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Discrete Tomography

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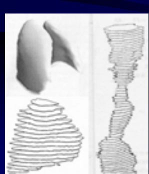
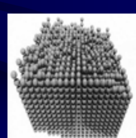
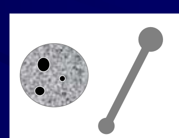
Outline

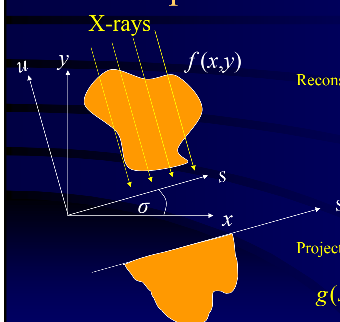
- What is Computerized Tomography (CT)
- What is Discrete Tomography (DT)
- Binary tomography using 2 projections
- Ambiguity and complexity problems
- A priori information
- Reconstruction as optimization
- Open questions

Tomography

- A technique for imaging the 2D cross-sections of 3D objects
  - parts of human body with X-rays
  - structure of molecules or crystals
  - obtaining shape information of industrial parts

Computerized Tomography



Reconstruct $f(x,y)$ from its projections

Projection of angle σ is a line integral:

$$g(s,\sigma) = \int_{-\infty}^{\infty} f(x,y)du$$

Discrete Tomography

- in CT we need a few hundred projections
  - time consuming
  - expensive
  - sometimes we do not have many projections
- in many applications the range of the function to be reconstructed is discrete and known → DT (only few 2-10 projections are needed)
- $f: R^2 \rightarrow D$, if $D=\{0,1\}$ then f has only binary values (presence or absence of material) → binary tomography

Binary tomography

- discrete set: a finite subset of the 2-dimensional integer lattice

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- reconstruct a discrete set from its projections

Reconstruction from 2 projections

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Reconstruction from 2 projections

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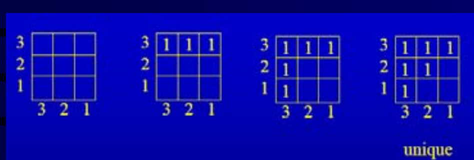
Examples



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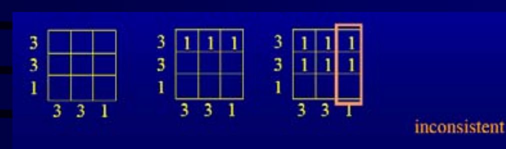
Examples



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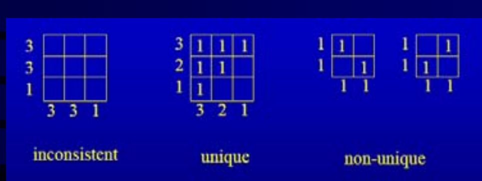
Classification



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Main Problems

1) Consistency
2) Uniqueness
3) Reconstruction

3) → 1)

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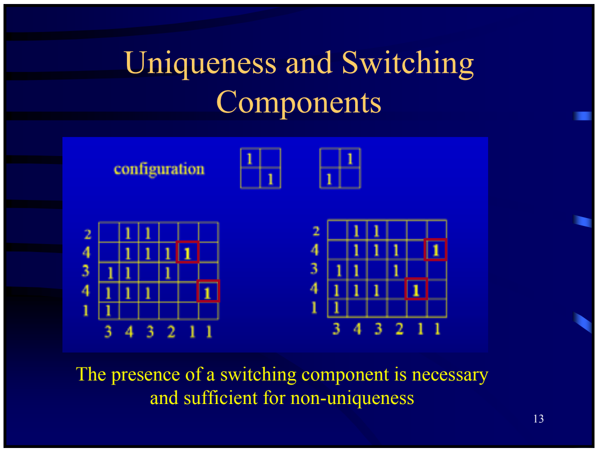
Uniqueness and Switching Components
configuration
The presence of a switching component is necessary and sufficient for non-uniqueness
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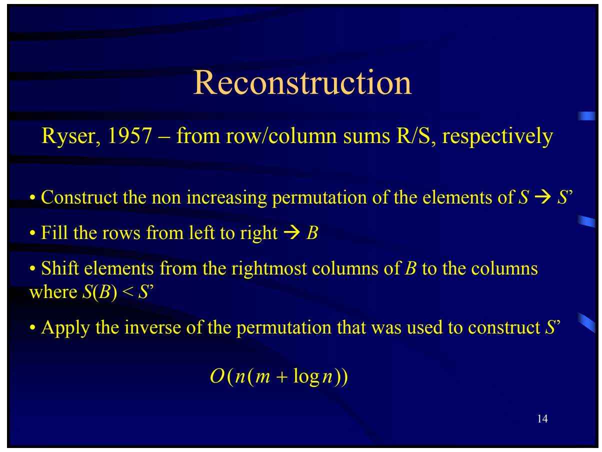
Reconstruction
Ryser, 1957 – from row/column sums R/S, respectively
• Construct the non increasing permutation of the elements of S → S'
• Fill the rows from left to right → B
• Shift elements from the rightmost columns of B to the columns where S(B) < S'
• Apply the inverse of the permutation that was used to construct S'
$O(n(m + \log n))$
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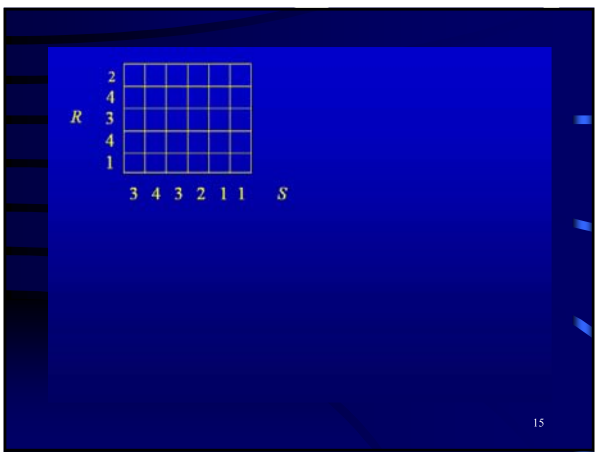
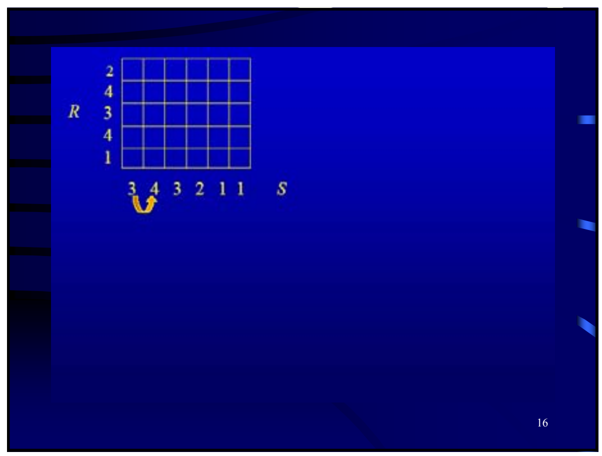

R
3 4 3 2 1 1 S
15


R
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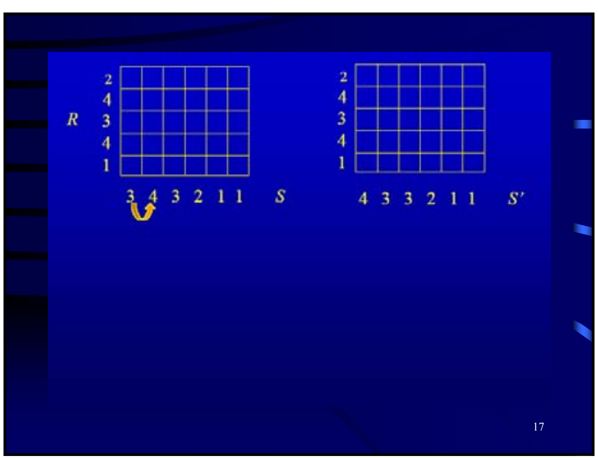
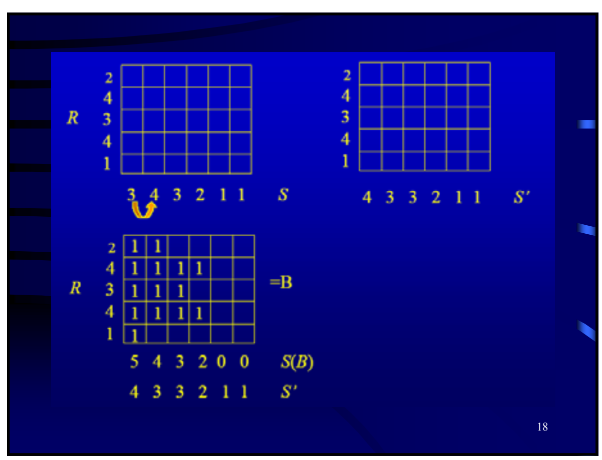
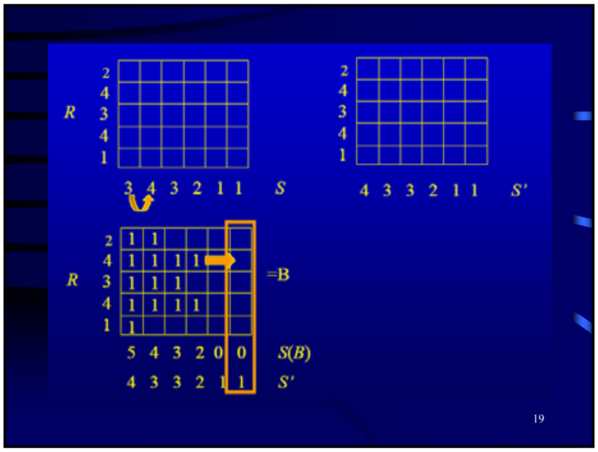

R
3 4 3 2 1 1 S
4 3 3 2 1 1 S'
17


R
3 4 3 2 1 1 S
4 3 3 2 1 1 S'
=B
5 4 3 2 0 0 S(B)
4 3 3 2 1 1 S'
18


R
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4 3 3 2 1 1 S'
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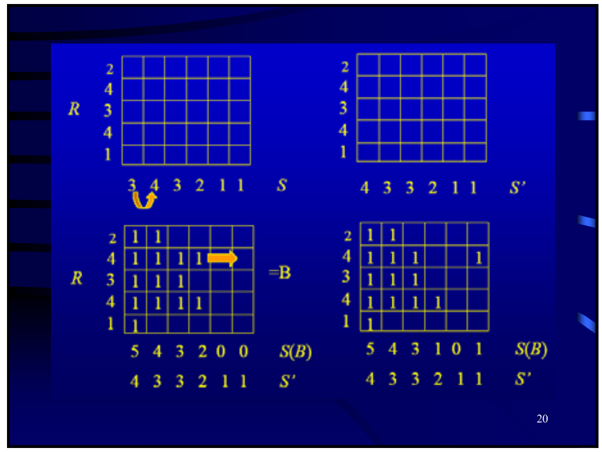
19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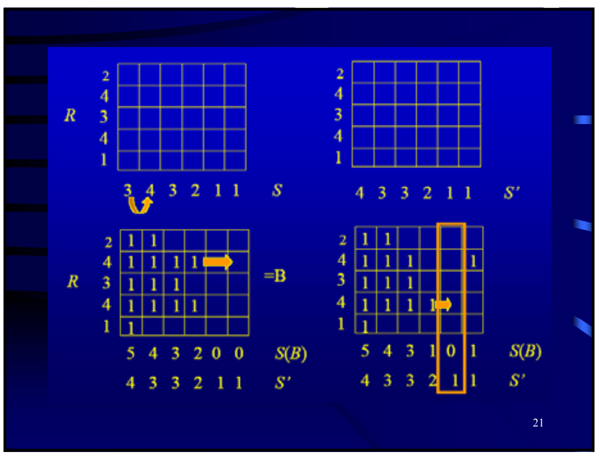
4 3 3 2 1 1 S'
20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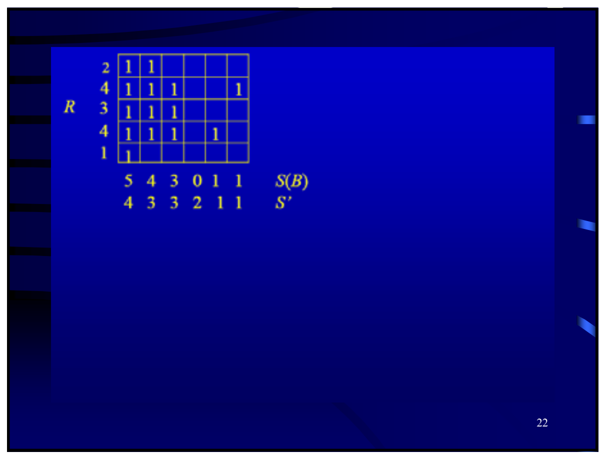
21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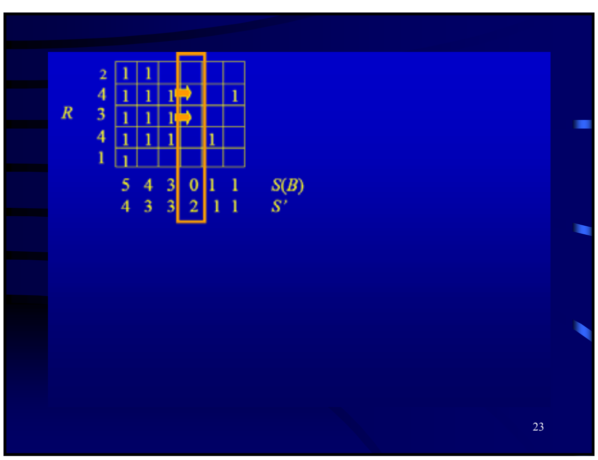
22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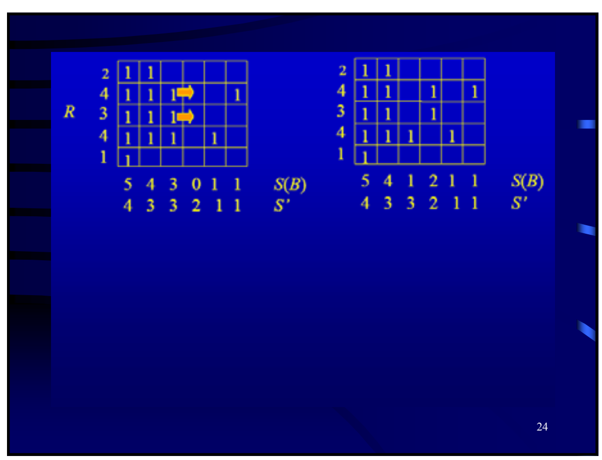
R
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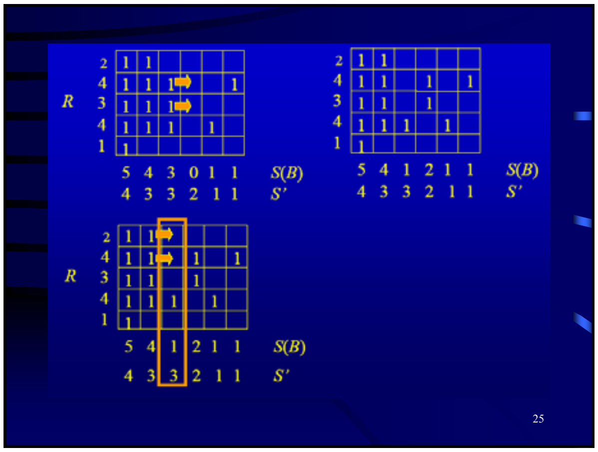
24


R
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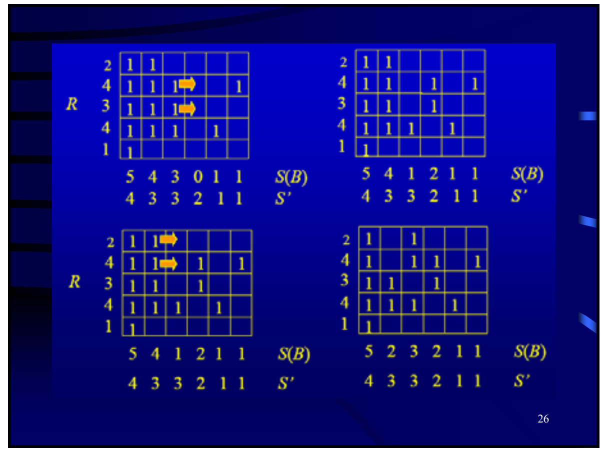
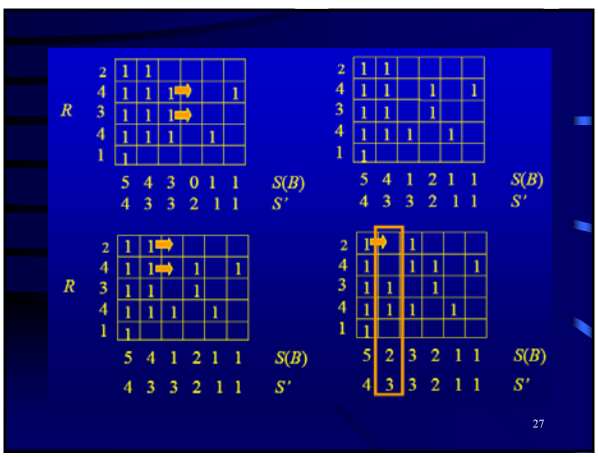
4 3 3 2 1 1 S'
5 4 1 2 1 1 S(B)
4 3 3 2 1 1 S'
R
5 4 1 2 1 1 S(B)
4 3 3 2 1 1 S'
25


R
5 4 3 0 1 1 S(B)
4 3 3 2 1 1 S'
5 4 1 2 1 1 S(B)
4 3 3 2 1 1 S'
R
5 4 1 2 1 1 S(B)
4 3 3 2 1 1 S'
5 2 3 2 1 1 S(B)
4 3 3 2 1 1 S'
26


R
5 4 3 0 1 1 S(B)
4 3 3 2 1 1 S'
5 4 1 2 1 1 S(B)
4 3 3 2 1 1 S'
R
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4 3 3 2 1 1 S'
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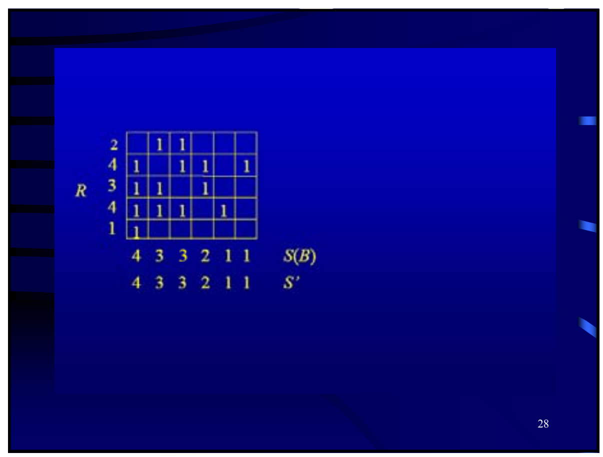
27


R
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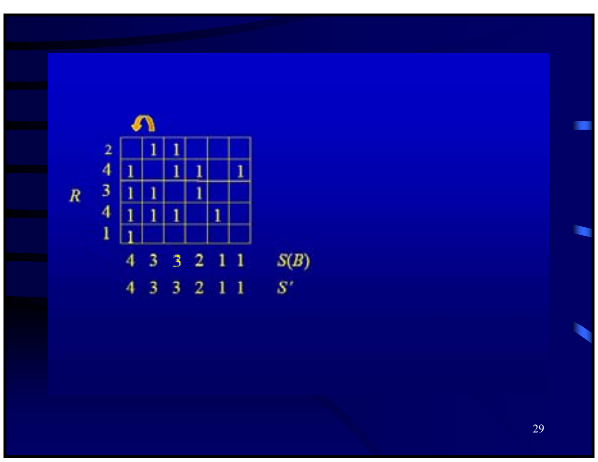
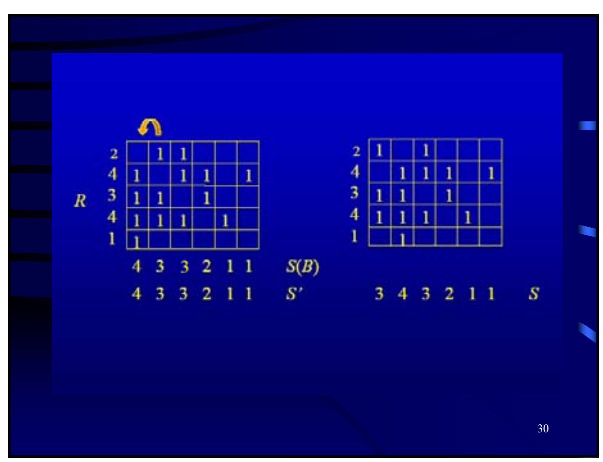
4 3 3 2 1 1 S'
28


R
4 3 3 2 1 1 S(B)
4 3 3 2 1 1 S'
29


R
4 3 3 2 1 1 S(B)
4 3 3 2 1 1 S'
3 4 3 2 1 1 S
30

Consistency

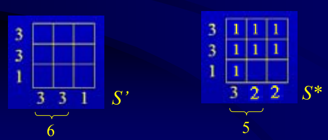
- Necessary condition: compatibility

$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$$

$$r_i \le n \ (i = 1,\dots,m),\ s_j \le m \ (j = 1,\dots,n)$$

- Gale, Ryser, 1957: there exist a solution iff

$$\sum_{j=1}^{k} s'_j \le \sum_{j=1}^{k} s^*_j \quad k = 1,\dots,n$$



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Ambiguity

Due to the presence of switching components there can be many solutions with the same two projections

Solutions:

1. Further projections can be taken along lattice directions
2. A priori information of the set to be reconstructed can be used

In the case of more than 2 projections uniqueness, consistency and reconstruction problems are NP-hard

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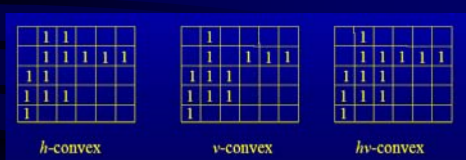
Convexity



hv-convex: NP-complete, Woeginger, 1996

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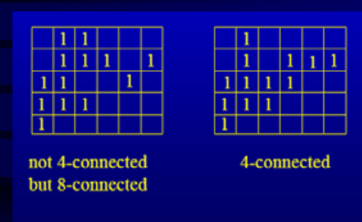
Connectedness



4-connected: NP-complete, Woeginger, 1996

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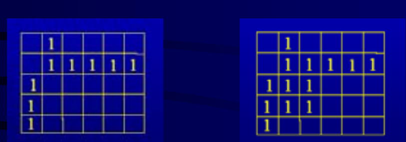
hv-Convex and Connected Sets



hv-convex 4-connected: $O(mn \cdot \min\{m^2, n^2\})$ - Chrobak, Dürr, 1999

hv-convex 8-connected: $O(mn \cdot \min\{m^2, n^2\})$ - Kuba, 1999

hv-convex 8- but not 4-connected: $O(mn \cdot \min\{m, n\})$

- Balázs, Balogh, Kuba, 2005

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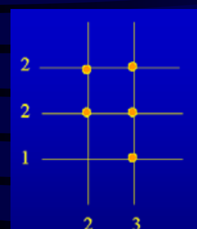
Reconstruction as Optimization



$$\left.\begin{aligned} x_1 + x_2 \qquad\qquad\qquad &= 2 \\ x_3 + x_4 \qquad\quad &= 2 \\ x_5 + x_6 &= 1 \\ x_1 \quad + x_3 \quad + x_5 \quad &= 2 \\ x_2 \quad + x_4 \quad + x_6 &= 3 \end{aligned}\right\} b$$

Px

$$P = \begin{pmatrix} 1 & 1 & & & & \\ & & 1 & 1 & & \\ & & & & 1 & 1 \\ 1 & & 1 & & 1 & \\ & 1 & & 1 & & 1 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

Optimization

$Px = b \quad x \in \{0,1\}^{m \times n}$

Problems:

- binary variables
- big system
- underdetermined (#equations < #unknowns)
- inconsistent (if there is noise)

$$x \in \{0,1\}^{m \times n}$$

$$\Phi(x) = \|Px - b\|^2 + g(x) \rightarrow \min$$

Term for prior information: convexity, similarity to a model image, etc.

optimization method: e.g., simulated annealing

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Open Problems

- New kinds of prior information (e.g., model image, special convexity properties, …)
- Efficient heuristics for NP-hard reconstruction
- Guessing geometrical properties from projections
- Tomography in higher dimensions
- Stability

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Thank you for your attention!