23th SSIP Summer School on Image Processing
14 July, 2015, Szeged, Hungary

Continuous and Discrete Image Reconstruction

Péter Balázs
Department of Image Processing and Computer Graphics
University of Szeged, HUNGARY

Steps of Machine Vision

- Image acquisition
- Preprocessing
- Segmentation
- Feature extraction
- Classification, interpretation
- Actuation

Image Acquisition

by visible light

by X-rays

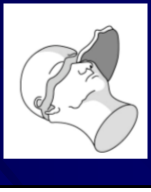
X-rays

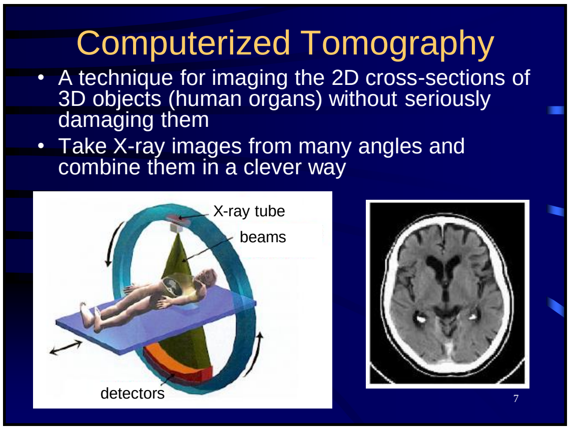
- 1895 - Wilhelm Conrad Röntgen describes the properties of X-rays
- Kind of electromagnetic radiation (similar to light but having more energy)
- Attenuation of X-rays depends on tissue → „Shadow" of the object from one direction

X-rays are Useful in Radiology (in some cases)

Tomography

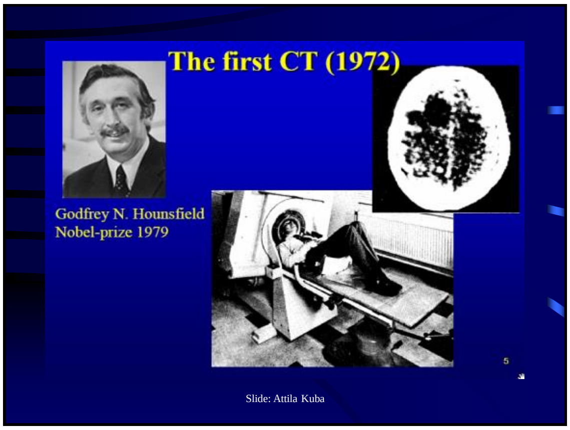
Tomos = part, section
Grapho = to write
Tomos + Grapho ≈ imaging by cross-sections (slices)





Computerized Tomography

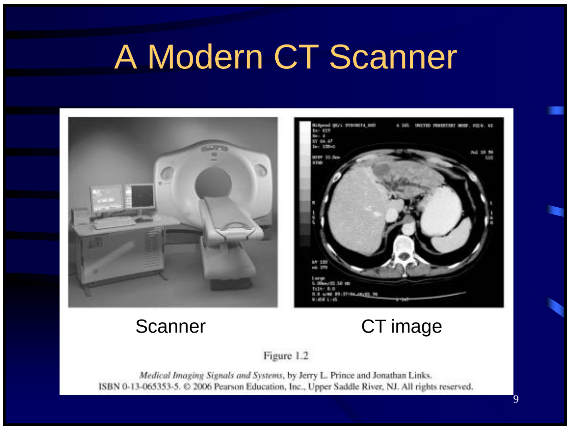
- A technique for imaging the 2D cross-sections of 3D objects (human organs) without seriously damaging them
- Take X-ray images from many angles and combine them in a clever way



The first CT (1972)

Godfrey N. Hounsfield
Nobel-prize 1979

Slide: Attila Kuba



A Modern CT Scanner

Figure 1.2

Medical Imaging Signals and Systems, by Jerry L. Prince and Jonathan Links.
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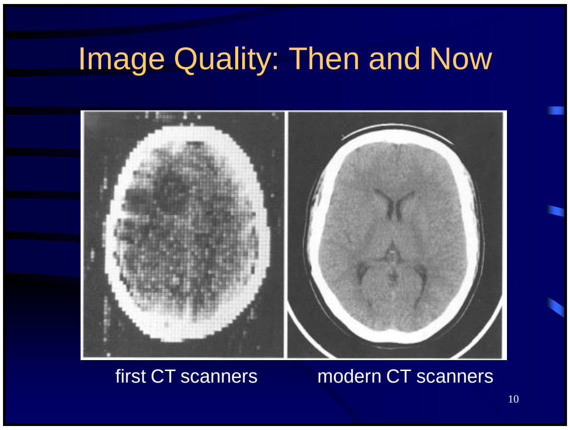
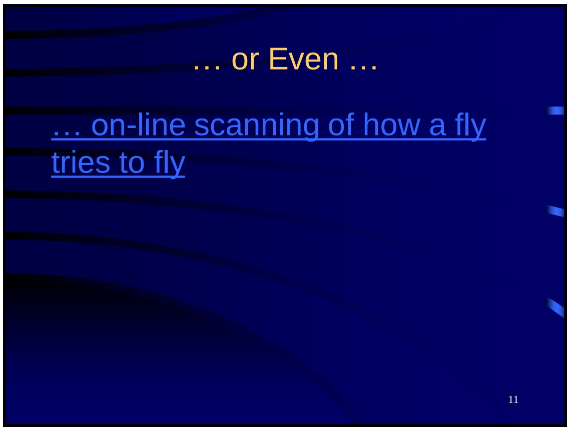
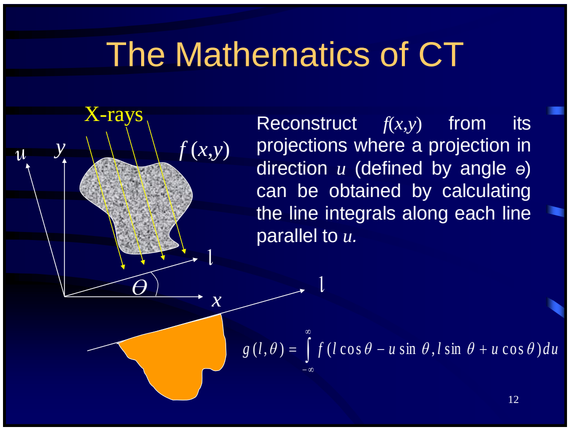


Image Quality: Then and Now



… or Even …

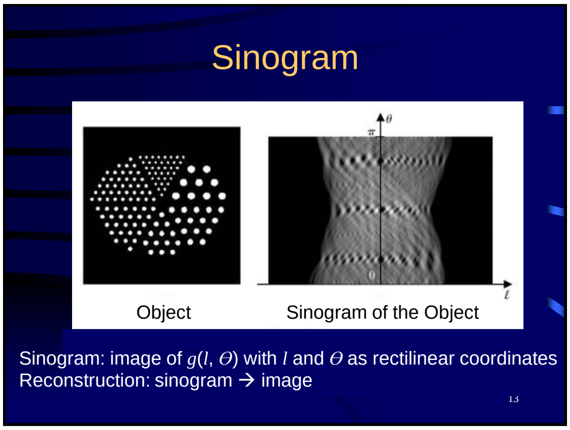
… on-line scanning of how a fly tries to fly



The Mathematics of CT

Reconstruct $f(x,y)$ from its projections where a projection in direction u (defined by angle θ) can be obtained by calculating the line integrals along each line parallel to u.

$$g(l,\theta) = \int_{-\infty}^{\infty} f(l\cos\theta - u\sin\theta, l\sin\theta + u\cos\theta)\,du$$

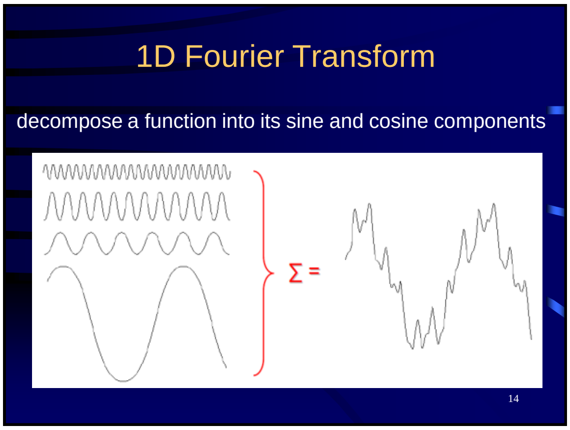


Sinogram

Object

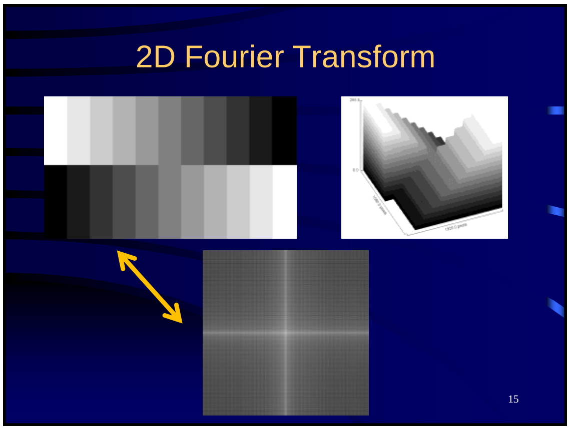
Sinogram of the Object

Sinogram: image of $g(l, \Theta)$ with l and Θ as rectilinear coordinates
Reconstruction: sinogram → image

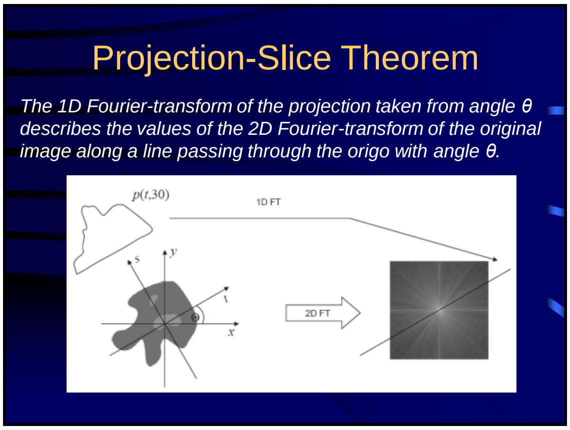


1D Fourier Transform

decompose a function into its sine and cosine components

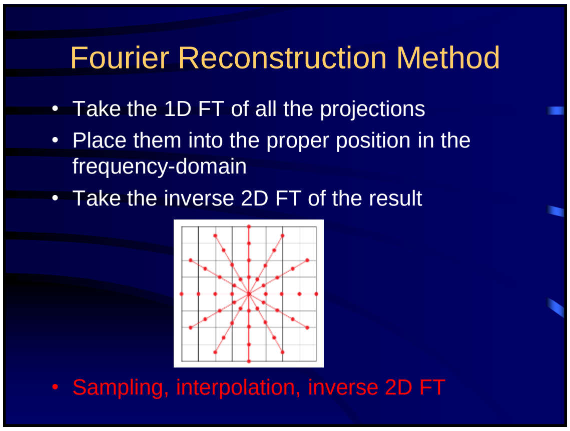


2D Fourier Transform



Projection-Slice Theorem

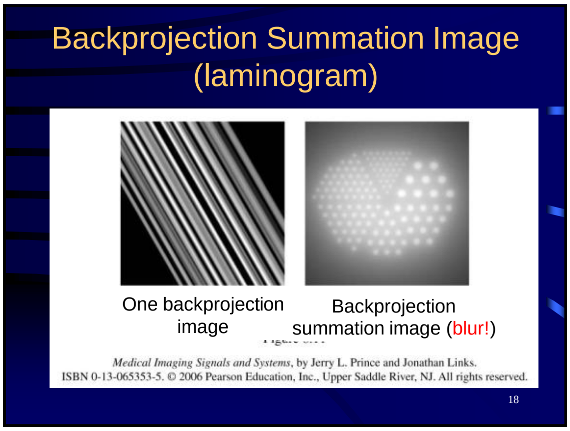
The 1D Fourier-transform of the projection taken from angle θ describes the values of the 2D Fourier-transform of the original image along a line passing through the origo with angle θ.



Fourier Reconstruction Method

- Take the 1D FT of all the projections
- Place them into the proper position in the frequency-domain
- Take the inverse 2D FT of the result

- Sampling, interpolation, inverse 2D FT



Backprojection Summation Image (laminogram)

One backprojection image

Backprojection summation image (blur!)

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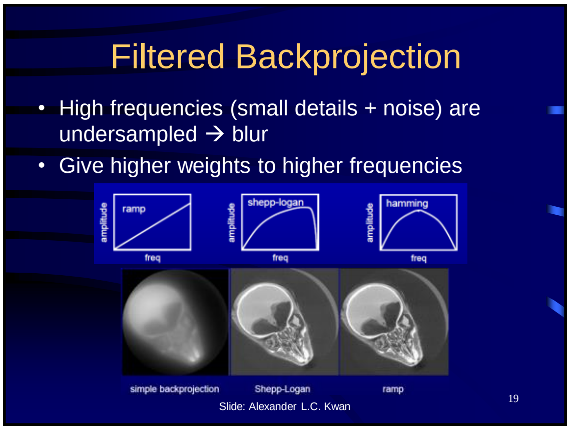
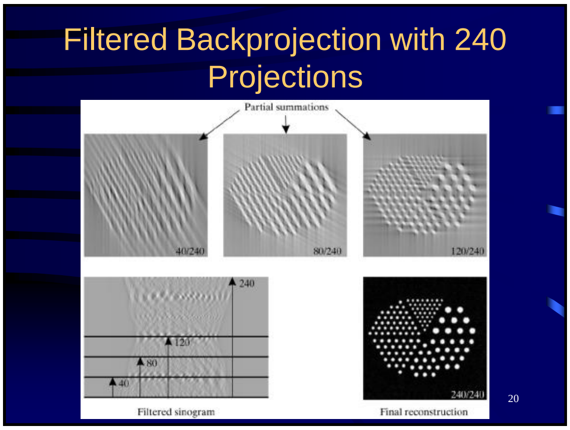


Filtered Backprojection

- High frequencies (small details + noise) are undersampled → blur
- Give higher weights to higher frequencies

simple backprojection Shepp-Logan ramp

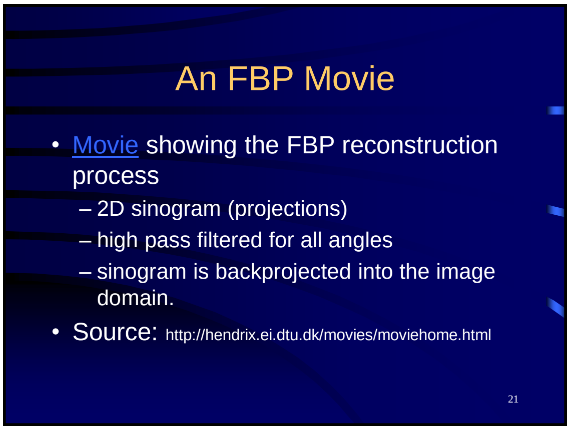
Slide: Alexander L.C. Kwan



Filtered Backprojection with 240 Projections

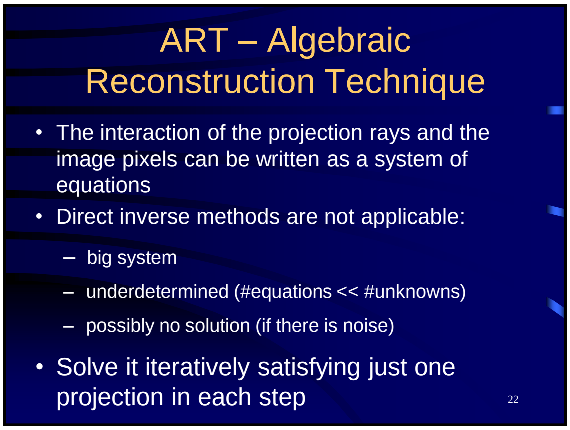
Filtered sinogram

Final reconstruction



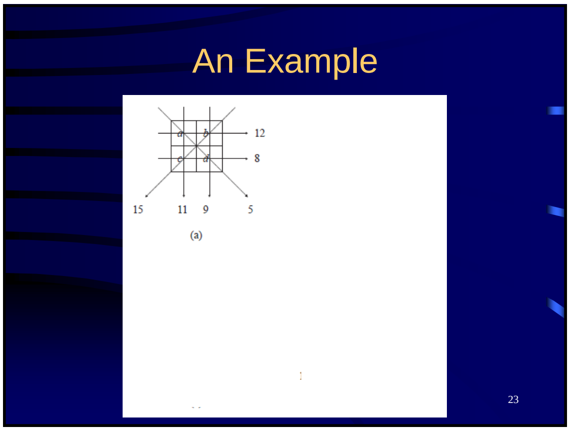
An FBP Movie

- Movie showing the FBP reconstruction process
  - 2D sinogram (projections)
  - high pass filtered for all angles
  - sinogram is backprojected into the image domain.
- Source: http://hendrix.ei.dtu.dk/movies/moviehome.html



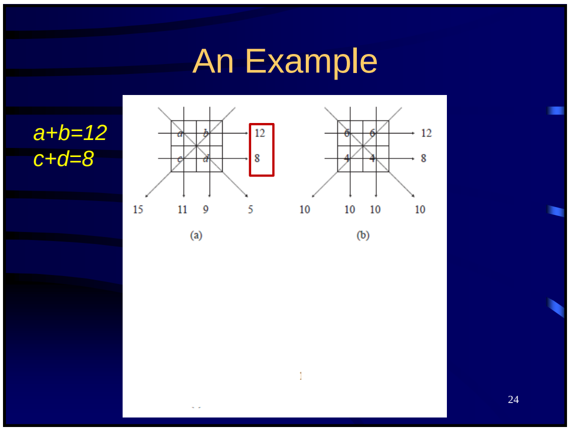
ART – Algebraic Reconstruction Technique

- The interaction of the projection rays and the image pixels can be written as a system of equations
- Direct inverse methods are not applicable:
  - big system
  - underdetermined (#equations << #unknowns)
  - possibly no solution (if there is noise)
- Solve it iteratively satisfying just one projection in each step



An Example

(a)



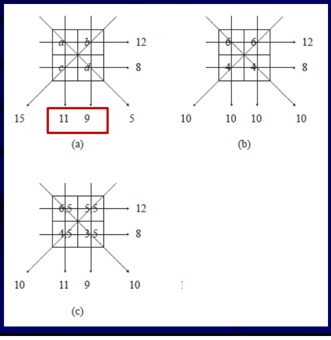
An Example

a+b=12
c+d=8

(a) (b)

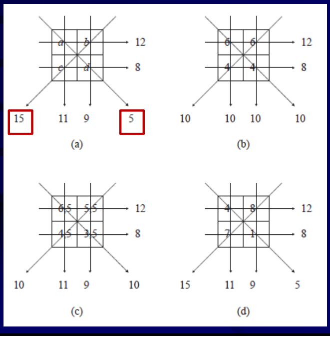
An Example

a+c=11
b+d=9



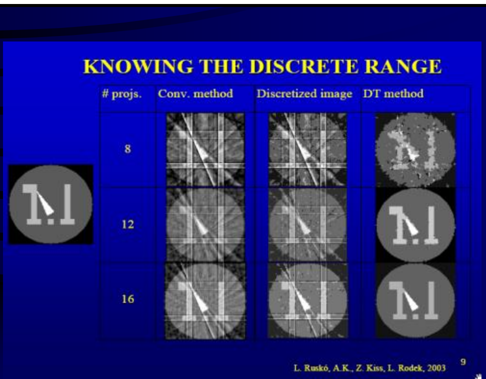
An Example

a+d=5
b+c=15



Discrete/Binary Tomography

- FBP and ART need several hundreds of projections
  - time consuming
  - expensive
  - may damage the object
  - not possible
- In certain applications the range of the function to be reconstructed is discrete and known → DT (only few (2-10) projections are needed)
- Binary Tomography: the range of the function is {0,1} (absence or presence of material)



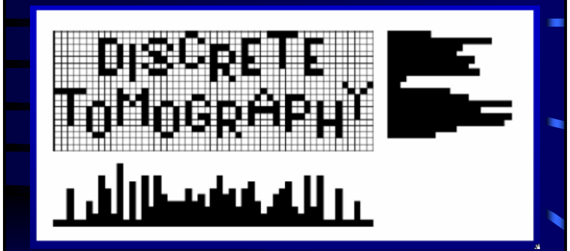
KNOWING THE DISCRETE RANGE

L. Ruskó, A.K., Z. Kiss, L. Rodek, 2003

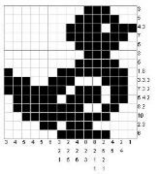
Source: Attila Kuba

Binary Reconstruction from 2 Projections

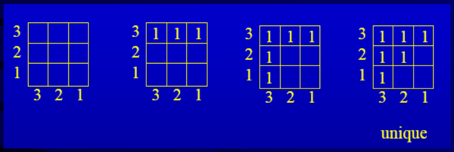
Binary Reconstruction from 2 Projections



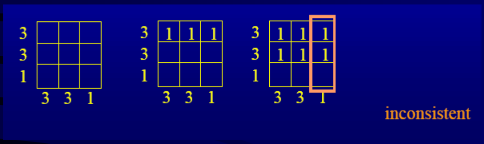
Nonograms



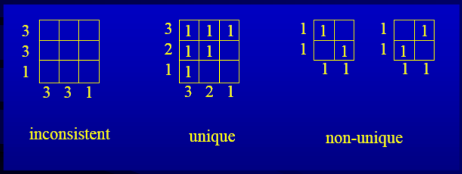
Example for Uniqueness



Example for Inconsistency



Classification



Two Main Problems

Reconstruction: Construct a discrete set from its projections.

Uniqueness: Is a discrete set uniquely determined by a given set of projections?

Reconstruction

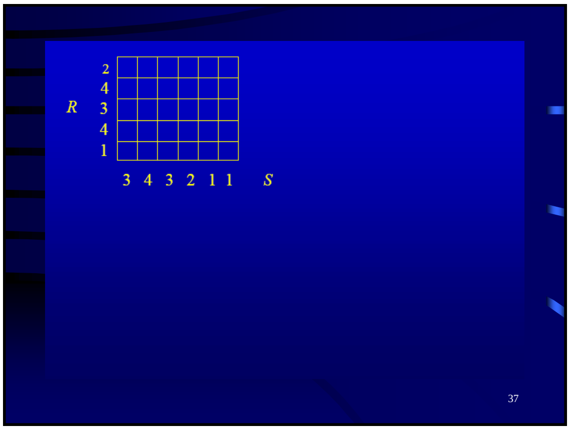
Ryser, 1957 – from row sums *R* and column sums *S*

Order the elements of *S* in a non-increasing way by $\pi \rightarrow S'$

Fill the rows from left to right → *B* (canonical matrix)

Shift elements from the rightmost columns of *B* to the columns where $S(B) < S'$

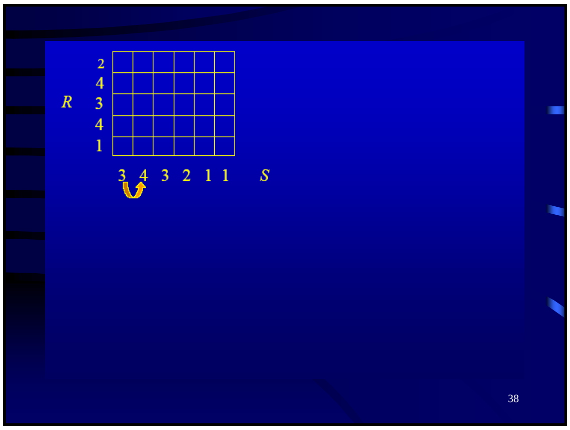
Reorder the colums by applying the inverse of π



R

2						
4						
3						
4						
1						

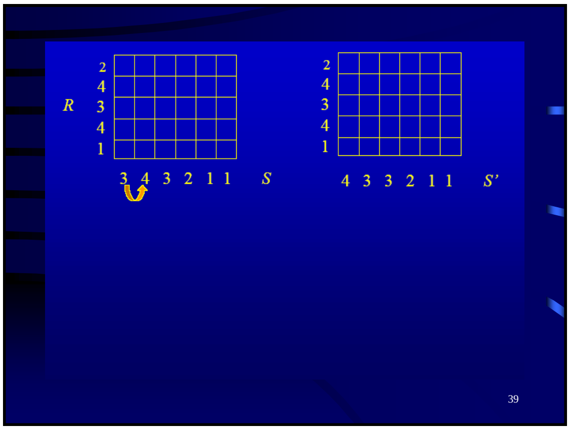
3 4 3 2 1 1 S



R

2						
4						
3						
4						
1						

3 4 3 2 1 1 S



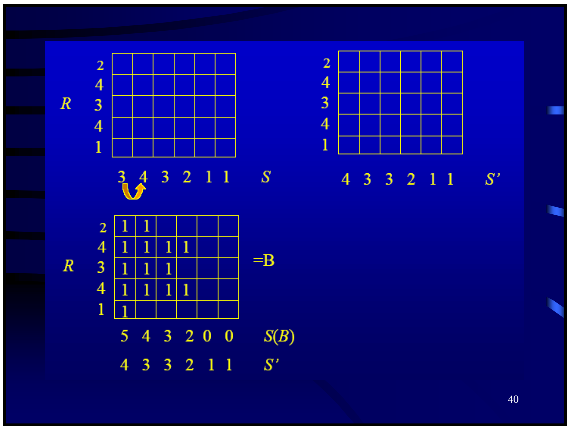
R

2						
4						
3						
4						
1						

3 4 3 2 1 1 S

2						
4						
3						
4						
1						

4 3 3 2 1 1 S’



R

2						
4						
3						
4						
1						

3 4 3 2 1 1 S

2						
4						
3						
4						
1						

4 3 3 2 1 1 S’

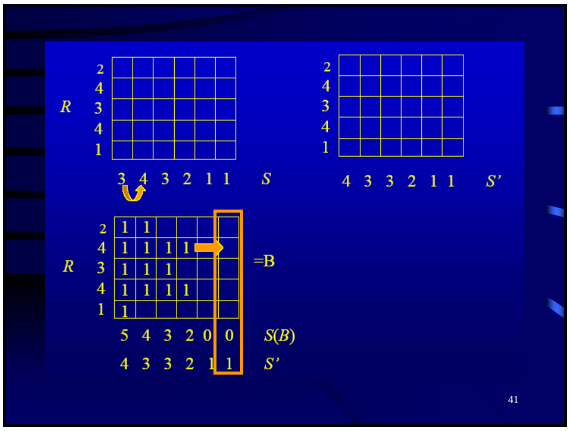
R

2	1	1				
4	1	1	1	1		
3	1	1	1			
4	1	1	1	1		
1	1					

=B

5 4 3 2 0 0 S(B)

4 3 3 2 1 1 S’



R

2						
4						
3						
4						
1						

3 4 3 2 1 1 S

2						
4						
3						
4						
1						

4 3 3 2 1 1 S’

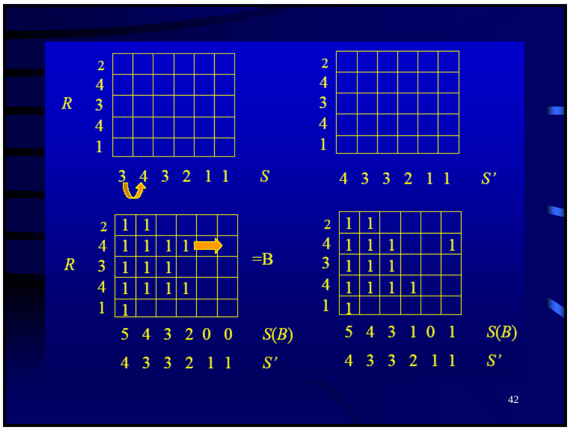
R

2	1	1				
4	1	1	1	1		
3	1	1	1			
4	1	1	1	1		
1	1					

=B

5 4 3 2 0 0 S(B)

4 3 3 2 1 1 S’



R

2						
4						
3						
4						
1						

3 4 3 2 1 1 S

2						
4						
3						
4						
1						

4 3 3 2 1 1 S’

R

2	1	1				
4	1	1	1	1		
3	1	1	1			
4	1	1	1	1		
1	1					

=B

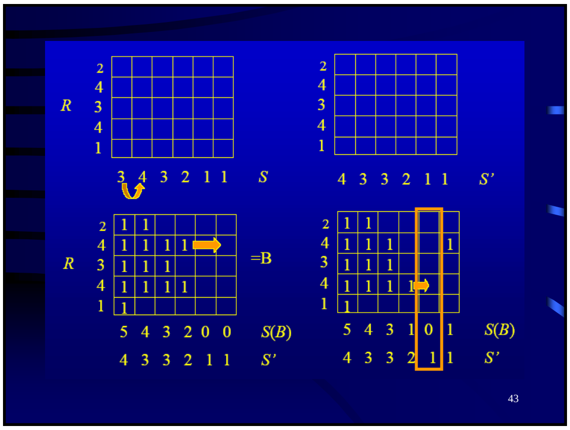
5 4 3 2 0 0 S(B)

4 3 3 2 1 1 S’

2	1	1				
4	1	1	1			1
3	1	1	1			
4	1	1	1	1		
1	1					

5 4 3 1 0 1 S(B)

4 3 3 2 1 1 S’

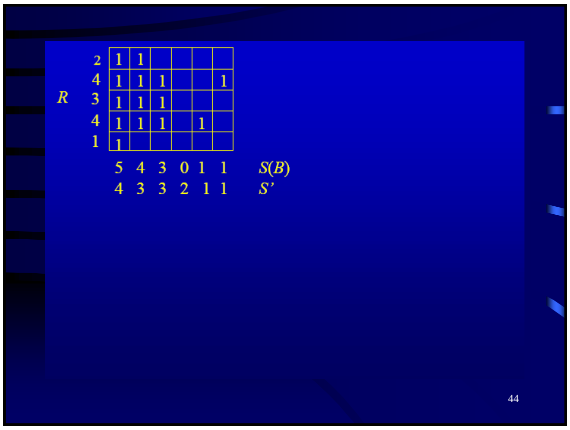


	2							
	4							
R	3							
	4							
	1							
		3	4	3	2	1	1	S

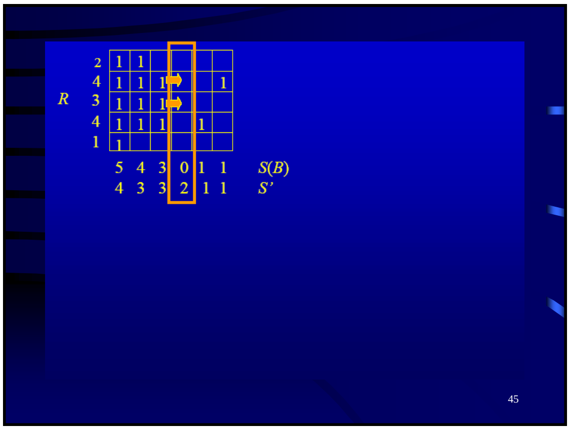
	2							
	4							
	3							
	4							
	1							
		4	3	3	2	1	1	S’

	2	1	1					
	4	1	1	1	1	➡		
R	3	1	1	1				=B
	4	1	1	1	1			
	1	1						
		5	4	3	2	0	0	S(B)
		4	3	3	2	1	1	S’

	2	1	1					
	4	1	1	1			1	
	3	1	1	1				
	4	1	1	1	1 ➡			
	1	1						
		5	4	3	1	0	1	S(B)
		4	3	3	2	1	1	S’



	2	1	1					
	4	1	1	1			1	
R	3	1	1	1				
	4	1	1	1		1		
	1	1						
		5	4	3	0	1	1	S(B)
		4	3	3	2	1	1	S’



	2	1	1					
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R	3	1	1	1 ➡				
	4	1	1	1		1		
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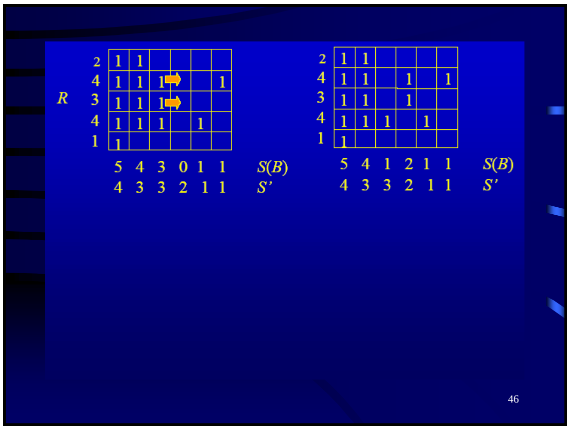
		4	3	3	2	1	1	S’



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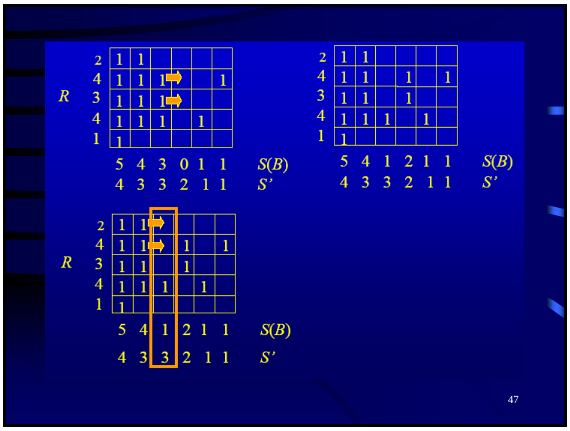
R	3	1	1	1 ➡				
	4	1	1	1		1		
	1	1						
		5	4	3	0	1	1	S(B)
		4	3	3	2	1	1	S’

	2	1	1					
	4	1	1		1		1	
	3	1	1		1			
	4	1	1	1		1		
	1	1						
		5	4	1	2	1	1	S(B)
		4	3	3	2	1	1	S’



	2	1	1					
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R	3	1	1	1 ➡				
	4	1	1	1		1		
	1	1						
		5	4	3	0	1	1	S(B)
		4	3	3	2	1	1	S’

	2	1	1					
	4	1	1		1		1	
	3	1	1		1			
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	1	1						
		5	4	1	2	1	1	S(B)
		4	3	3	2	1	1	S’

	2	1	1 ➡					
	4	1	1 ➡		1		1	
R	3	1	1		1			
	4	1	1	1		1		
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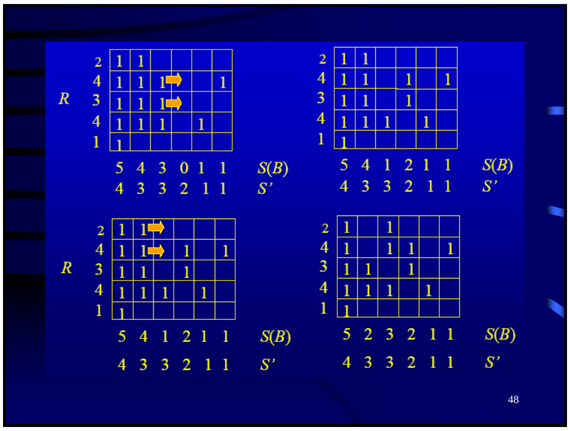

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	4	1	1	1		1		
	1	1						
		5	4	3	0	1	1	S(B)
		4	3	3	2	1	1	S’

	2	1	1					
	4	1	1		1		1	
	3	1	1		1			
	4	1	1	1		1		
	1	1						
		5	4	1	2	1	1	S(B)
		4	3	3	2	1	1	S’

	2	1	1 ➡					
	4	1	1 ➡		1		1	
R	3	1	1		1			
	4	1	1	1		1		
	1	1						
		5	4	1	2	1	1	S(B)
		4	3	3	2	1	1	S’

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	4	1		1	1		1	
	3	1	1		1			
	4	1	1	1		1		
	1	1						
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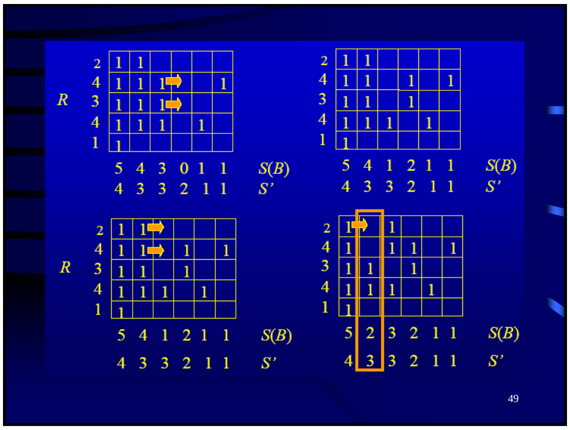
		4	3	3	2	1	1	S’



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	4	1	1	1		1		
	1	1						
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		4	3	3	2	1	1	S’

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3	1	1		1			
4	1	1	1		1		
1	1						
	5	4	1	2	1	1	S(B)
	4	3	3	2	1	1	S’

	2	1	1					
	4	1	1		1		1	
R	3	1	1		1			
	4	1	1	1		1		
	1	1						
		5	4	1	2	1	1	S(B)
		4	3	3	2	1	1	S’

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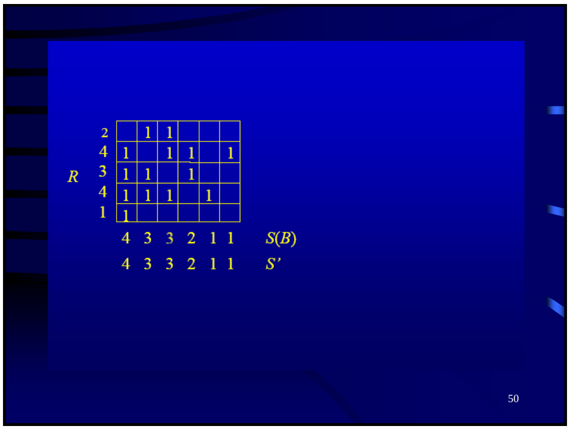
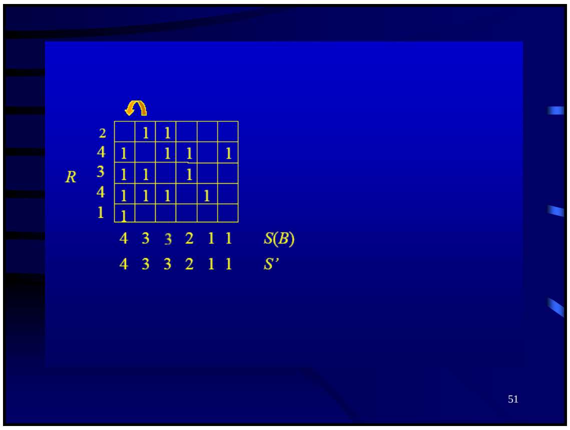
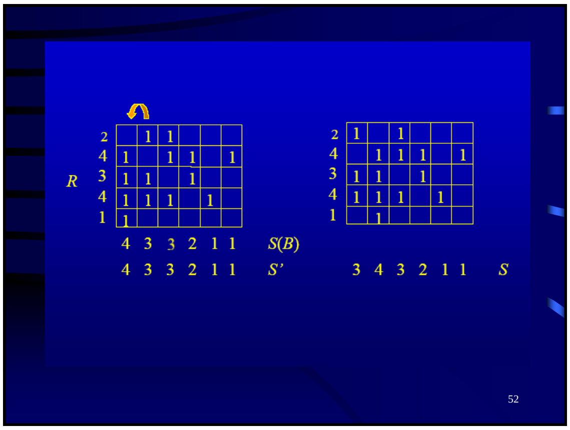
4	1		1	1		1	
3	1	1		1			
4	1	1	1		1		
1	1						
	5	2	3	2	1	1	S(B)
	4	3	3	2	1	1	S’



	2		1	1				
	4	1		1	1		1	
R	3	1	1		1			
	4	1	1	1		1		
	1	1						
		4	3	3	2	1	1	S(B)
		4	3	3	2	1	1	S’

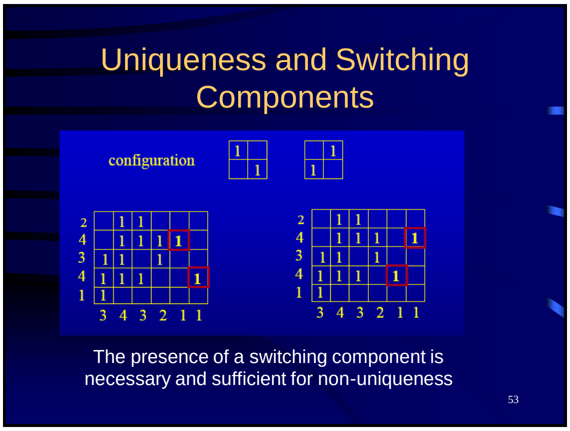


	2		1	1				
	4	1		1	1		1	
R	3	1	1		1			
	4	1	1	1		1		
	1	1						
		4	3	3	2	1	1	S(B)
		4	3	3	2	1	1	S’



	2		1	1				
	4	1		1	1		1	
R	3	1	1		1			
	4	1	1	1		1		
	1	1						
		4	3	3	2	1	1	S(B)
		4	3	3	2	1	1	S’

2	1		1				
4		1	1	1		1	
3	1	1		1			
4	1	1	1		1		
1		1					
	3	4	3	2	1	1	S



Uniqueness and Switching Components

configuration

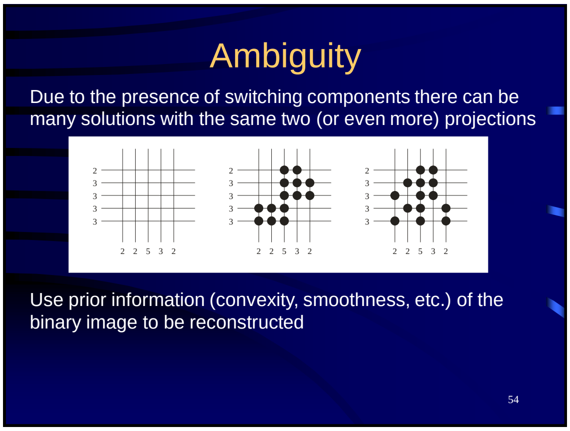
1	
	1

	1
1	

2		1	1			
4		1	1	1	1	
3	1	1		1		
4	1	1	1			1
1	1					
	3	4	3	2	1	1

2		1	1			
4		1	1	1		1
3	1	1		1		
4	1	1	1		1	
1	1					
	3	4	3	2	1	1

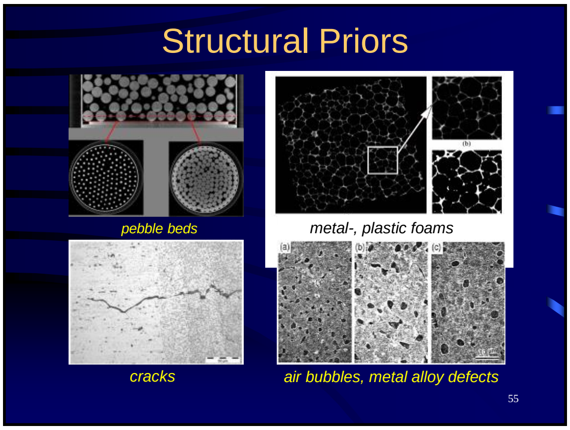
The presence of a switching component is necessary and sufficient for non-uniqueness



Ambiguity

Due to the presence of switching components there can be many solutions with the same two (or even more) projections

Use prior information (convexity, smoothness, etc.) of the binary image to be reconstructed



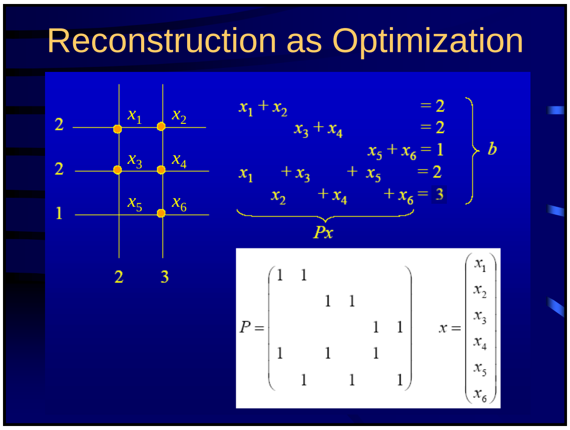
Structural Priors

pebble beds

metal-, plastic foams

cracks

air bubbles, metal alloy defects



Reconstruction as Optimization

$$\underbrace{\begin{array}{llllll} x_1 + x_2 & & & & & = 2 \\ & x_3 + x_4 & & & & = 2 \\ & & & & x_5 + x_6 & = 1 \\ x_1 & + x_3 & & + x_5 & & = 2 \\ & x_2 & + x_4 & & + x_6 & = 3 \end{array}}_{Px} \left.\vphantom{\begin{array}{l}1\\1\\1\\1\\1\end{array}}\right\} b$$

$$P = \begin{pmatrix} 1 & 1 & & & & \\ & & 1 & 1 & & \\ & & & & 1 & 1 \\ 1 & & 1 & & 1 & \\ & 1 & & 1 & & 1 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

Optimization

$$Px = b \qquad x \in \{0,1\}^{m \times n}$$

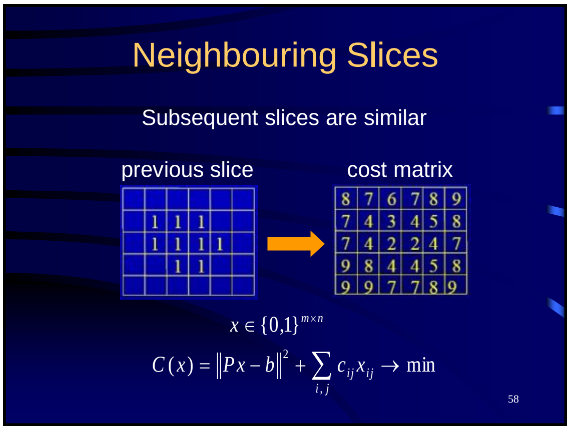
Problems:

- binary variables
- big system
- underdetermined (#equations << #unknowns)
- possibly no solution (if there is noise)

$$x \in \{0,1\}^{m \times n}$$

$$C(x) = \|Px - b\|^2 + g(x) \to \min$$

Term for prior information: structure, similarity to a model image, etc.



Neighbouring Slices

Subsequent slices are similar

previous slice

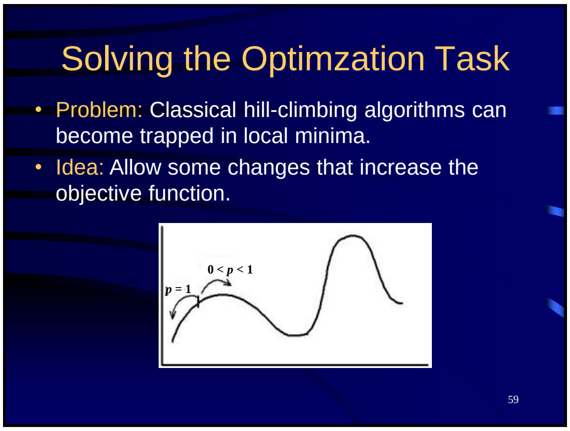
	1	1	1		
	1	1	1	1	
		1	1		

cost matrix

8	7	6	7	8	9
7	4	3	4	5	8
7	4	2	2	4	7
9	8	4	4	5	8
9	9	7	7	8	9

$$x \in \{0,1\}^{m \times n}$$

$$C(x) = \|Px - b\|^2 + \sum_{i,j} c_{ij} x_{ij} \to \min$$

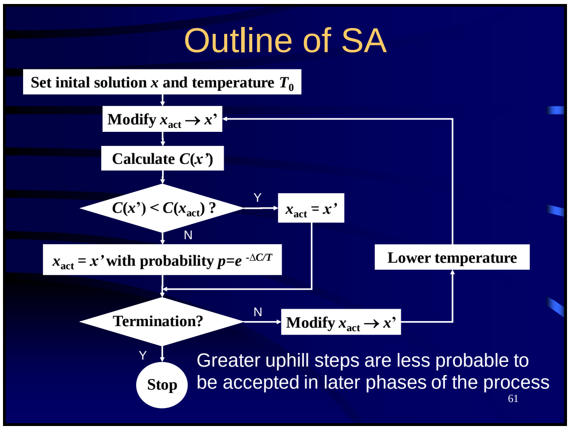


Solving the Optimzation Task

- Problem: Classical hill-climbing algorithms can become trapped in local minima.
- Idea: Allow some changes that increase the objective function.

Simulated Annealing

- Annealing: a thermodinamical process in which a metal cools and freezes.
- Due to the thermical noise the energy of the liquid in some cases grows during the annealing.
- By carefully controlling the cooling temperature the fluid freezes into a minimum energy crystalline.
- Simulated annealing: a random-search technique based on the above observation.



Outline of SA

Set inital solution x and temperature T_0

Modify $x_{act} \rightarrow x'$

Calculate $C(x')$

$C(x') < C(x_{act})$?

Y: $x_{act} = x'$

N: $x_{act} = x'$ with probability $p=e^{-\Delta C/T}$

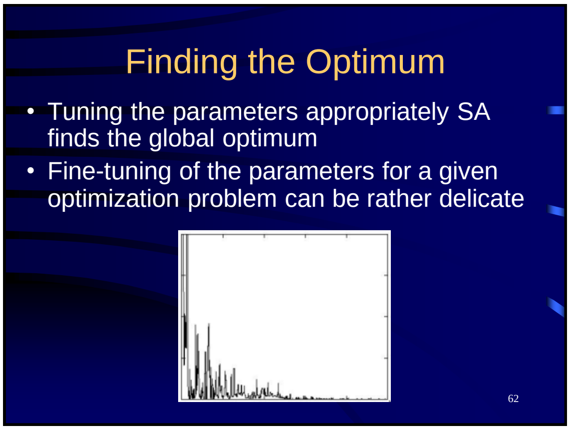
Termination?

N: Modify $x_{act} \rightarrow x'$

Lower temperature

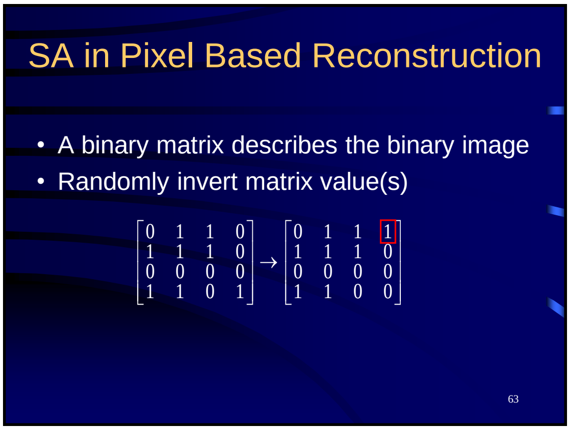
Y: Stop

Greater uphill steps are less probable to be accepted in later phases of the process



Finding the Optimum

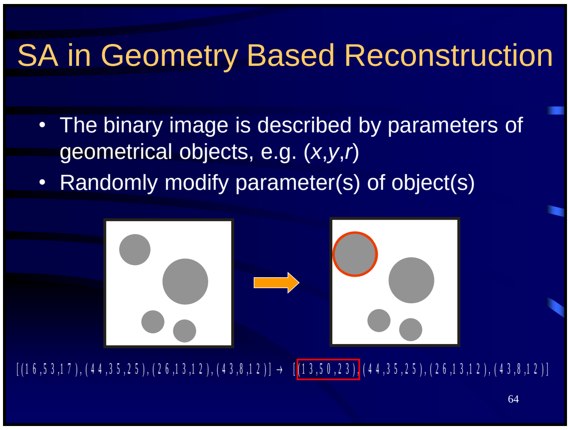
- Tuning the parameters appropriately SA finds the global optimum
- Fine-tuning of the parameters for a given optimization problem can be rather delicate



SA in Pixel Based Reconstruction

- A binary matrix describes the binary image
- Randomly invert matrix value(s)

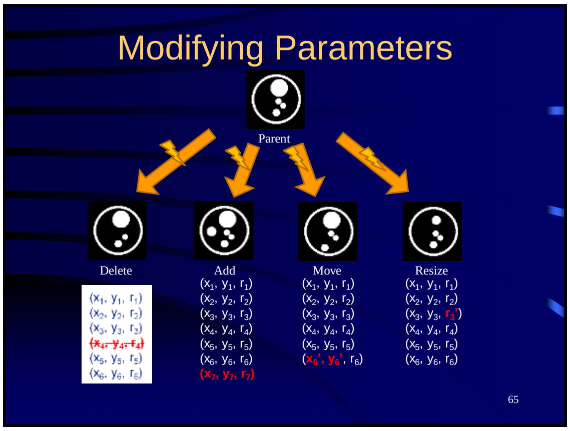
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 & \boxed{1} \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$



SA in Geometry Based Reconstruction

- The binary image is described by parameters of geometrical objects, e.g. (x,y,r)
- Randomly modify parameter(s) of object(s)

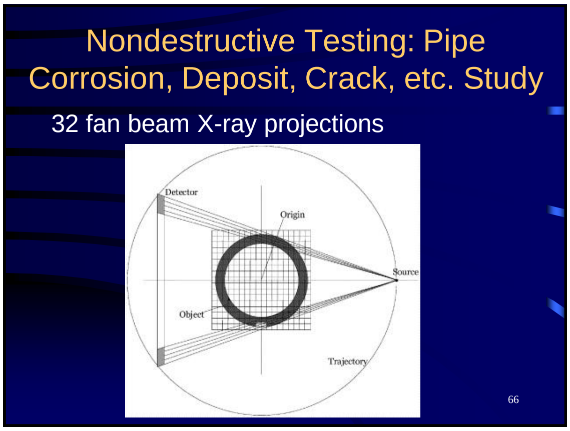
$[(16,53,17),(44,35,25),(26,13,12),(43,8,12)] \rightarrow [\boxed{(13,50,23),}(44,35,25),(26,13,12),(43,8,12)]$



Modifying Parameters

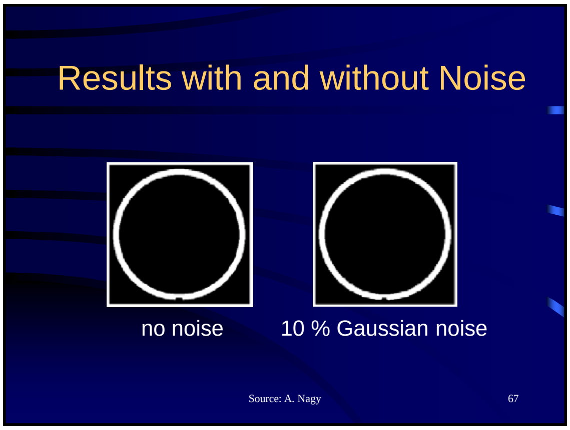
Parent

Delete	Add	Move	Resize
(x_1, y_1, r_1)	(x_1, y_1, r_1)	(x_1, y_1, r_1)	(x_1, y_1, r_1)
(x_2, y_2, r_2)	(x_2, y_2, r_2)	(x_2, y_2, r_2)	(x_2, y_2, r_2)
(x_3, y_3, r_3)	(x_3, y_3, r_3)	(x_3, y_3, r_3)	(x_3, y_3, r_3')
~~(x_4, y_4, r_4)~~	(x_4, y_4, r_4)	(x_4, y_4, r_4)	(x_4, y_4, r_4)
(x_5, y_5, r_5)	(x_5, y_5, r_5)	(x_5, y_5, r_5)	(x_5, y_5, r_5)
(x_6, y_6, r_6)	(x_6, y_6, r_6)	(x_6', y_6', r_6)	(x_6, y_6, r_6)
	(x_7, y_7, r_7)		



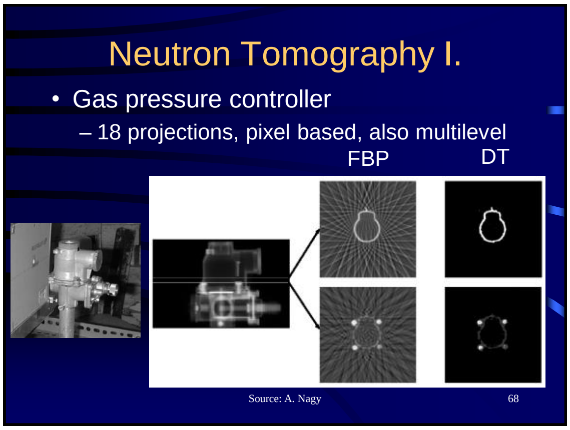
Nondestructive Testing: Pipe Corrosion, Deposit, Crack, etc. Study

32 fan beam X-ray projections



Results with and without Noise

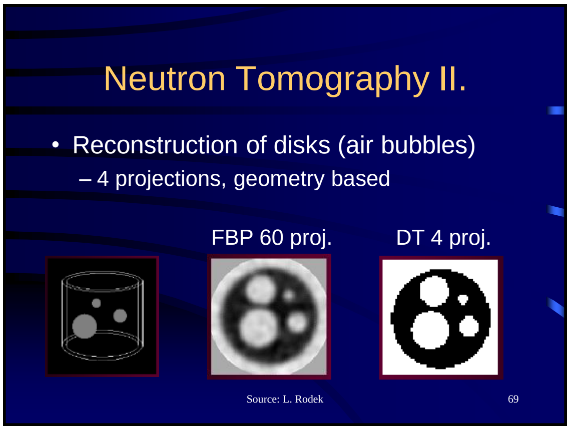
Source: A. Nagy



Neutron Tomography I.

- Gas pressure controller
  - 18 projections, pixel based, also multilevel

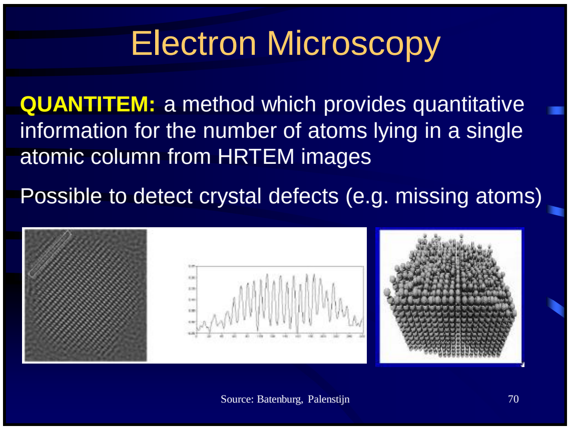
Source: A. Nagy



Neutron Tomography II.

- Reconstruction of disks (air bubbles)
  - 4 projections, geometry based

Source: L. Rodek



Electron Microscopy

QUANTITEM: a method which provides quantitative information for the number of atoms lying in a single atomic column from HRTEM images

Possible to detect crystal defects (e.g. missing atoms)

Source: Batenburg, Palenstijn