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Binary Tomography

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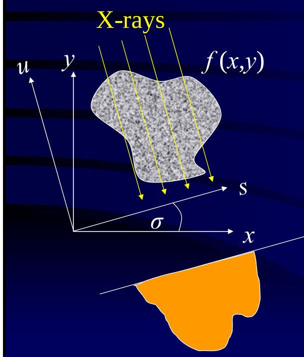
Outline

- What is Computerized Tomography (CT)
- What is Discrete Tomography (DT)
- Binary tomography using 2 projections
- Ambiguity and complexity problems
- A priori information
- Reconstruction as optimization
- Open questions

Computerized Tomography



- A technique for imaging the 2D cross-sections of 3D objects
- Reconstruct $f(x,y)$ from its projections
- Projection in direction u (defined by the angle σ) can be obtained by calculating the line integrals along each line parallel to u

$$g(s,\sigma)=\int_{-\infty}^{\infty} f(x,y)\,du$$

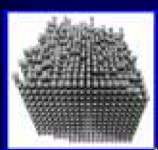
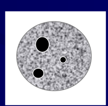
Discrete Tomography

- in CT we need a few hundred projections
  - time consuming
  - expensive
  - the aquisition of many projections may damage the object
- in many applications the range of the function to be reconstructed is discrete and known → DT (only few (2-10) projections are needed)

Binary Tomography

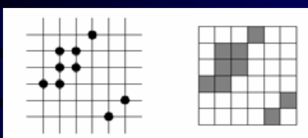
the range of the function to be reconstructed is {0,1} (absence or presence of material)

- angiography: parts of human body with X-rays
- electron microscopy: structure of molecules or crystals
- industrial applications: obtaining shape information of objects

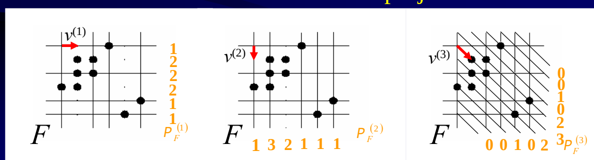
Binary tomography

- discrete set: a finite subset of the 2D integer lattice

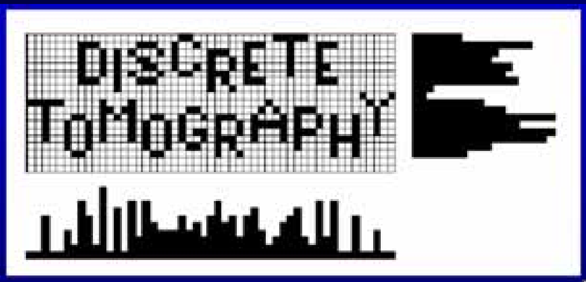


$$\begin{bmatrix}0&0&0&1&0&0\\0&1&1&0&0&0\\0&1&1&0&0&0\\1&1&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{bmatrix}$$

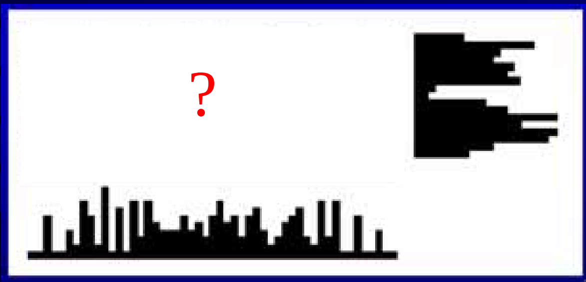
- reconstruct a discrete set from its projections



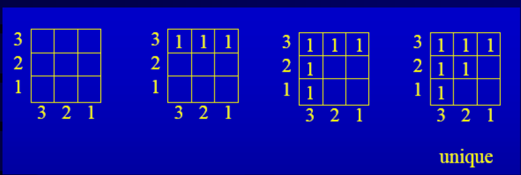
Reconstruction from 2 Projections



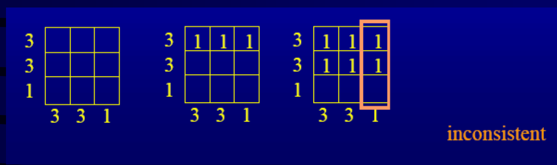
Reconstruction from 2 Projections



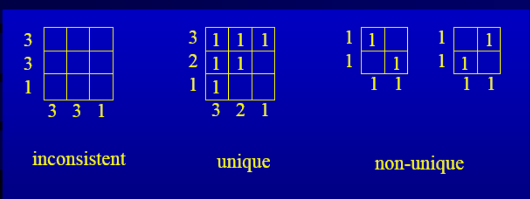
Examples



Examples



Classification

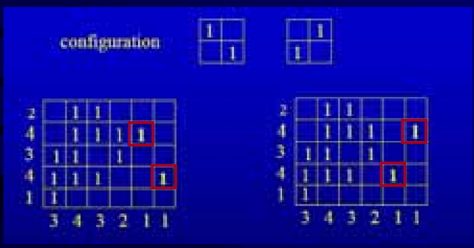


Main Problems

1) Consistency
2) Uniqueness
3) Reconstruction

   3) → 1)

Uniqueness and Switching Components



The presence of a switching component is necessary and sufficient for non-uniqueness

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Reconstruction

Ryser, 1957 – from row/column sums R/S, respectively

- Construct the non increasing permutation of the elements of $S \rightarrow S'$
- Fill the rows from left to right $\rightarrow B$
- Shift elements from the rightmost columns of B to the columns where $S(B) < S'$
- Apply the inverse of the permutation that was used to construct S'

$$O(n(m + \log n))$$

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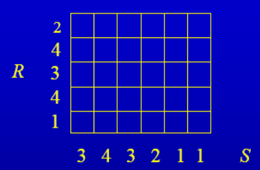

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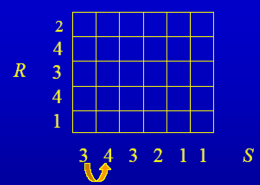

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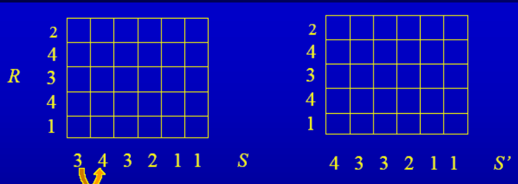

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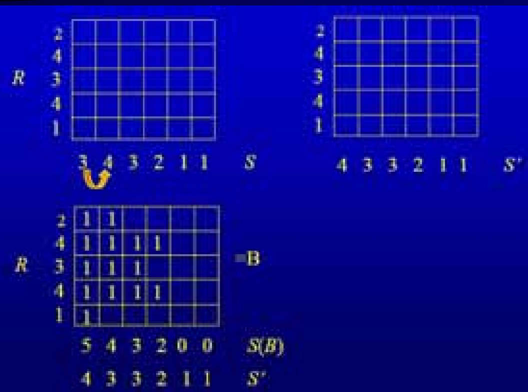

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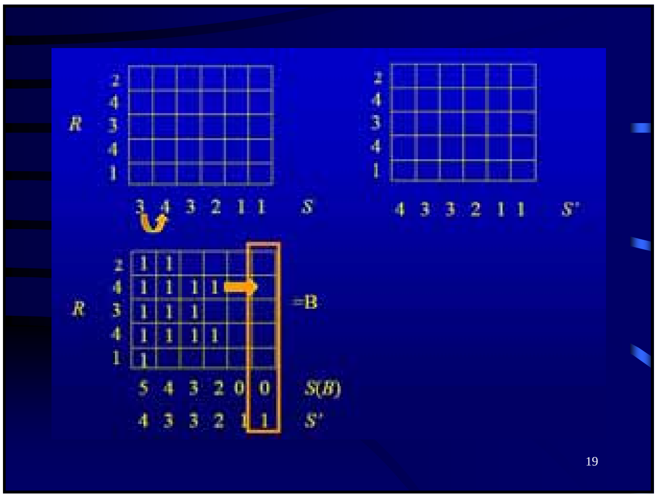
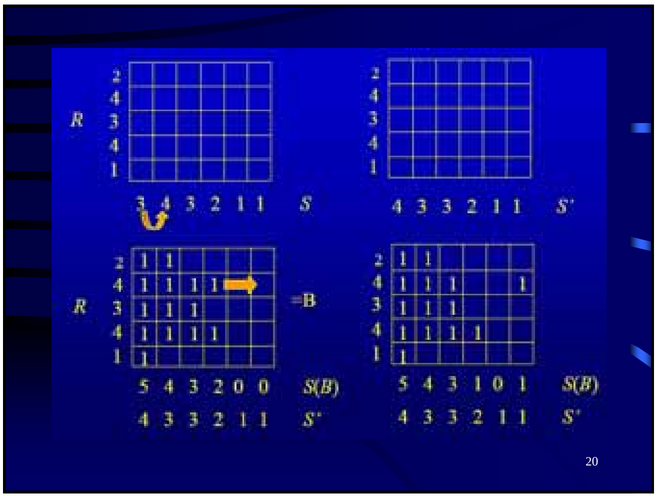

R
3 4 3 2 1 1 S
4 3 3 2 1 1 S'
=B
5 4 3 2 0 0 S(B)
4 3 3 2 1 1 S'
19


R
3 4 3 2 1 1 S
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=B
5 4 3 2 0 0 S(B)
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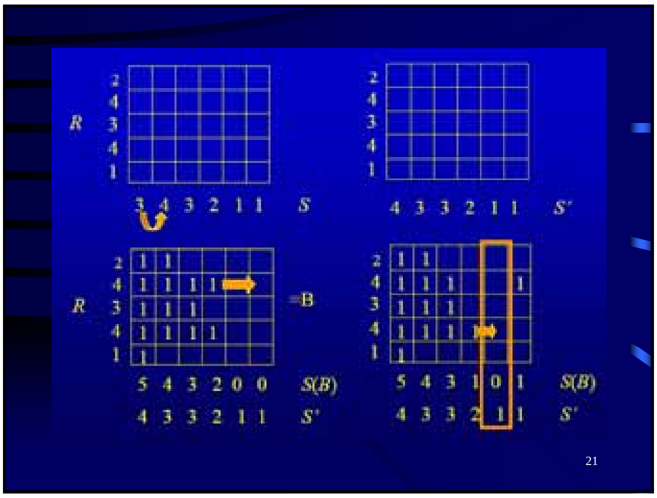
20


R
3 4 3 2 1 1 S
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=B
5 4 3 2 0 0 S(B)
4 3 3 2 1 1 S'
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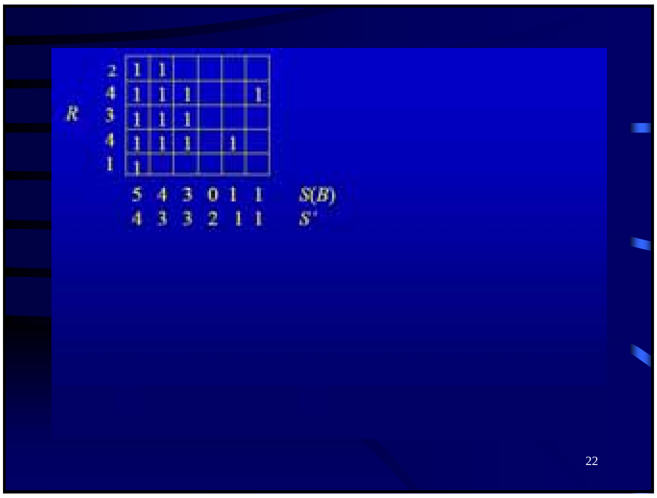
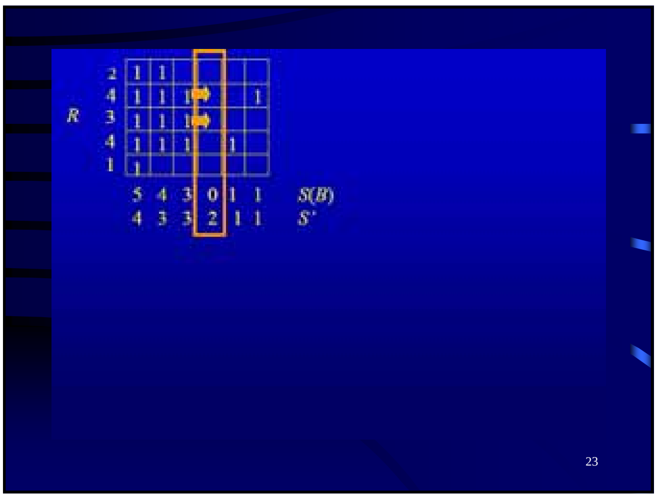
4 3 3 2 1 1 S'
21


R
5 4 3 0 1 1 S(B)
4 3 3 2 1 1 S'
22


R
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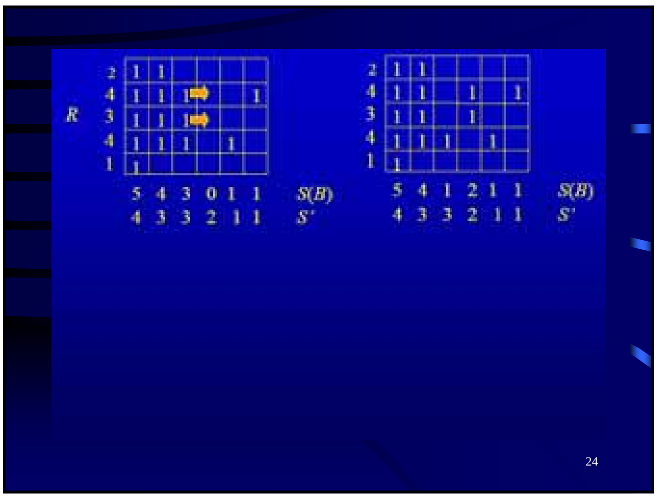
23


R
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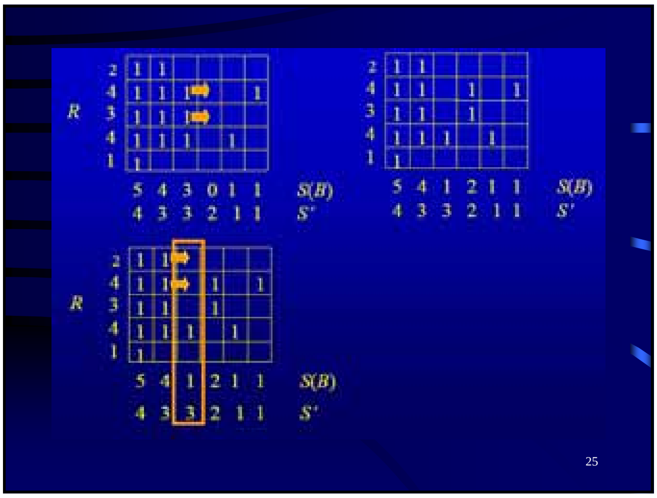
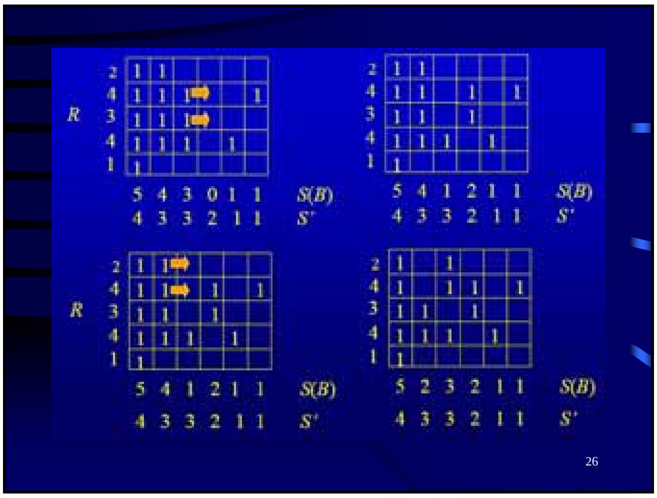
4 3 3 2 1 1 S'
5 4 1 2 1 1 S(B)
4 3 3 2 1 1 S'
24


R
2 4 3 4 1
5 4 3 0 1 1 S(B)
4 3 3 2 1 1 S'
5 4 1 2 1 1 S(B)
4 3 3 2 1 1 S'
5 4 1 2 1 1 S(B)
4 3 3 2 1 1 S'
25


R
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4 3 3 2 1 1 S'
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4 3 3 2 1 1 S'
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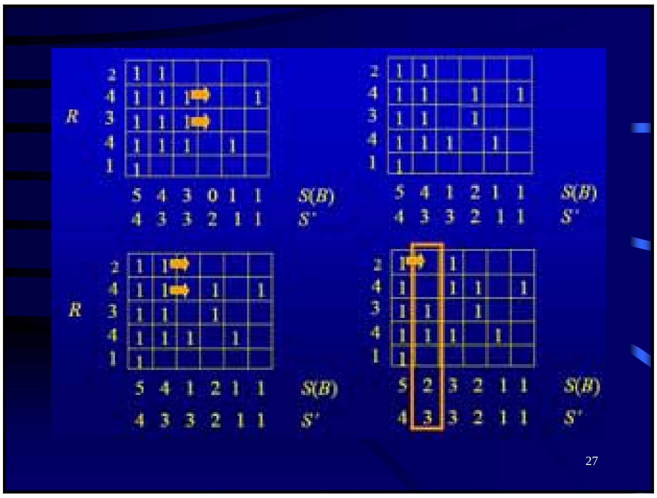
R
2 4 3 4 1
5 4 3 0 1 1 S(B)
4 3 3 2 1 1 S'
5 4 1 2 1 1 S(B)
4 3 3 2 1 1 S'
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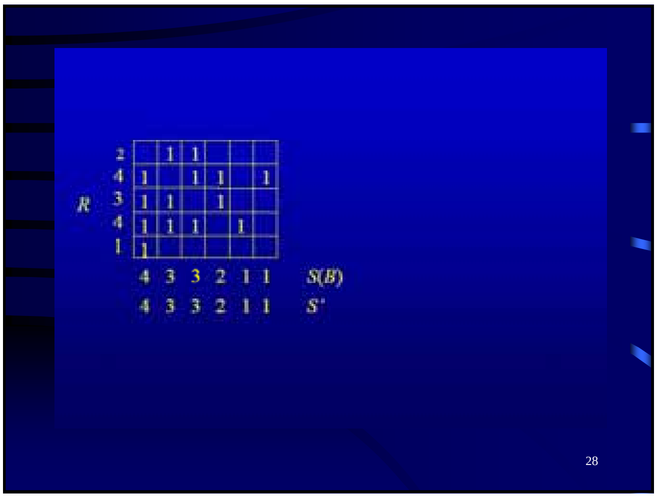
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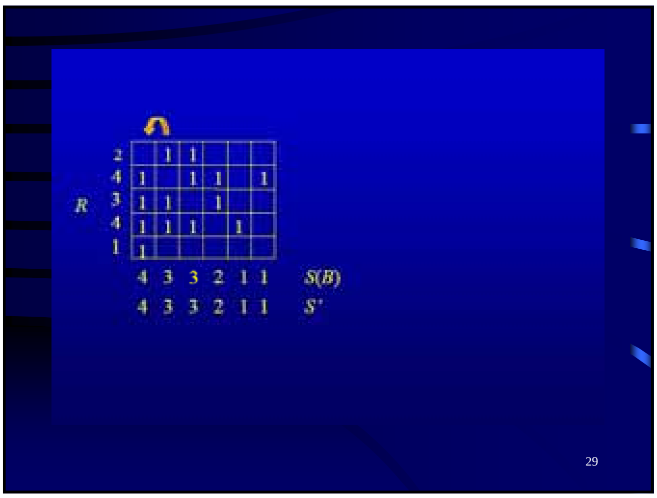
28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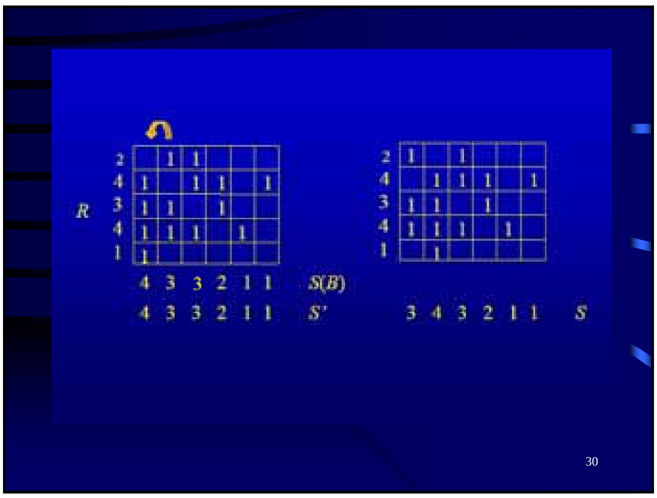
29


R
2 4 3 4 1
4 3 3 2 1 1 S(B)
4 3 3 2 1 1 S'
3 4 3 2 1 1 S
30

Consistency

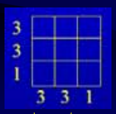
- Necessary condition: compatibility

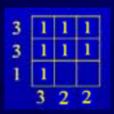
$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$$

$$r_i \le n \; (i = 1,\dots,m), \;\; s_j \le m \; (j = 1,\dots,n)$$

- Gale, Ryser, 1957: there exist a solution iff

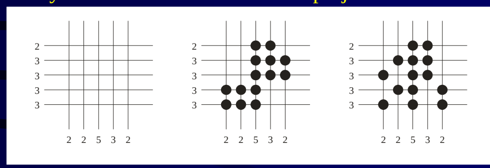
$$\sum_{j=1}^{k} s'_j \le \sum_{j=1}^{k} s^*_j \quad k = 1,\dots,n$$





Ambiguity

Due to the presence of switching components there can be many solutions with the same two projections



Solutions:

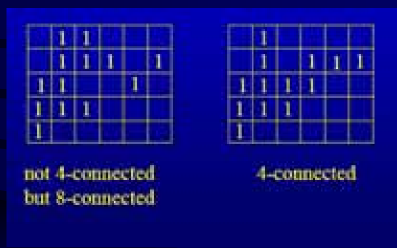
1. Further projections can be taken along lattice directions
   - In the case of more than 2 projections uniqueness, consistency and reconstruction problems are in general NP-hard – Gardner, Gritzmann 1999
2. A priori information of the set to be reconstructed can be used

Convexity

h-convex or *v*-convex: NP-complete - Barcucci et al., 1996

hv-convex: NP-complete - Woeginger, 1996

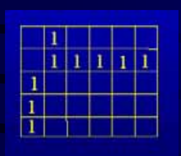
Connectedness

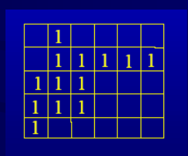


4-connected: NP-complete - Woeginger, 1996

h-convex or *v*-convex, 4-connected: NP-complete - Barcucci *et al.*, 1996

hv-Convex and Connected Sets



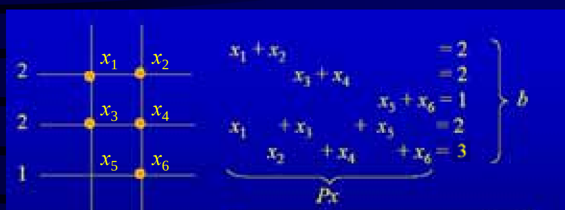


hv-convex 4-connected: $O(mn \cdot \min\{m^2, n^2\})$ - Chrobak, Dürr, 1999

hv-convex 8-connected: $O(mn \cdot \min\{m^2, n^2\})$ - Kuba, 1999

hv-convex 8- but not 4-connected: $O(mn \cdot \min\{m, n\})$ - Balázs, Balogh, Kuba, 2005

Reconstruction as Optimization



$$P = \begin{pmatrix} 1 & 1 & & & & \\ & & 1 & 1 & & \\ & & & & 1 & 1 \\ 1 & & 1 & & 1 & \\ & 1 & & 1 & & 1 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

Optimization

$$Px = b \qquad x \in \{0,1\}^{m \times n}$$

Problems:

- binary variables
- big system
- underdetermined (#equations << #unknowns)
- inconsistent (if there is noise)

$$x \in \{0,1\}^{m \times n}$$

$$\Phi(x) = \|Px - b\|^2 + g(x) \rightarrow \min$$

Term for prior information: convexity, similarity to a model image, etc.

optimization method: e.g., simulated annealing

Open Problems

- New kinds of prior information, e.g. special geometrical properties
- Efficient heuristics for NP-hard reconstruction
- Tomography in higher dimensions
- Stability (do small changes in the projections of F cause very dissimilar reconstructions to F?)

Thank you for your attention!