|^2 + g(x) \to \min$$



a priory information:

the neighboring sections are similar

then let $g(x)$ such that

it gives high values if x is not similar to the neighboring section

„SIMILAR” NEIGHBOR SECTION

	1	1	1		
	1	1	1	1	
		1	1		

x'

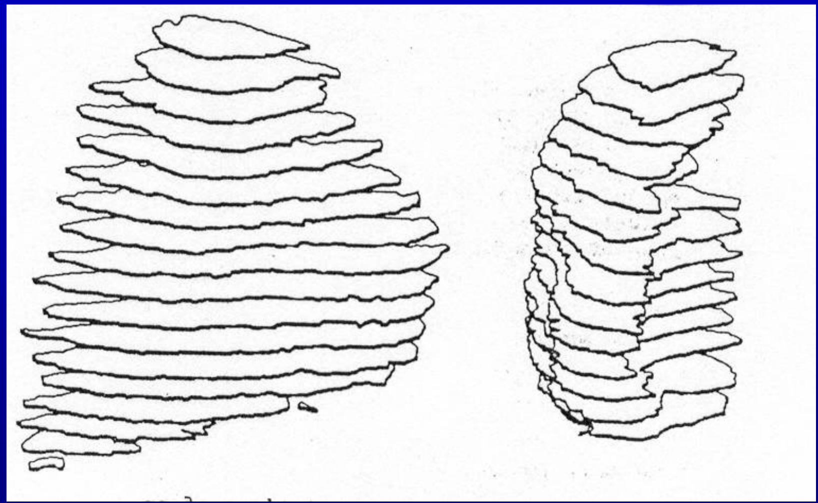
$$x \in \{0,1\}^{m \times n}$$

$$\Phi(x) = \|Px - b\|^2 + c'x \to \min$$

8	7	6	7	8	9
7	4	3	4	5	8
7	4	2	2	4	7
9	8	4	4	5	8
9	9	7	7	8	9

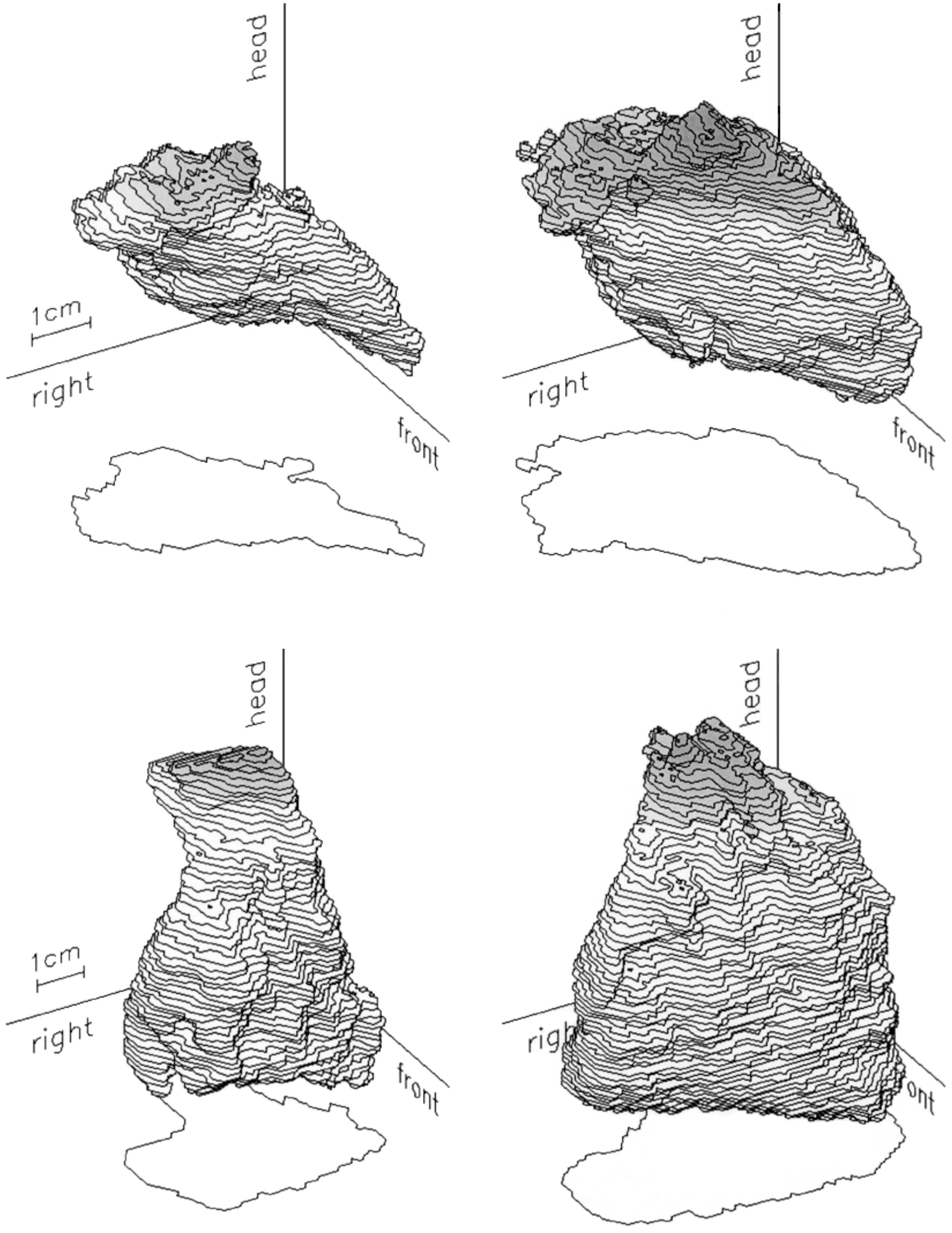
c

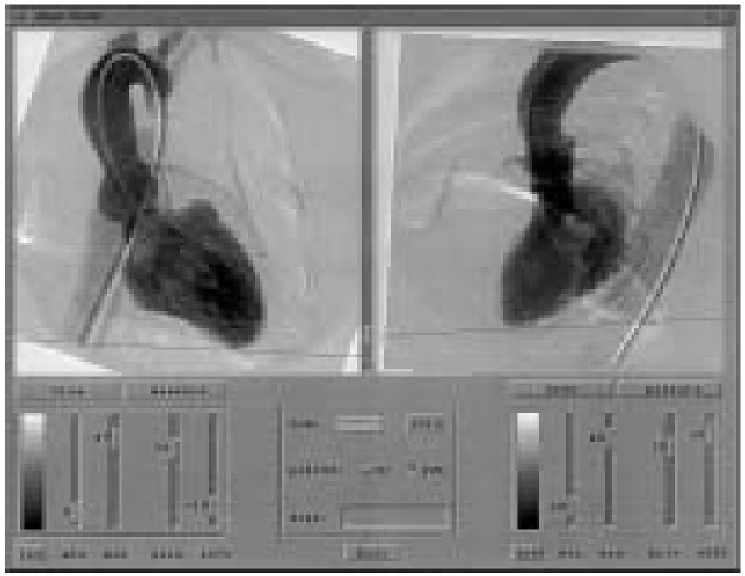
ANGIOGRAPHY



Onnasch, Prause, 1999

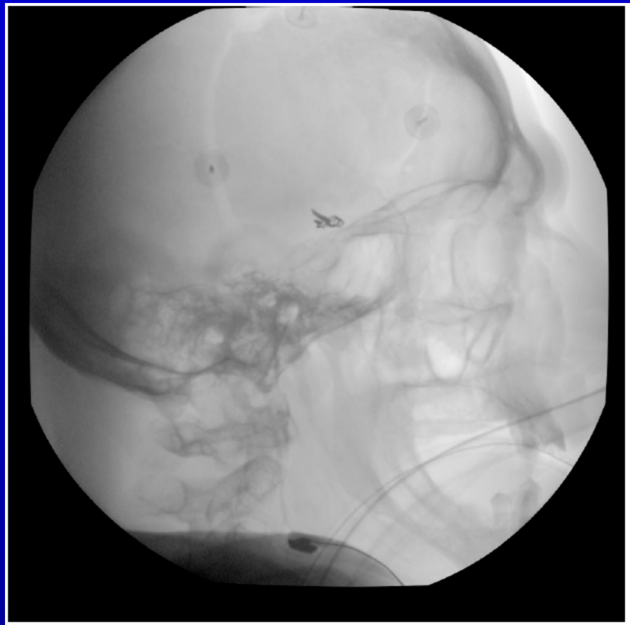
ANGIOGRAPHY





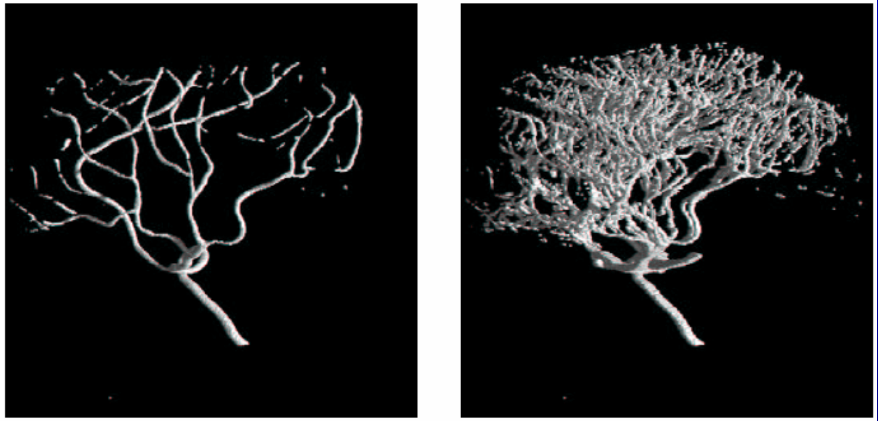
Onnasch, Prause, 1999

ANGIOGRÁFIA

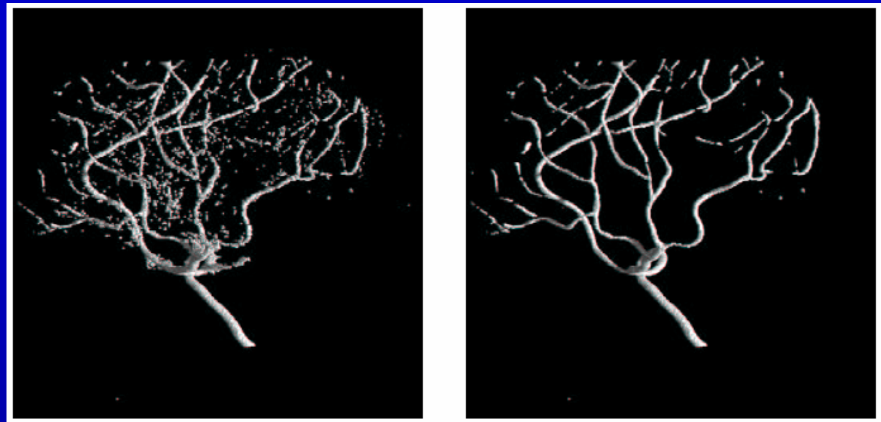


T. Schüle, 2003

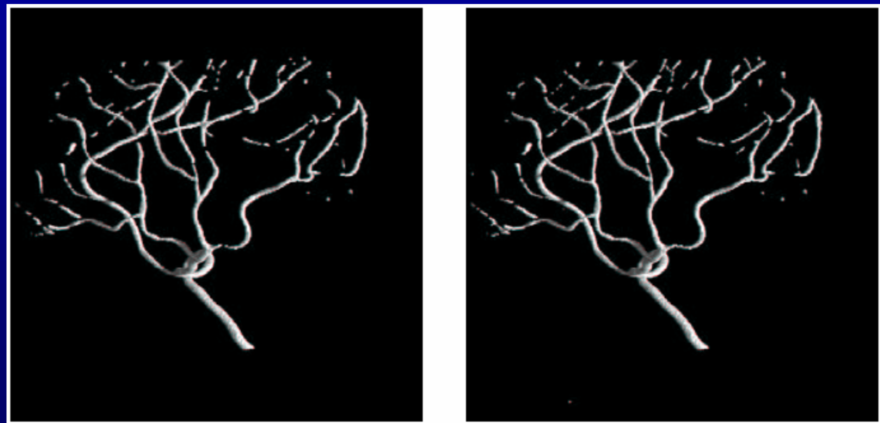
ANGIOGRÁFIA



original volume peel volume



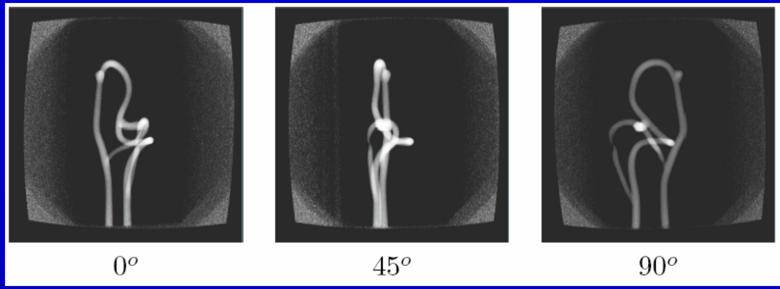
(BIF) (LSA)



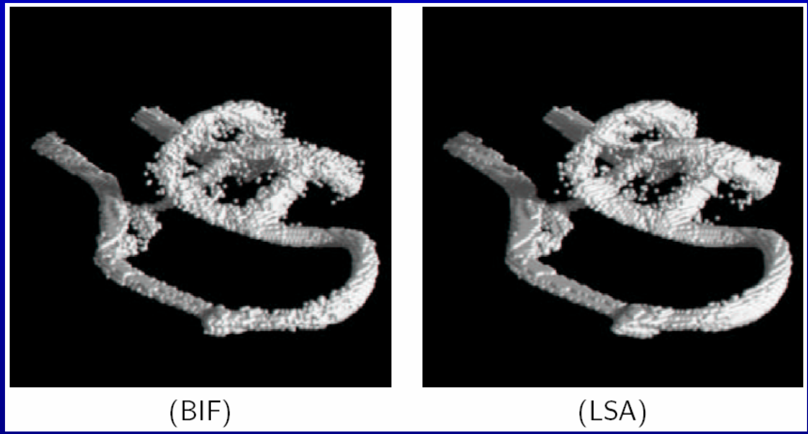
(BIF)+(FIRST) (LSA)+(FIRST)

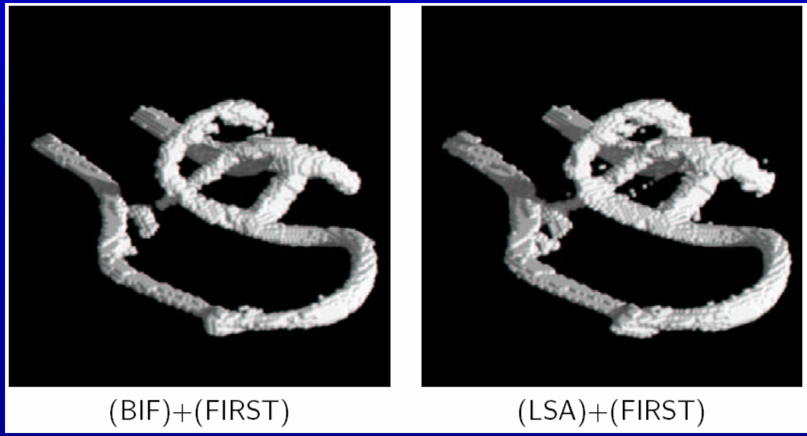
T. Schüle, 2003

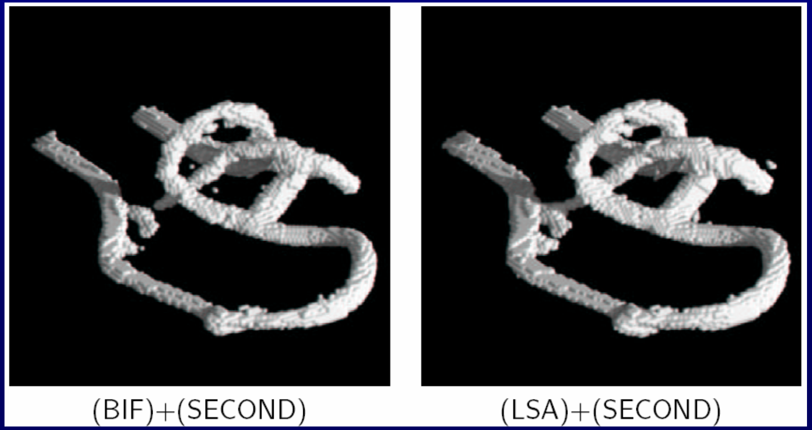
ANGIOGRÁFIA



5 vetület



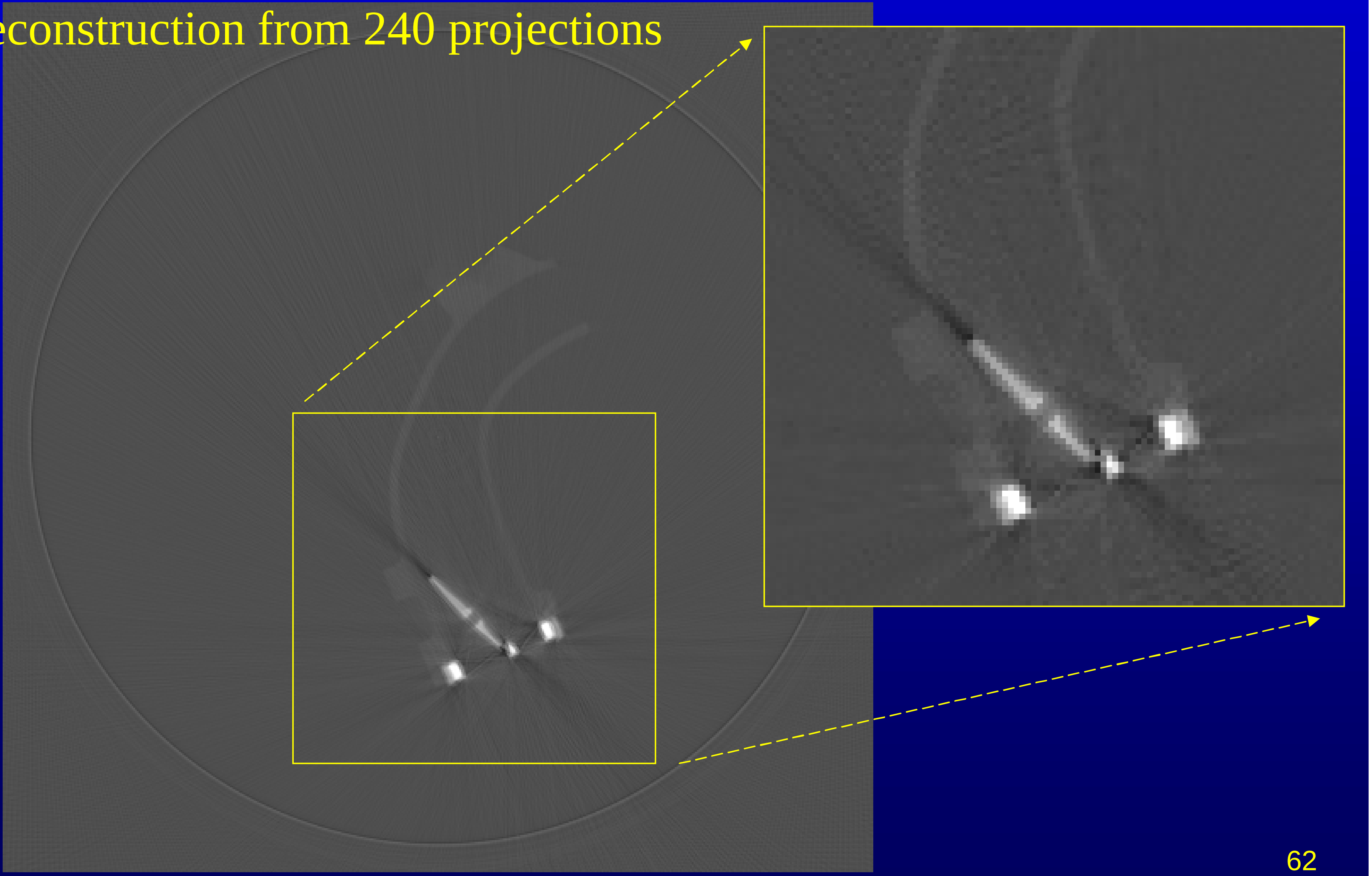




T. Schüle, 2003

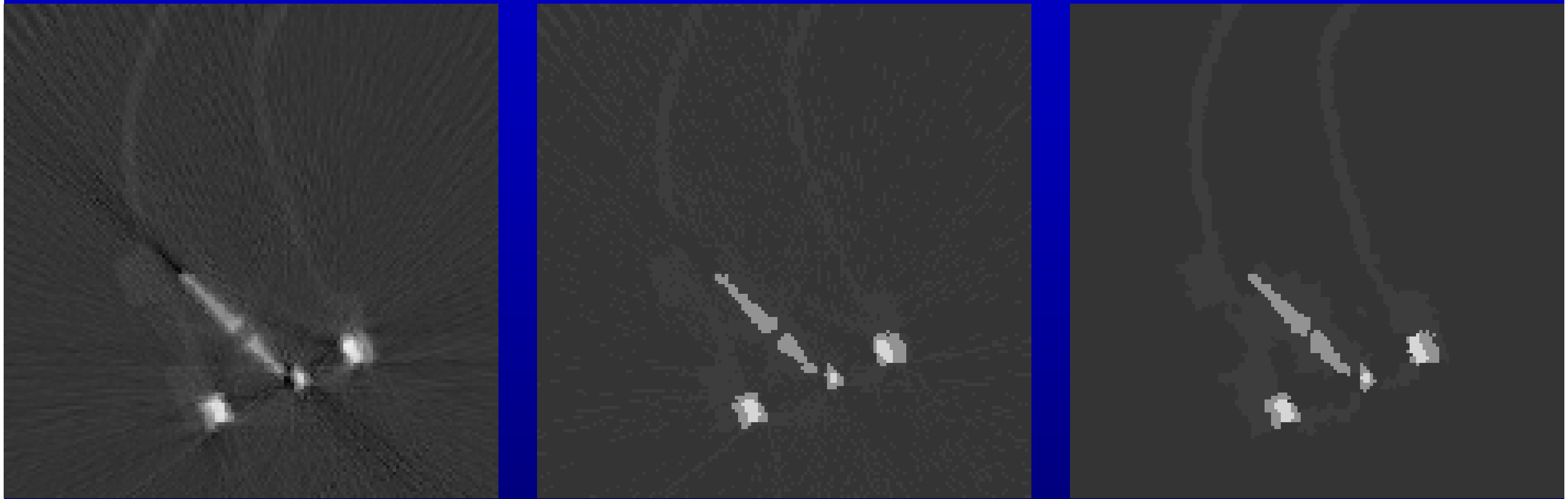
EXPERIMENT 3

reconstruction from 240 projections



EXPERIMENT 3

reconstruction from 80 projections



filtered back-proj. | filtered back-proj. + discretization | DT

L. Ruskó, A.K., Z. Kiss, L. Rodek, 2003

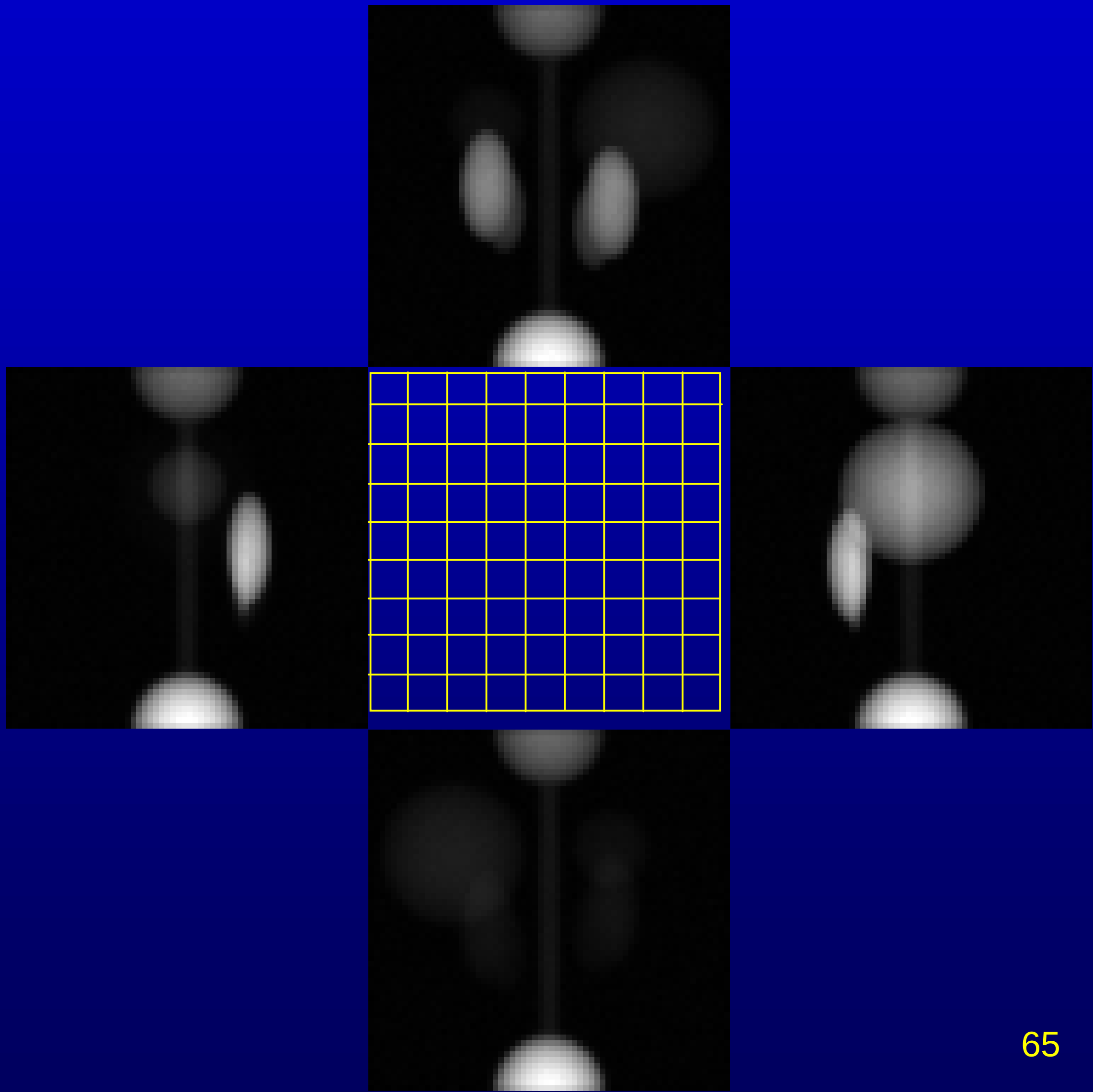
- Function of 3D dynamic object
  - can be expressed as a linear combination of binary valued functions and noise

$$f(r,t) = c_1(t) \cdot f_1(r) + c_2(t) \cdot f_2(r) + \cdots + c_K(t) \cdot f_K(r) + \eta(r,t)$$

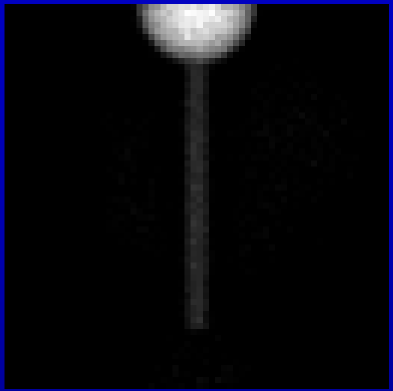
  - to reconstruct the function from its absorbed projections

Projections

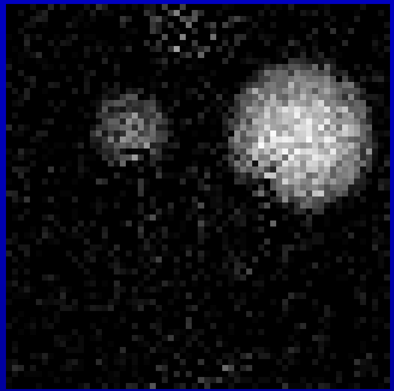
- 4 views changing in time
  - attenuation,
  - scatter,
  - depth dependent resolution,
  - partial volume effects,
  - Poisson noise



Factor analysis result (Up projections)

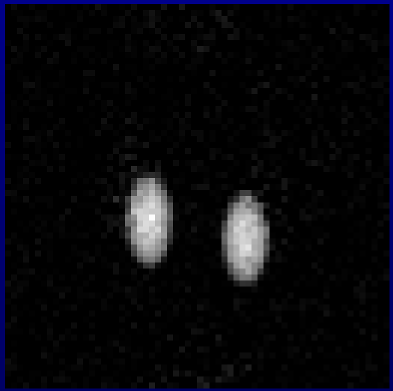


Heart & aorta

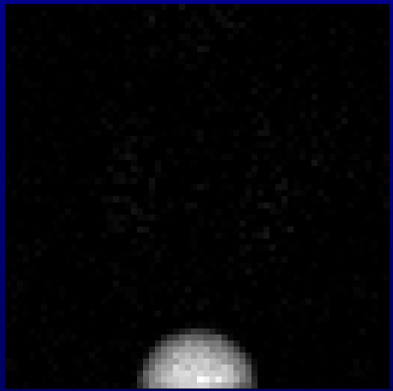


Liver & spleen

Renal parenchymas

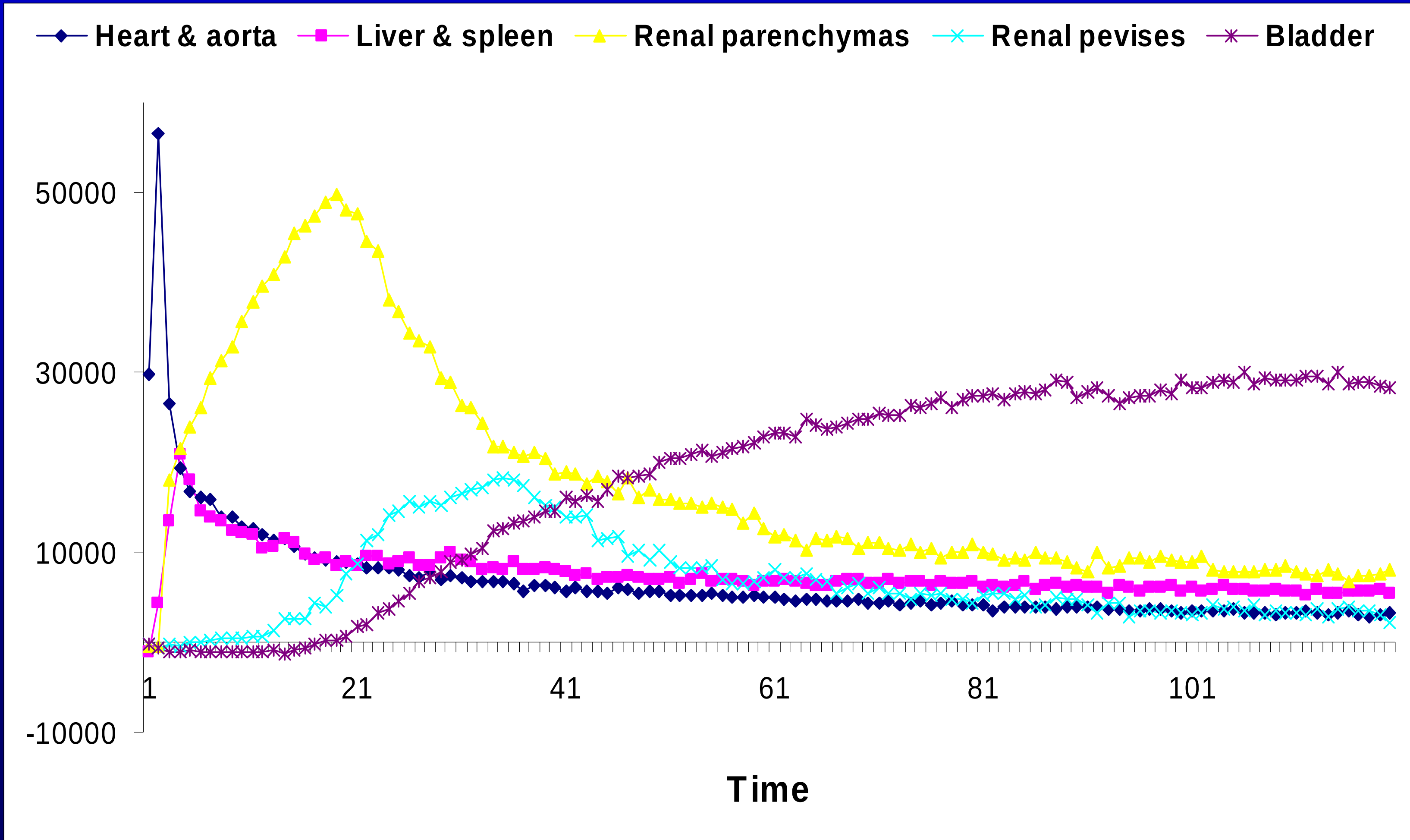


Renal pelvises



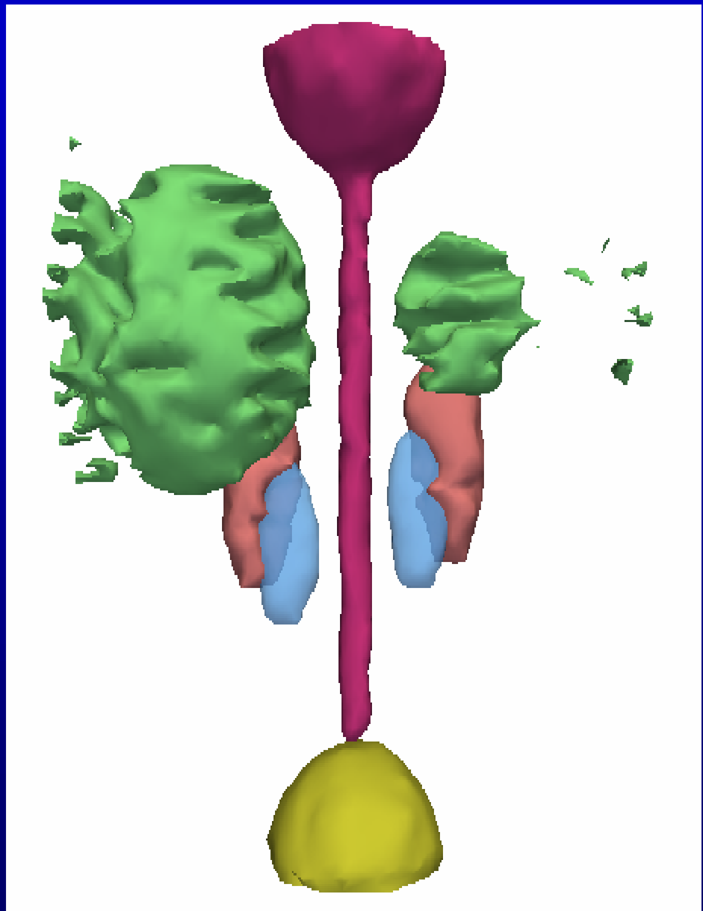
Bladder

Factor analysis result



Curves of the weighting coefficients of the Up projections

Reconstructed structures I.

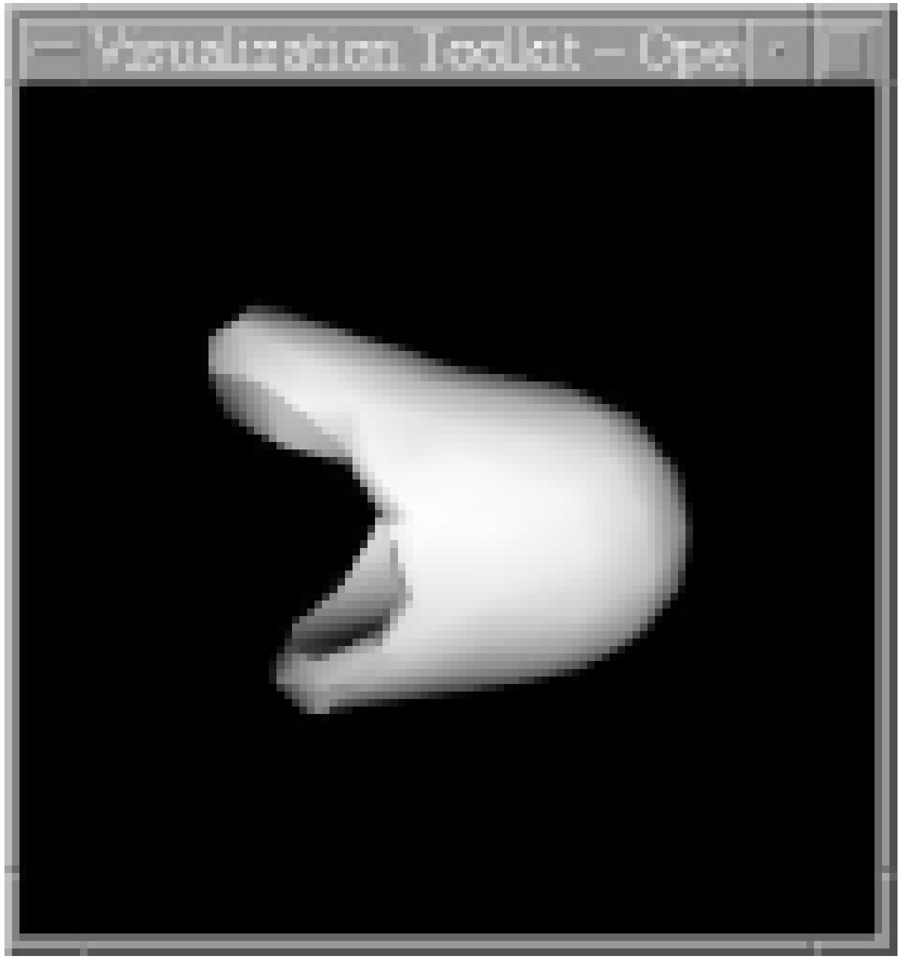


Result of the reconstruction

Structure name	Rec. volume
Heart and aorta	96%
Liver and spleen	90%
Renal parenchymas	107%
Renal pelvises	85%
Urinary bladder	92%

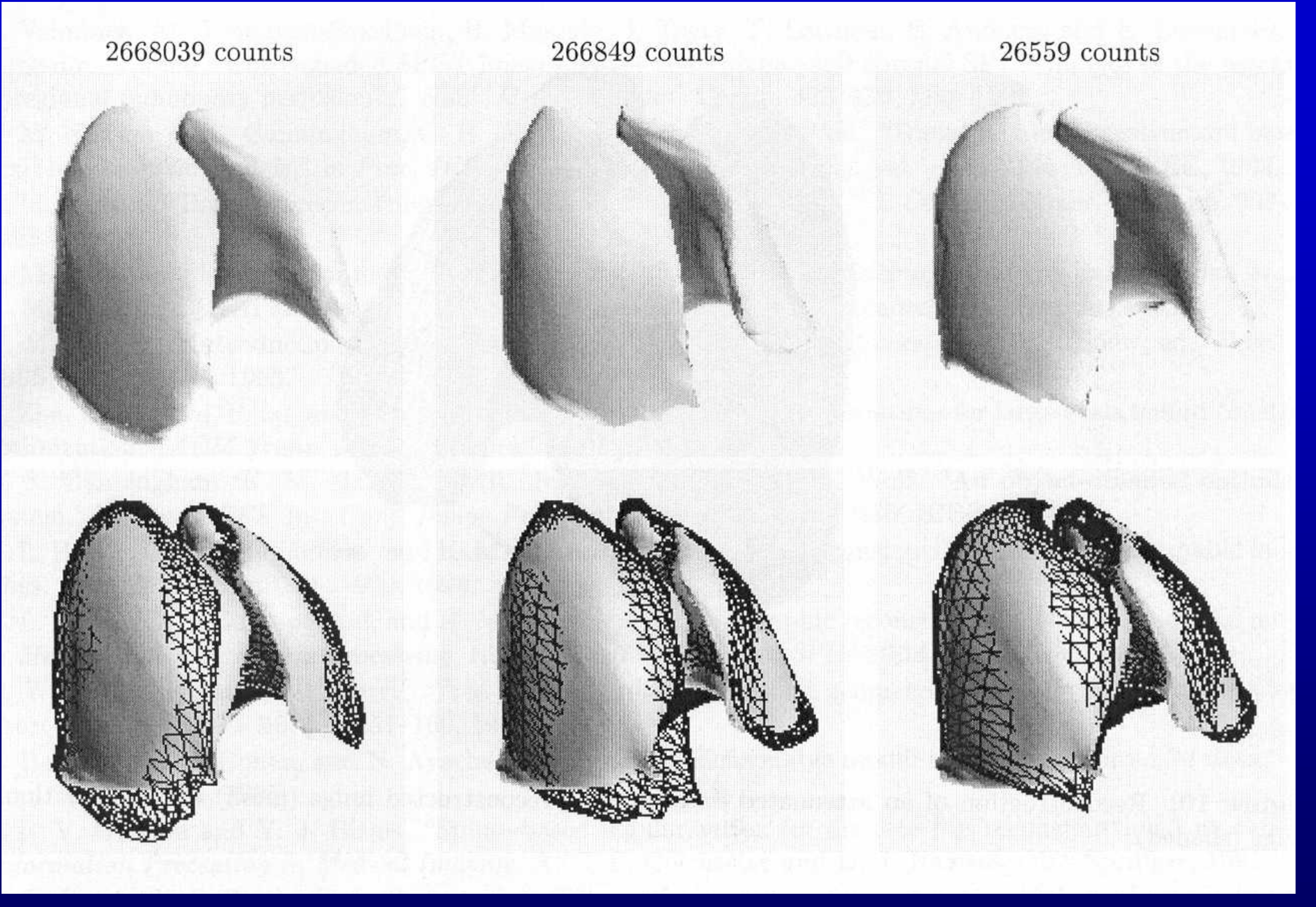
A. Nagy, A.K., M. Samal, 2005

SPECT



MAP reconstructions of the bolus boundary surface
Cunningham, Hanson, Battle, 1998

SPECT

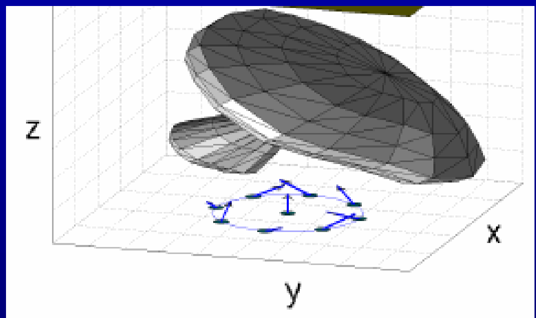


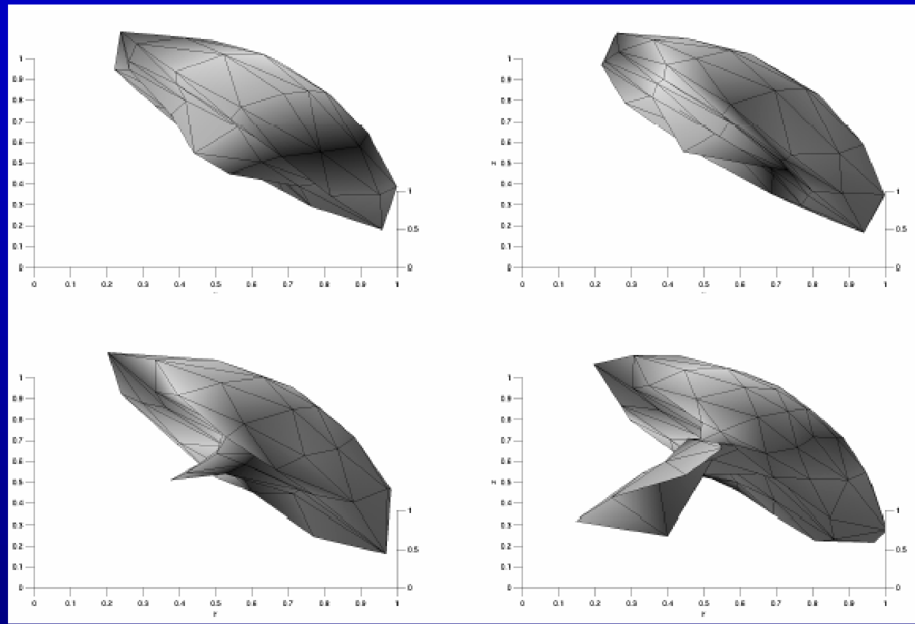
Tomographic reconstruction using free-form deformation models
FFDs reconstructions for different levels of noise
Battle, 1999

OBJECT TO BE RECONSTRUCTED

deformable geometric models (parametric models):

- quadrics,
- superquadrics,
- harmonic surfaces,
- splines





Polyhedral reconstructions with various initializations, which are the results of quadric, superquadric, or harmonic methods. First row: initialization by the ellipsoid and super-ellipsoid. Second row: initialization by the harmonic surfaces.

C. Soussen, A. Mohammad-Djafari, 2000