A Universal and Exact Linear Framework for Estimation, Registration and Recognition of Deformable Objects

Joseph M. Francos, Rami Hagege,
Shahar Kovalsky and Nadav Geva
EE Dept. Ben-Gurion University

The Simplest Possible Case

g

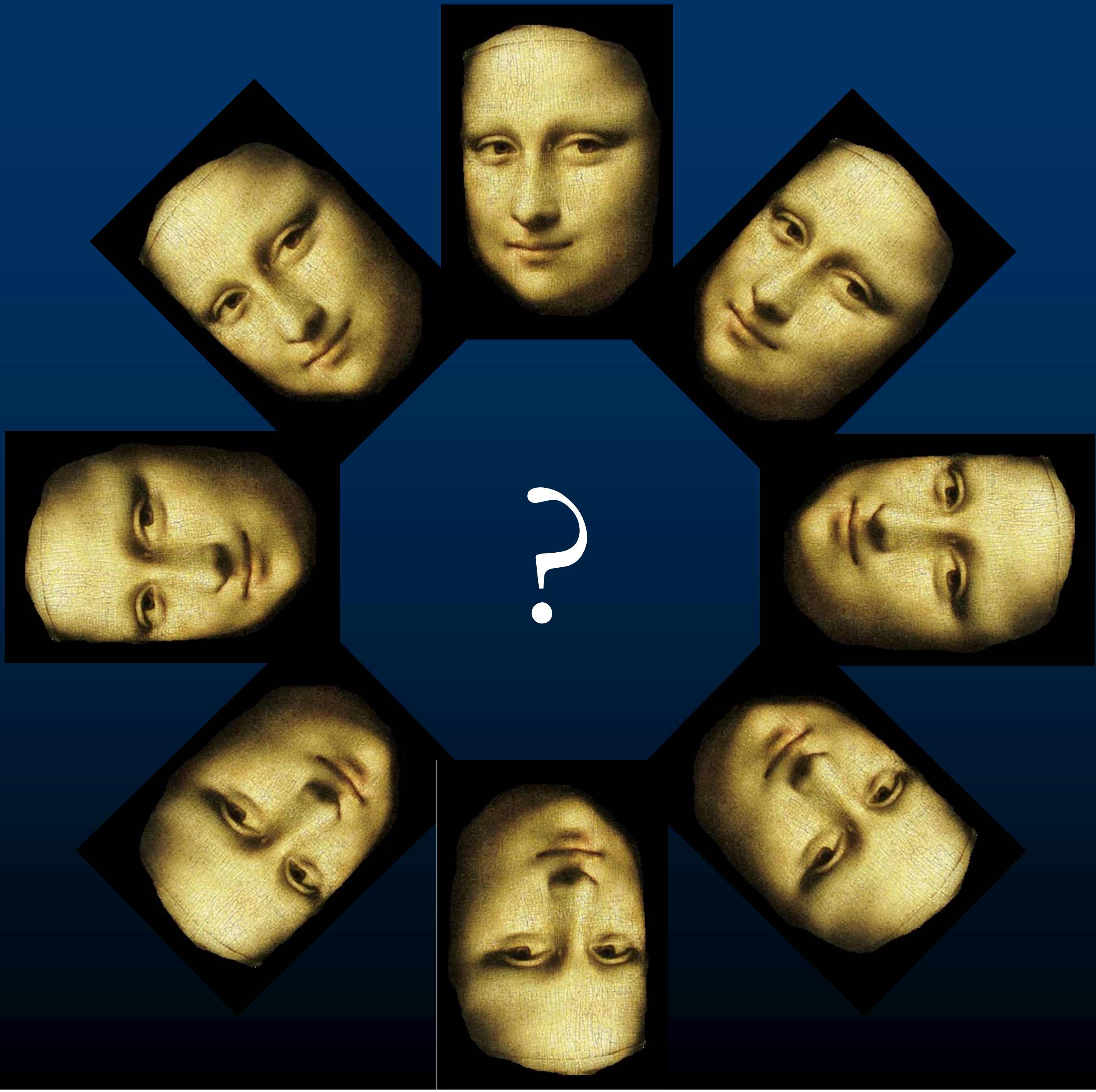
$$h(\mathbf{x}) = g(\mathbf{A}\mathbf{x})$$

A

Given two affine related images, find the parameters of the transformation that relates the two → A

h

The Problem is not Linear

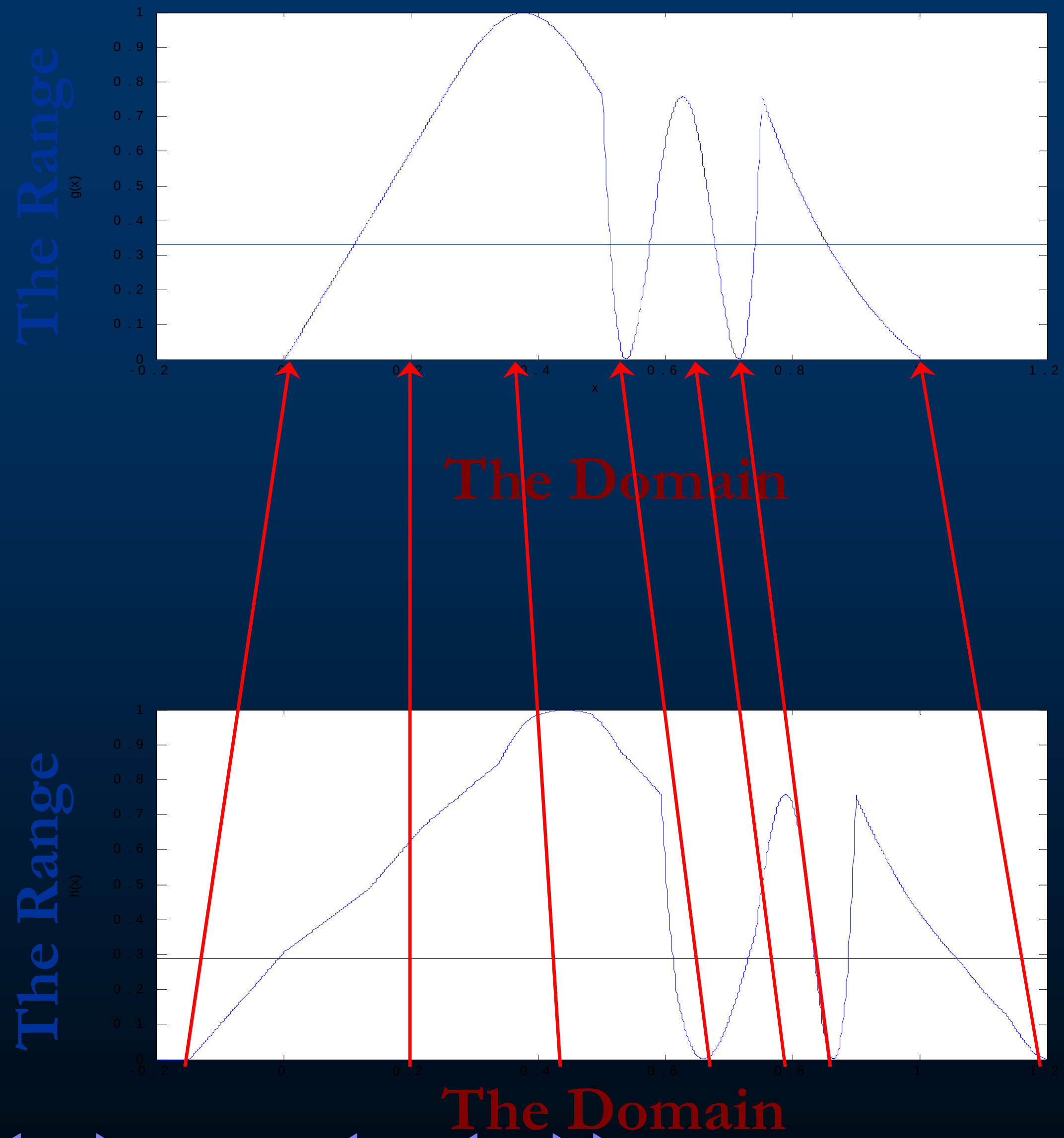


The Geometric Mathematical Problem

The Template $g(x)$

The Deformation $\varphi(x)$

The Observation $h(x)$



The Basic Relation $h(x) = g(\varphi(x))$

$$The\ Problem - Knowing\ \ h(x)\ \ and\ \ g(x)\ what\ is\ \ \varphi(x)\ ?$$

Existing Art

- *Apply each of the possible deformations to the template in search for the deformed template that minimizes a cost*
  - *Unlimited resources: time and computations*
  - *The "solution": Optimization. Implies local minima problems.*

The Implicit Solution

$$\textit{The Basic Relation} \;-\; h(x) = g(\varphi(x))$$

$$\hat{\varphi}(x) = argmin_{\varphi(x)}\left(D(h(x), g(\varphi(x))) + L(\varphi(x))\right)$$

D – *A functional measuring the dissimilarity between* $h(x)$ *and* $g(\varphi(x))$

L – *A functional describing prior knowledge on* $\varphi(x)$

Existing Art

- *Apply each of the possible deformations to the template in search for the deformed template that minimizes a cost*

  \- *Unlimited resources: time and computations*

  \- *The "solution": Optimization. Implies local minima problems.*

Alternatively: Attempts to bypass the difficulties:

  \- *Landmarks and their extensions: Local descriptors, e.g. SIFT, MSER…..*

Existing Art

- *Apply each of the possible deformations to the template in search for the deformed template that minimizes a cost*

  - *Unlimited resources: time and computations*

  - *The "solution": Optimization. Implies local minima problems.*

Alternatively: Attempts to bypass the difficulties:

- *Assuming the deformation is very small and the object is simple various* ad-hoc *approximations are made :*

  - *Landmarks*

  - *Linearization*

  High sensitivity in the presence of large deformations or noise.

Existing Art

- *Apply each of the possible deformations to the template in search for the deformed template that minimizes a cost*

  - *Unlimited resources: time and computations*

  - *The "solution": Optimization. Implies local minima problems.*

Alternatively: Attempts to bypass the difficulties:

- *Assuming the deformation is very small and the object is simple various* ad-hoc *approximations are made :*

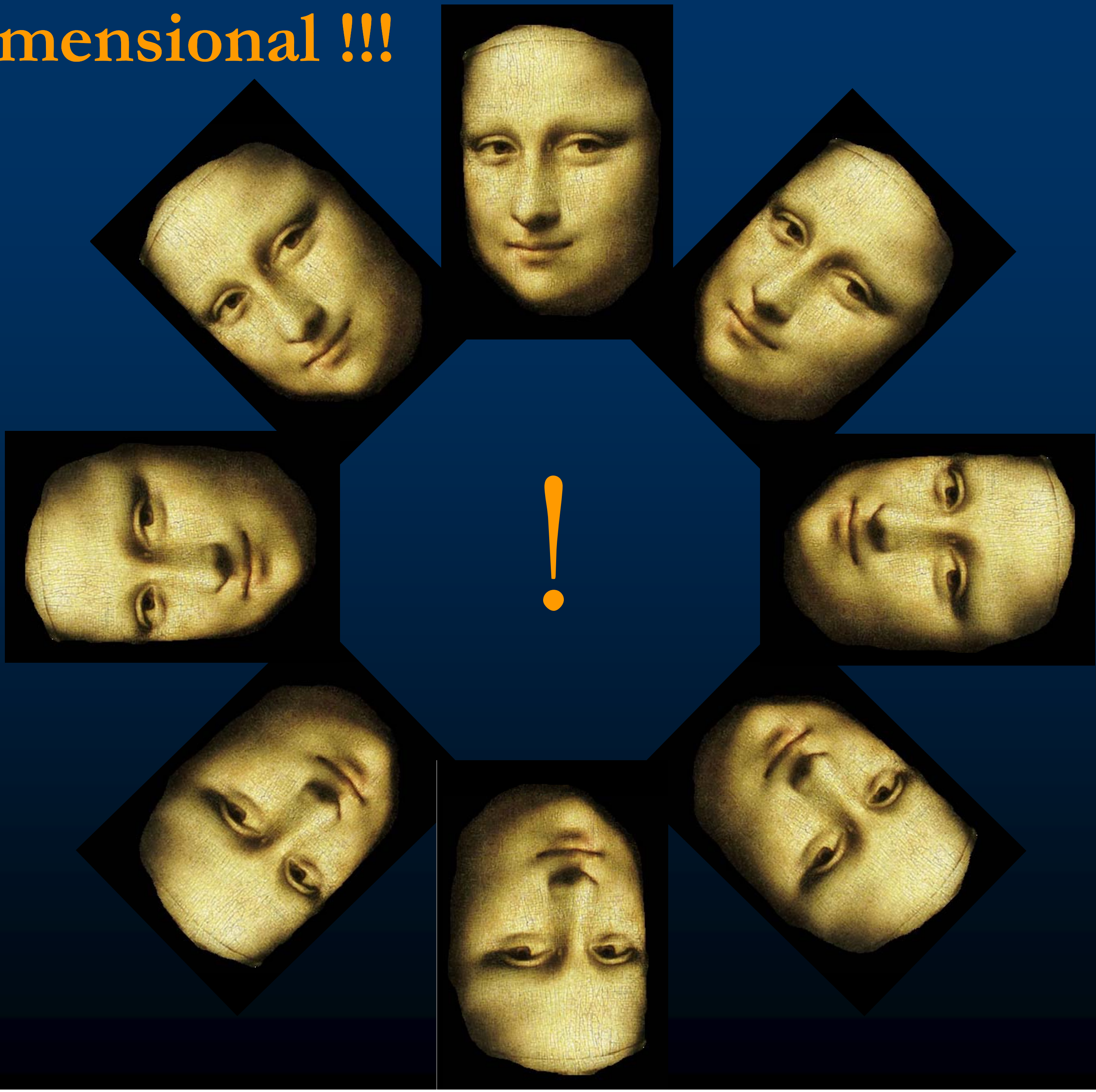
  - *Landmarks and their extensions: Local descriptors*

  - *Linearization*

  High sensitivity in the presence of large deformations or noise. Poor repeatability and distinctiveness of the feature detectors

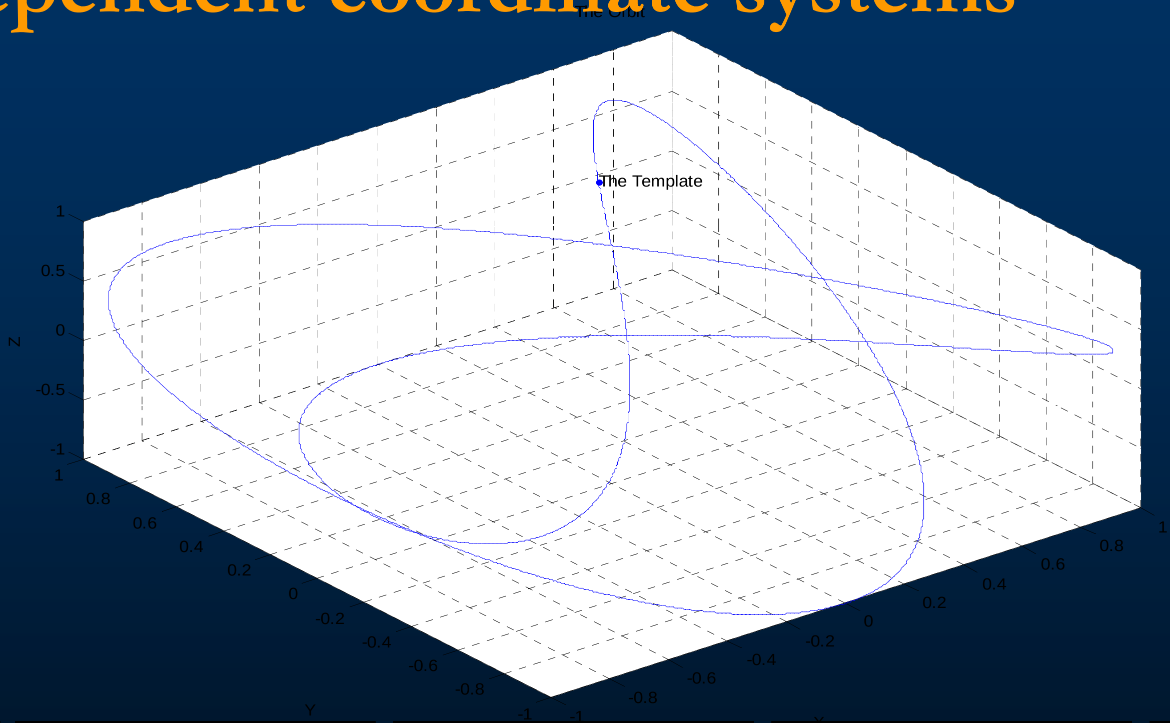
No explicit solution exists even for the simplest sub-problem.

The Space of Allowed Deformations is Low Dimensional !!!



Non Linear Structure

Non linear structure remains non linear in all linearly dependent coordinate systems







The Simplest Possible Case

g

A

h

$$h(\mathbf{x}) = g(\mathbf{A}\mathbf{x})$$

Given two affine related images, find the parameters of the transformation that relates the two → A

First step: Jacobian Evaluation

Let $w:\mathbb{R}\to\mathbb{R}$ *be some measurable function. Let* $\mathbf{y}=\mathbf{Ax}$. *Since*

$$h(\mathbf{x}) = g(\mathbf{Ax})$$

$$\int_{\mathbb{R}^n} w(h(\mathbf{x})) = \int_{\mathbb{R}^n} w(g(\mathbf{Ax})) = \left|\mathbf{A}^{-1}\right| \int_{\mathbb{R}^n} w(g(\mathbf{y}))$$

$$\left|\mathbf{A}^{-1}\right| = \frac{\int_{\mathbb{R}^n} w(h(\mathbf{x}))}{\int_{\mathbb{R}^n} w(g(\mathbf{y}))}$$

Second Step: The Elements of A

$$\int_{\mathbb{R}^n} \mathbf{x} w_l(h(\mathbf{x})) = \int_{\mathbb{R}^n} \mathbf{x} w_l(g(\mathbf{A}\mathbf{x}))$$

$$= \left|\mathbf{A}^{-1}\right| \int_{\mathbb{R}^n} \left(\mathbf{A}^{-1}\mathbf{y}\right) w_l(g(\mathbf{y}))$$

$$= \left|\mathbf{A}^{-1}\right| \mathbf{A}^{-1} \int_{\mathbb{R}^n} \mathbf{y} w_l(g(\mathbf{y}))$$

which is a linear equation expressed in terms of the unknown transformation parameters. In a matrix form

$$\left|\mathbf{A}\right| [\int_{\mathbb{R}^n} \mathbf{x} w_1(h(\mathbf{x})),\ldots,\int_{\mathbb{R}^n} \mathbf{x} w_p(h(\mathbf{x}))] = \mathbf{A}^{-1} [\int_{\mathbb{R}^n} \mathbf{y} w_1(g(\mathbf{y})),\ldots,\int_{\mathbb{R}^n} \mathbf{y} w_p(g(\mathbf{y}))]$$

$$\left|\mathbf{A}\right| H_p = \mathbf{A}^{-1} G_p$$

Elastic Deformations

The Problem:

$$h(x) = g(\varphi(x)) \qquad \varphi^{-1}(z)' = \sum_{i=1}^{N} a_i e_i(z)$$

The operation of the fundamental functional:

$$\int_{-\infty}^{\infty} w(h(x))dx = \int_{-\infty}^{\infty} w(g(\varphi(x)))dx =$$

$$\left(z = \varphi(x)\right)$$

$$\int_{-\infty}^{\infty} \varphi^{-1}(z)' w(g(z))dz = \sum_{i=1}^{N} a_i \int_{-\infty}^{\infty} e_i(z) w(g(z))dz$$

A Linear Constraint !

Independent of the template ‘g’ !

Independent of the geometric deformation model !

The Basic Solution

$$\begin{bmatrix} \int w_1 \circ h \\ \vdots \\ \int w_m \circ h \end{bmatrix} = \begin{bmatrix} \int e_1 w_1 \circ g & \cdots & \int e_m w_1 \circ g \\ \vdots & \ddots & \vdots \\ \int e_1 w_m \circ g & \cdots & \int e_m w_m \circ g \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$$

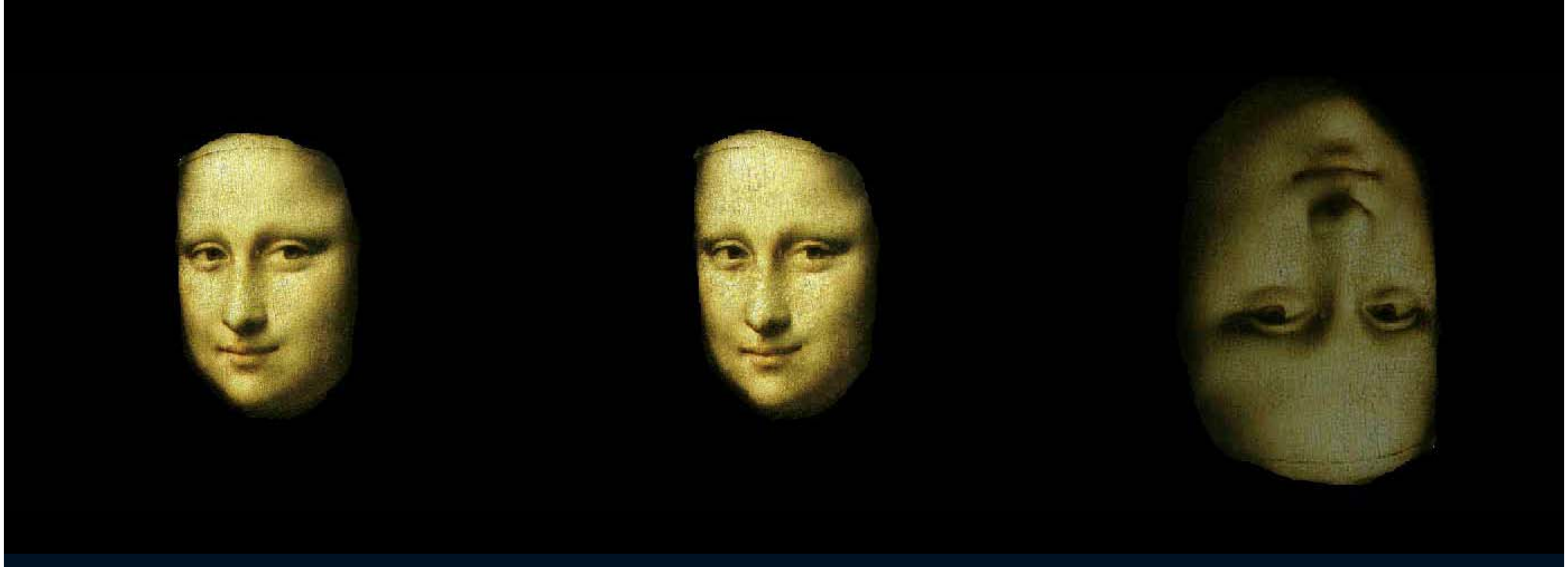
$$V = Ma$$

M is invertible for almost any template $g(x)$

$$a = M^{-1} V$$

A universal, linear, explicit and exact solution!

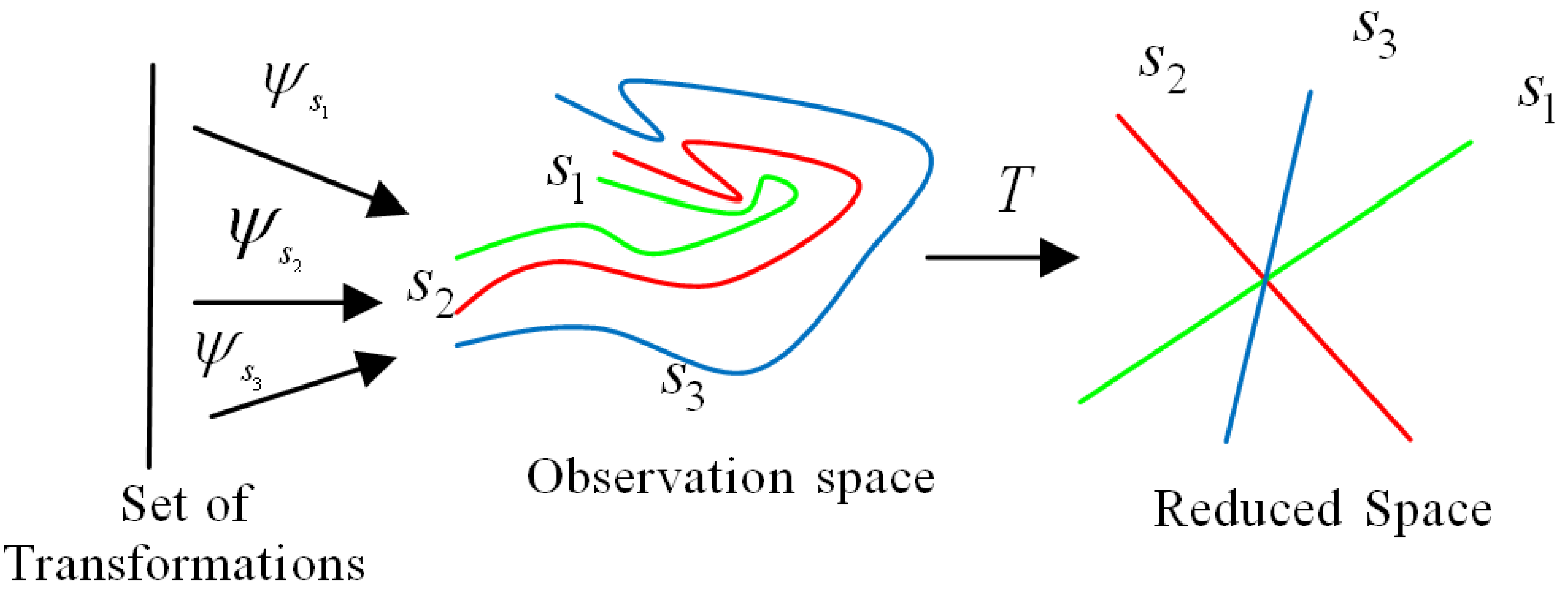
Elastic Deformations



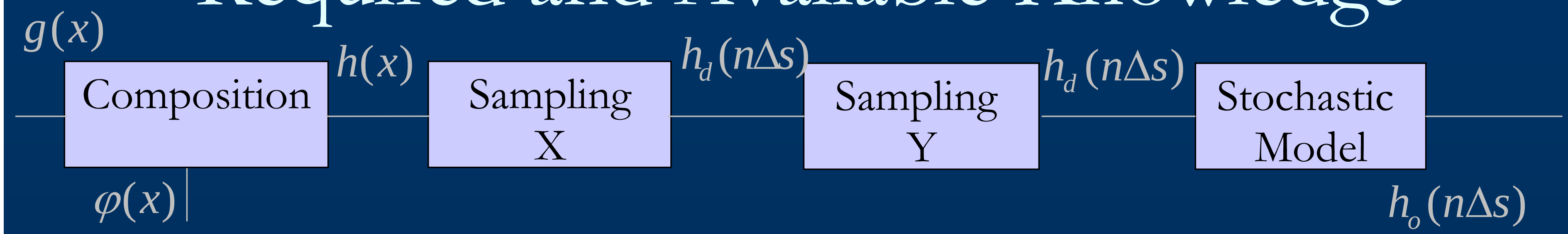
Universal Manifold Embedding

- If the set of deformations, G, admits a finite dimensional representation, there is a mapping from the space of observations to a low dimensional linear space.
- The manifold corresponding to each object is mapped to a linear subspace with the same dimension as that of the manifold.
- The embedding of the space of observations depends on the deformation model, and is independent of the specific observed object, hence it is universal.

Universal Manifold Embedding



Required and Available Knowledge



Knowing $V(h(x)) = \begin{pmatrix} \int_X w_1(h(x))dx \\ \vdots \\ \int_X w_{N_Y}(h(x))dx \end{pmatrix}$ **the problem is solved!**

Unfortunately, all we can actually calculate however is

$$Y(h_o(n\Delta s)) = \begin{pmatrix} \sum w_1(h_o(n\Delta s))\Delta s \\ \vdots \\ \sum w_{N_f}(h_o(n\Delta s))\Delta s \end{pmatrix}$$

The Basic Stochastic Solution

$$V(h) = Ma + n$$

The LMMSE estimator

$$\hat{a} = E(a) + [\mathrm{cov}(V(h))]^{-1}\mathrm{cov}(V(h), a)[V(h) - E(V(h))]$$

Where this is leading us ?

A Random Sets Framework for Error Analysis in Estimating Geometric Transformations

Two Representations of Functions

A Point to Point Map

Each point in X is mapped to a unique point in Y

Decomposition of Space

Each value of Y is represented as a subset of X

Standard Error Models

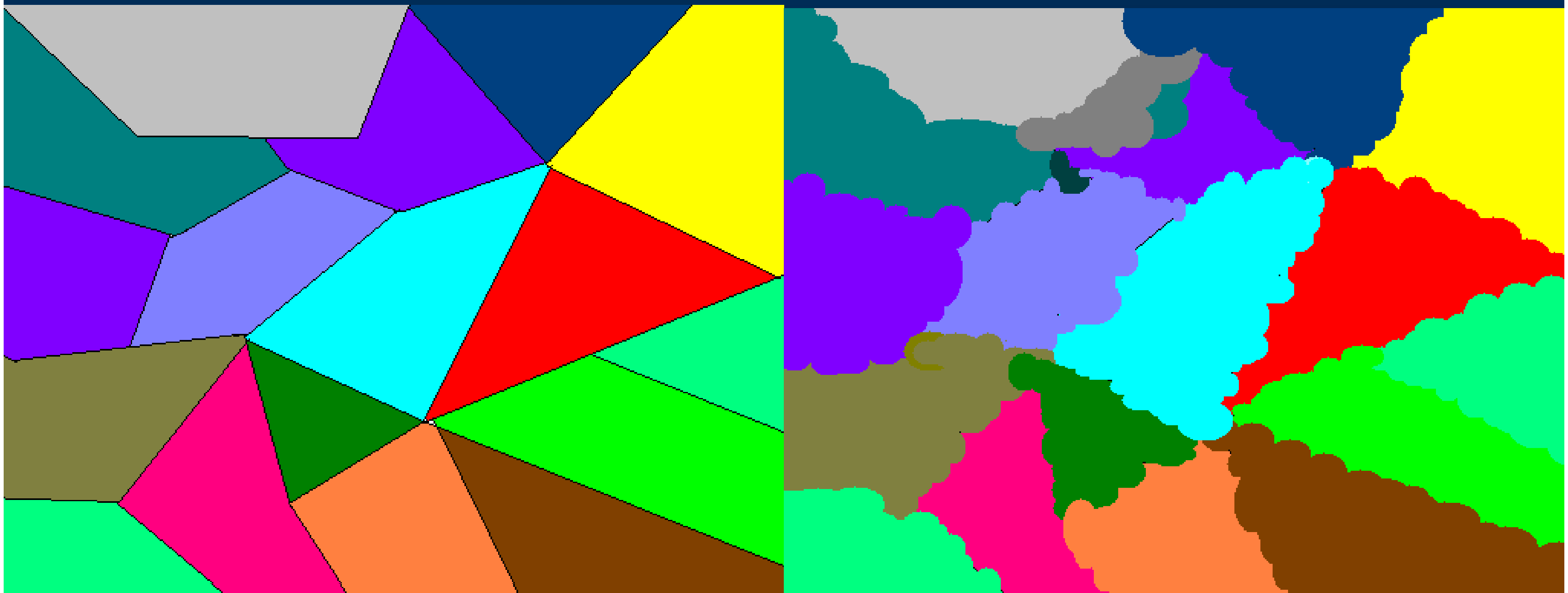
- The most common models, e.g., additive-, or multiplicative noise models provide a point-wise description of the noise effect by stochastically defining how the amplitude of the observed signal at every point in time, or space, is affected by the noise.
- The global effects of the noise contribution are measured by evaluating moments of the observed signal, or by estimating its probability distribution.

Natural Error Model – A Set Decomposition Perspective

From this point of view for each value of Y we have a subset in X

Together these sets decompose X completely

When errors occur the decomposition changes



Error Analysis – The Concept

- Since our goal is to estimate the geometric transformation, the appropriate noise model for the problem is a model that explicitly relates the presence of noise and the measures of the geometric entities in the observed image.
- These entities are the above zero- and first-order moments

Quantized Range

The observed signals are quantized. Hence the effect of noise (of any kind) is the random mapping of a sample (pixel) whose measured quantized amplitude is j in the template to some other level k in the observation.

Definitions

- Let S be the support of some function f whose range is {0,...,N}
- Let S_k denote the support of the points $(x_1,x_2)\in S$ such that $f(x_1,x_2)=k$
- S can be represented as a union of disjoint level sets S_k

$$S = \coprod_{k\in\{0,\ldots,N\}} S_k$$

- Let P be a probability transition matrix, such that $P_{j,k}$ denotes the transition probability from level j to level k due to the noise.
- Assuming independence of the noise samples, this probability measure is identical for every point in the domain.

The Methodology

- Let $w_k(z) = \begin{cases} 1 & z = k \\ 0 & z \neq k \end{cases} \qquad k = 1 \ldots N$

- Define the decomposition of the noise-free observation $h(x_1, x_2)$ and noisy observation $\tilde{h}(x_1, x_2)$ into their level sets by applying $\{w_k(z)\}_{k=1}^{N}$ to both functions to yield $s = \coprod_{k \in \{1,\ldots,N\}} s_k$ and $s = \coprod_{k \in \{1,\ldots,N\}} \tilde{s}_k$

- Recall that the solution is based on the zero- and first order moments of these sets

The Statistical Question

What are the statistical relations between the desired quantities computed from $h(x_1,x_2)$ and those computed from $\tilde{h}(x_1,x_2)$?

The Statistical Question

Define

$$\mu(S_k) = \int_S w_k\left(h(x_1, x_2)\right)$$

$$V(h) = [\mu(S_1), \ldots, \mu(S_N)]$$

$$V_1(h) = [\int_{S_1} x_1 w_k\left(h(x_1, x_2)\right), \ldots, \int_{S_N} x_1 w_k\left(h(x_1, x_2)\right)]$$

$$V_2(h) = [\int_{S_1} x_2 w_k\left(h(x_1, x_2)\right), \ldots, \int_{S_N} x_2 w_k\left(h(x_1, x_2)\right)]$$

and similarly for $\tilde{h}$

First Order Analysis of the Relations – Zero Order Moment

Theorem

$$E(V(\tilde{h})) = V(h)\mathbf{P}$$

From the analysis of the deterministic case

$$V(h) = \int_{\mathbb{R}^n} w(h(\mathbf{x})) = \left|\mathbf{A}^{-1}\right| \int_{\mathbb{R}^n} w(g(\mathbf{y})) = \left|\mathbf{A}^{-1}\right| V(g)$$

$$E(V(\tilde{h})) = V(h)\mathbf{P} = \left|\mathbf{A}^{-1}\right| V(g)\mathbf{P}$$

First Order Analysis of the Relations – First Order Moments

Theorem

$$V_l(h) = [\int_{S_1} x_l w_k \left(h(x_1, x_2)\right), \ldots, \int_{S_N} x_l w_k \left(h(x_1, x_2)\right)]$$

$$E(V_l(\tilde{h})) = V_l(h)\mathbf{P}$$

Substituting in the deterministic case solution

$$H_N = \left|\mathbf{A}^{-1}\right| \mathbf{A}^{-1} G_N$$

$$E(V_l(\tilde{h})) = V_l(h)\mathbf{P} = \left|\mathbf{A}^{-1}\right| [\mathbf{A}^{-1}\mathbf{G}_N](l,:)\mathbf{P}$$

Conclusions

Combining the results of the deterministic case and the first-order error analysis we have

$$\tilde{\mathbf{H}}_N = \left|\mathbf{A}^{-1}\right|\mathbf{A}^{-1}\mathbf{G}_N\mathbf{P} + \mathbf{E}$$

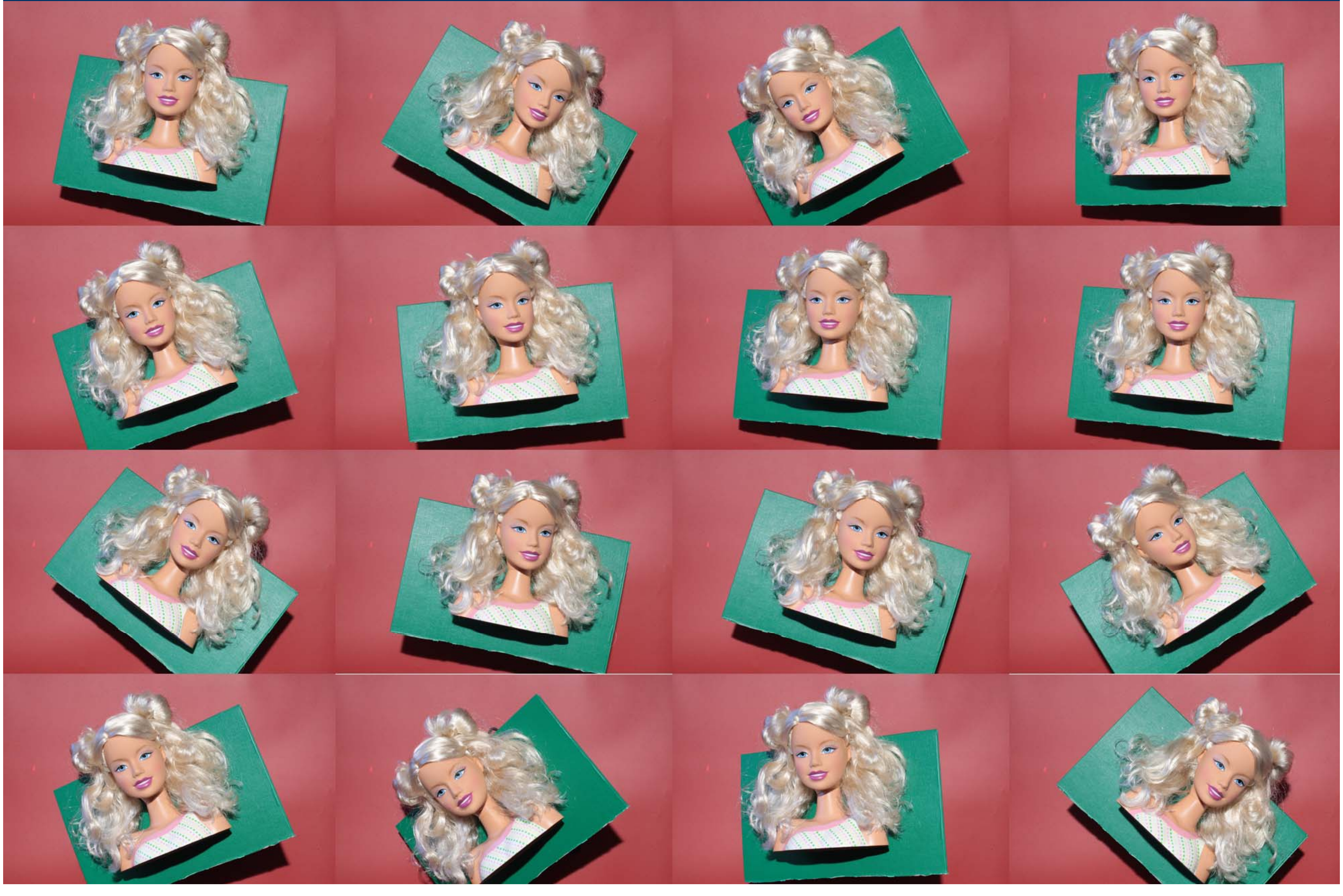
where E is a zero-mean random matrix.

Yields an unbiased LS solution for the deformation parameters

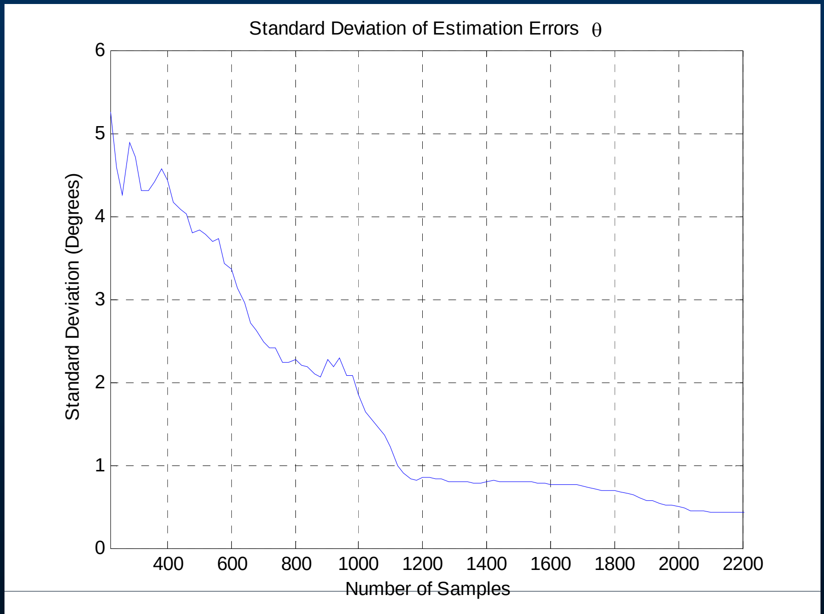
Discussion

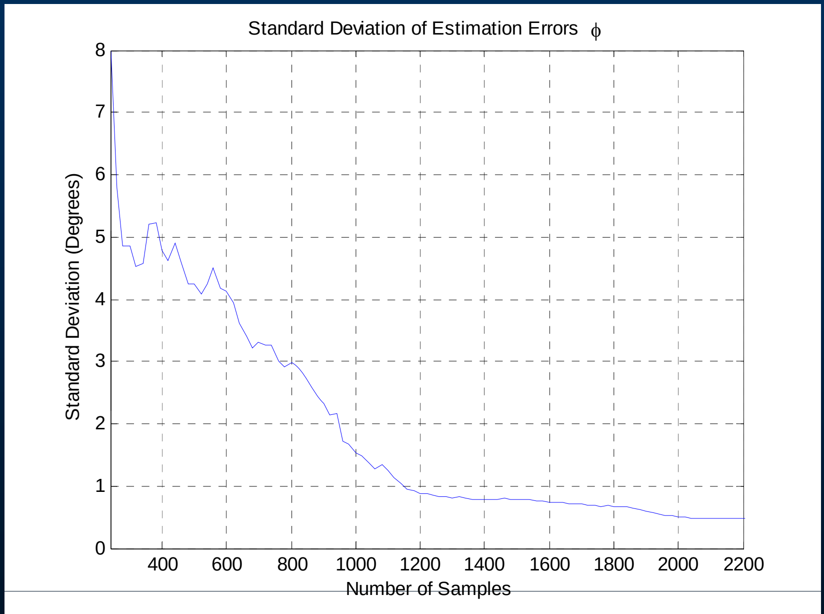
- The effect of the noise, as expressed by the operation of the transition matrix P, is to linearly combine the measures of the level sets in the observation.
- In the presence of noise, all that needs to be done in order to replace the deterministic linear system, by an unbiased linear LS solution is to multiply the right hand side of the system by P.

Single Frame Location/Orientation Estimation



LMMSE Performance





Estimator std as a function of the number of observations used to empirically estimate the covariances of $V_1(h), V_2(h)$ and their cross-covariance with the angles

Orientation Estimation Varying Illumination $std \approx 1.5^o$

Estimating the Basis Functions

We wish to estimate a minimal set of basis functions, analytically defined on the continuum of a specific geometric deformation caused by some physical phenomena

$$\phi_j \triangleq \left\{ (\phi_x^j(x_i^j, y_i^j), \phi_y^j(x_i^j, y_i^j)) \right\}_{i=1}^{N}$$

Interpolate the sampled deformations Φ_j on a regular, evenly spaced grid, using B-spline interpolation

$$\phi_x^j(x, y) = \sum_n \sum_m c_{n,m}^j f_B(x-n, y-m)$$

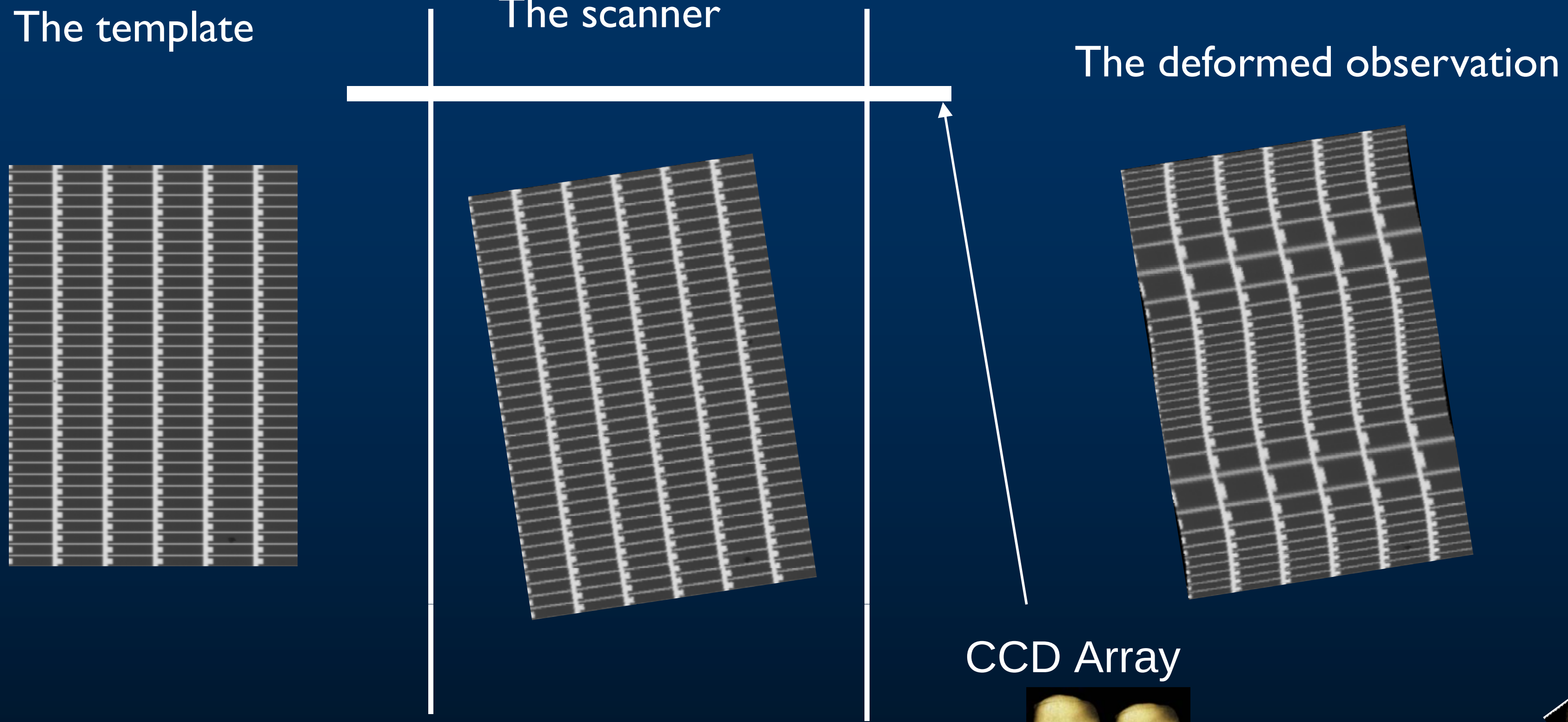
Applying PCA on the set of coefficients

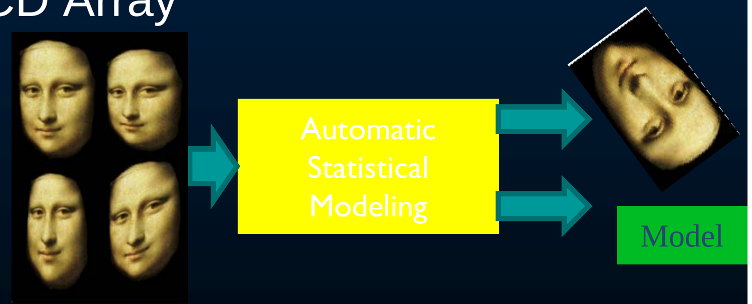
$$c_{n,m}^j = \sum_{i=1}^{N_x} a_k^j b_{n,m}^i$$

Estimating the Basis Functions

$$
\begin{aligned}
\phi_x^j(x,y) &= \sum_n \sum_m c_{n,m}^j \, f_B(x-n, y-m) \\
&= \sum_n \sum_m \sum_{i=1}^{N_x} a_i^j b_{n,m}^i \, f_B(x-n, y-m) \\
&= \sum_{i=1}^{N_x} a_i^j \underbrace{b_{n,m}^i \sum_n \sum_m \, f_B(x-n, y-m)}_{e_i^x(x,y)} \\
&= \sum_{i=1}^{N_x} a_i^j e_i^x(x,y)
\end{aligned}
$$

Change Detection in a Deforming Scanner

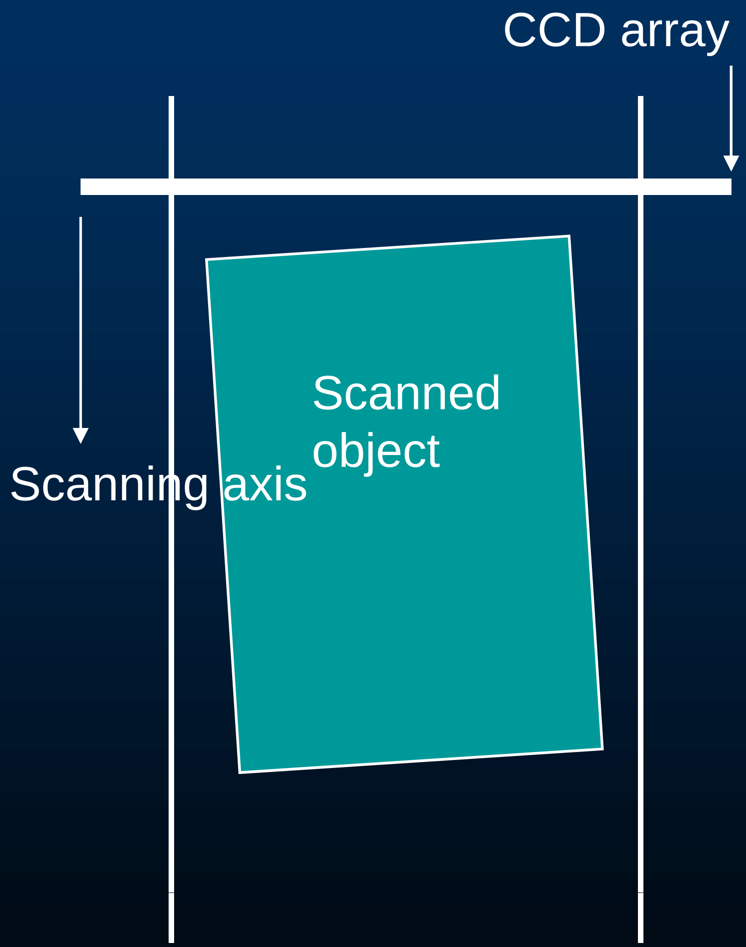




Preliminary step: Building a parametric geometric deformation model.

Estimation: Estimating the parameters which construct the deformation functions.

Registration of Line Scanner Deformations



Unknown geometric and radiometric deformations

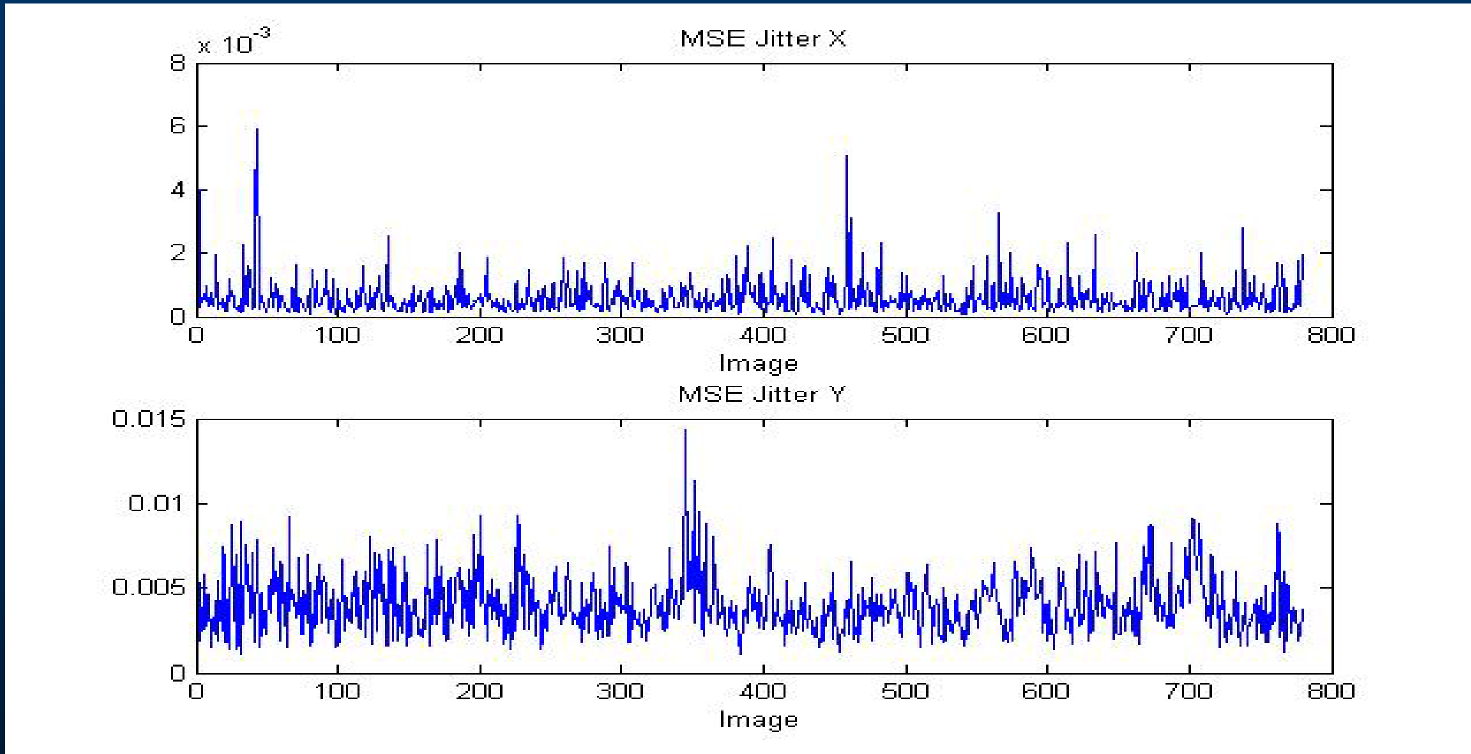
The need: Change detection (defect detection)

Overall Geometric Deformation

$$\phi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \phi_x(x,y) \\ \phi_y(x,y) \end{pmatrix} = A \cdot \begin{pmatrix} x + B(x) + \varphi_x(y) \\ y_N + \varphi_y(y) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x + B(x) + \varphi_x(y) \\ y_N + \varphi_y(y) \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}(x + B(x) + \varphi_x(y)) + a_{12}(y_N + \varphi_y(y)) \\ a_{21}(x + B(x) + \varphi_x(y)) + a_{22}(y_N + \varphi_y(y)) \end{pmatrix} \cong \begin{pmatrix} a_{11}(x + B(x) + \sum_i \alpha_i e_i^x) + a_{12}(y_N + \sum_i \beta_i e_i^y) \\ a_{21}(x + B(x) + \sum_i \alpha_i e_i^x) + a_{22}(y_N + \sum_i \beta_i e_i^y) \end{pmatrix}$$

Estimation Statistics (pixels)



$$mean(MSE\_Jitter\_X) = 0.0006$$

$$mean(MSE\_Jitter\_Y) = 0.004$$

•Two orders of magnitude accuracy improvement over standard correlator

•Replaces a tedious computation (many hours) by a real-time solution

Change Detection in Video – Static Camera

Simple problem !!!

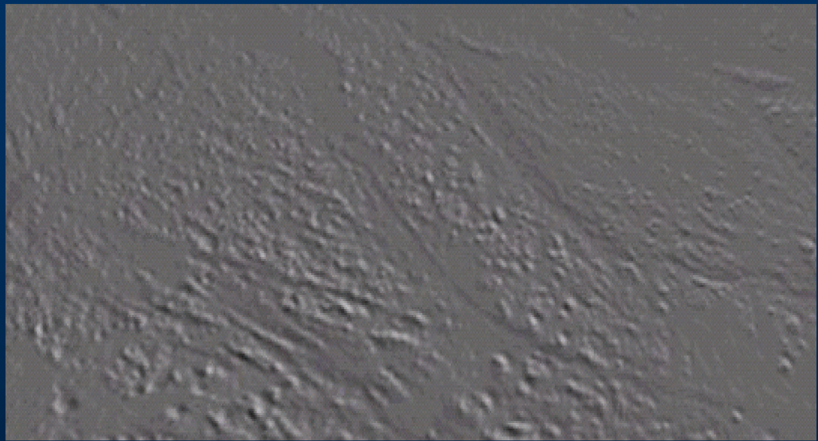
Frame N

$(Frame\ (N+1)) - (Frame\ N)$

Frame (N+1)

Change Detection – Dynamic Platform

$(Frame\ (N+1)) - (Frame\ N)$ *Dosen't work !*

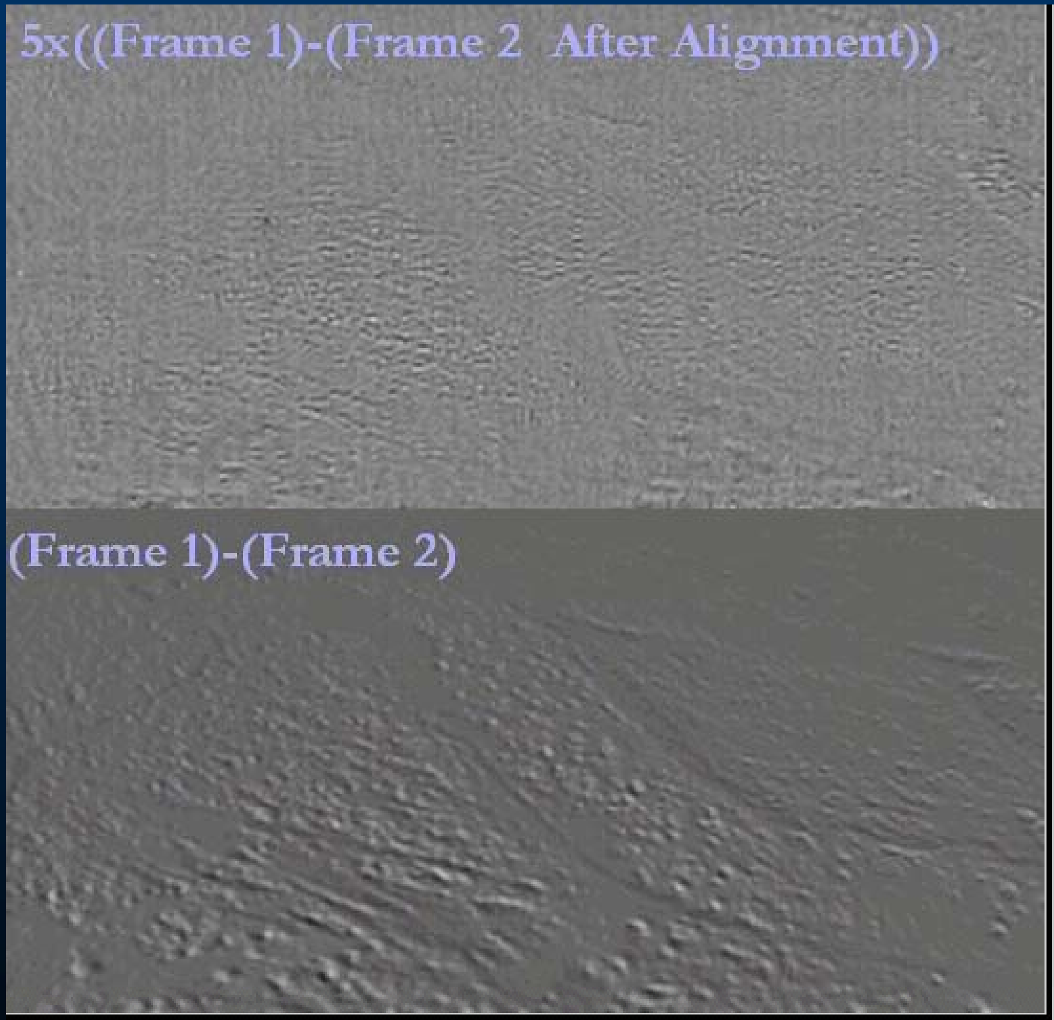


Most of the Energy is the result of the platform movement

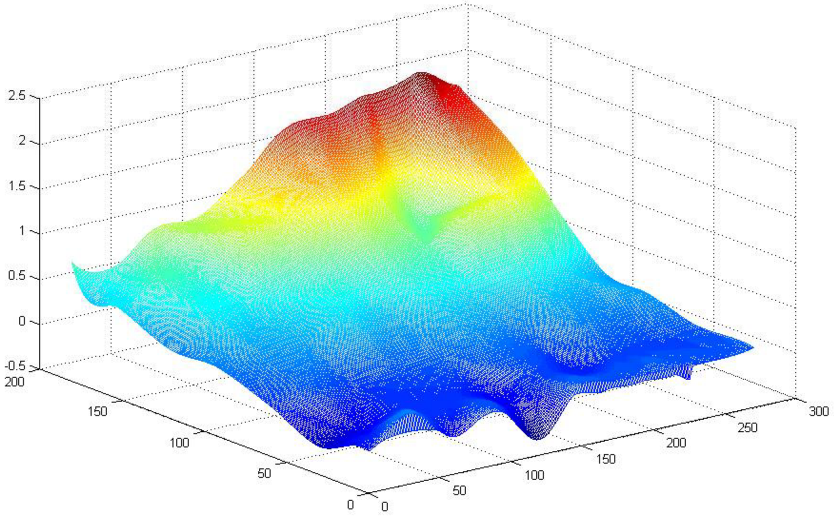
Suggested Solution – Simulate static platform by alignment

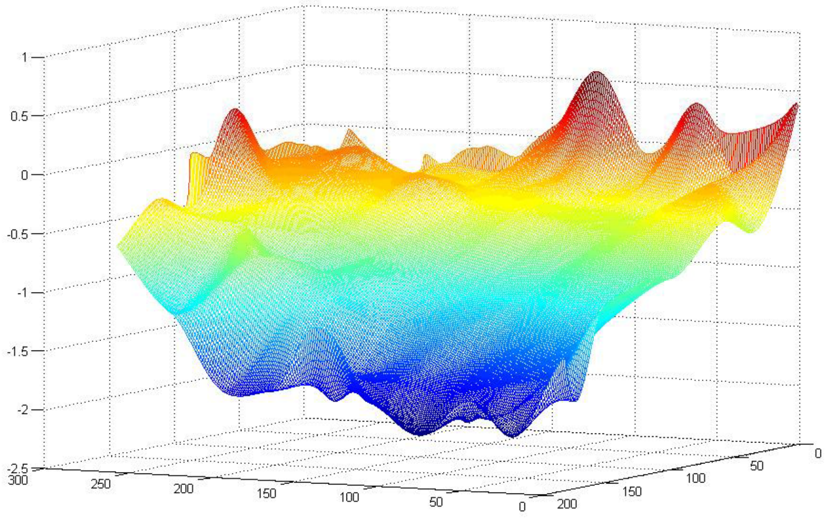
Change Detection – Dynamic Platform

Amplitude multiplied by 5



Registration of Images Taken from a Moving Platform- the Deformation





The Breakthrough

Explicit solution at the smallest possible computational complexity

- *The high dimensional nonlinear problem is formulated in terms of an **equivalent linear parameter estimation problem** – straightforward to solve.*
- ***Not** an approximation but an alternative, equivalent representation.*
- *The solution is **unique, exact** and **robust** to noise.*
- *Applicable to any deformation **regardless of its magnitude**.*
- ***Independence** of sensor **modality** and **object** type.*
- *Employs **all the information in the function**, rather than information of "zero measure" (points, contours).*