On the Number of *hv*-Convex Discrete Sets

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Abstract. One of the basic problems in discrete tomography is the reconstruction of discrete sets from few projections. Assuming that the set to be reconstructed fulfills some geometrical properties is a commonly used technique to reduce the number of possibly many different solutions of the same reconstruction problem. The class of *hv*-convex discrete sets and its subclasses have a well-developed theory. Several reconstruction algorithms as well as some complexity results are known for those classes. The key to achieve polynomial-time reconstruction of an *hv*-convex discrete set is to have the additional assumption that the set is connected as well. This paper collects several statistics on *hv*-convex discrete sets, which are of great importance in the analysis of algorithms for reconstructing such kind of discrete sets.

Keywords: discrete tomography, *hv*-convex discrete set, connectedness, analysis of algorithms.

1 Introduction

Discrete tomography (DT) [15,16] aims to reconstruct a discrete set (a finite subset of the two-dimensional integer lattice defined up to translation) from the number of its elements lying on the same line along several (usually horizontal, vertical, diagonal, and antidiagonal) directions, called projections. It has several applications in pattern recognition, image processing, electron microscopy, angiography, non-destructive testing, and so on. The main challenge in DT is that practical limitations every time reduce the number of available projections to at most about four – which results in a possibly extremely large number of solutions of the same reconstruction task. This can cause the reconstructed discrete set to be quite different from the original one. In addition, the reconstruction problem can be NP-hard, depending on the number and directions of the projections. One way of eliminating these problems is to suppose that the set to be reconstructed has some geometrical properties. In this way we can reduce the search space of the possible solutions and we can achieve fast and rare ambiguous reconstructions.

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The class of hv-convex discrete sets and its subclasses are very frequently studied in DT. The reconstruction in those classes has a well-developed theory including heuristic and exact reconstruction algorithms as well as some important complexity results. It turned out, that the key to achieve polynomial-time reconstruction of an hv-convex discrete set is to have the additional assumption that the set is connected as well.

In this paper we describe a method for counting elements of several subclasses of the hv-convex class. The paper is structured as follows. First, the necessary definitions are introduced in Section 2. In Section 3 we describe recursive formulas for counting hv-convex discrete sets, possibly with certain additional properties. After that, in Section 4 we collect some statistics that can affect the complexity of several reconstruction algorithms developed for the hv-convex class. Section 5 is for the conclusion.

2 Definitions

The finite subsets of the 2D integer lattice are called *discrete sets*. The *size* of a discrete set is defined by the size of its minimal bounding discrete rectangle (i.e. not the number of its elements). A discrete set F of size $m \times n$ is defined up to a translation and it is usually represented by a binary picture formed from unitary cells (see Fig. 1). We refer to the topmost row of the discrete set as the first row, and to the leftmost column of the set as the first column. Thus, the upper left corner of the minimal bounding rectangle of a discrete set is always the $(1, 1)$ position, and the remaining positions of the minimal bounding rectangle (and of the discrete set as well) are addressed consequently.

A discrete set F is *4-connected* (*8-connected*), if for any two positions $P \in F$ and $Q \in F$ of the set there exist a sequence of distinct positions $(i_0, j_0) = P, \ldots, (i_k, j_k) = Q$ such that $(i_l, j_l) \in F$ and $|i_l - i_{l+1}| + |j_l - j_{l+1}| = 1$ $(\max\{|i_l - i_{l+1}|, |j_l - j_{l+1}|\} = 1)$ for each $l = 0, \ldots, k-1$ (see Figs. 2a and 2b). The 4-connected sets are also called polyominoes [14]. If the discrete set is not 4-connected then it consists of several polyominoes. The maximal 4-connected subsets of a discrete set F are called the *components of F*. For, example the discrete set in Fig. 1 has three components: $\{(1, 4), (2, 4)\}$, $\{(2, 2), (3, 2), (4, 1), (4, 2)\}$, and $\{(3, 5), (4, 5), (4, 6), (5, 6), (6, 5), (6, 6)\}$. A discrete set is called *horizontally and vertically convex* (shortly, hv-convex) if all the rows and columns of the

Fig. 1. A discrete set represented by its elements (*left*) and a binary picture (*right*)

set are 4-connected (see Figs. 2c, 2d, and 2e). Let us introduce the following notations for some classes of *hv*-convex sets:

- $\mathcal{P}$ for the class of *hv*-convex polyominoes;
- $\mathcal{Q}$ for the class of *hv*-convex 8-connected discrete sets;
- $\mathcal{HV}'$ for the class of *hv*-convex discrete sets with nonempty rows and columns;
- $\mathcal{HV}$ for the class of *hv*-convex discrete sets which possibly can have empty rows and columns.

The following inclusions are trivial,

$$\mathcal{P} \subset \mathcal{Q} \subset \mathcal{HV}' \subset \mathcal{HV} \ . \tag{1}$$

A polyomino F is *northeast directed* (NE-directed for short) if there is a particular point $P \in F$ such that for each point $Q \in F$ there is a sequence $P_0 = P, \ldots, P_t = Q$ of distinct points of F such that each point P_l of the sequence is north or east of P_{l-1} for each $l = 1, \ldots, t$ (see Fig. 2f). Similar definitions can be given for SW-, SE-, and NW-directedness. An *hv*-convex polyomino is called *NW/NE-parallelogram polyomino* if it is both NW- and SE-directed or both NE- and SW-directed, respectively (see Fig. 2g).

3 Enumeration of *hv*-Convex Discrete Sets

The class of *hv*-convex discrete sets is one of the most important classes in discrete tomography. Although the reconstruction from two projections in this class is NP-hard [20] several methods can solve this problem by applying some heuristic [17], metaheuristic [8] or optimization [11] technique. Besides, for *hv*-convex polyominoes and *hv*-convex 8-connected sets different polynomial-time reconstruction algorithms have been developed. One of them approximates the solution iteratively by a nondecreasing sequence of so-called kernel sets and by a nonincreasing sequence of so-called shell sets (see [6,5,9]). This algorithm has a worst case time complexity of $O(mn \cdot \log mn \cdot \min\{m^2, n^2\})$. An other approach is based on an observation that the reconstruction task can be transformed into a 2SAT task that is solvable in polynomial time [10,18]. This latter algorithm has a worst case time complexity of $O(mn \cdot \min\{m^2, n^2\})$. In [4] the two algorithms were compared, and the observations concerning the average execution times of the two reconstruction approaches led to the design of a hybrid reconstruction algorithm that has the same worst case time complexity of $O(mn \cdot \min\{m^2, n^2\})$ and remains fast in the average case as well. Recently, an algorithm has been also published that can perform the reconstruction in the class of *hv*-convex 8-connected but not 4-connected discrete sets in $O(mn \cdot \min\{m, n\})$ time [3].

Summarizing the above-mentioned contributions we can say that the reconstruction of an *hv*-convex discrete set is in general a difficult problem but the additional information that the set satisfies some connectedness properties as well can adequately facilitate the reconstruction. Now, one can naturally pose the question whether an *hv*-convex discrete set – chosen randomly using a uniform distribution – often fulfills some connectedness properties as well. To answer

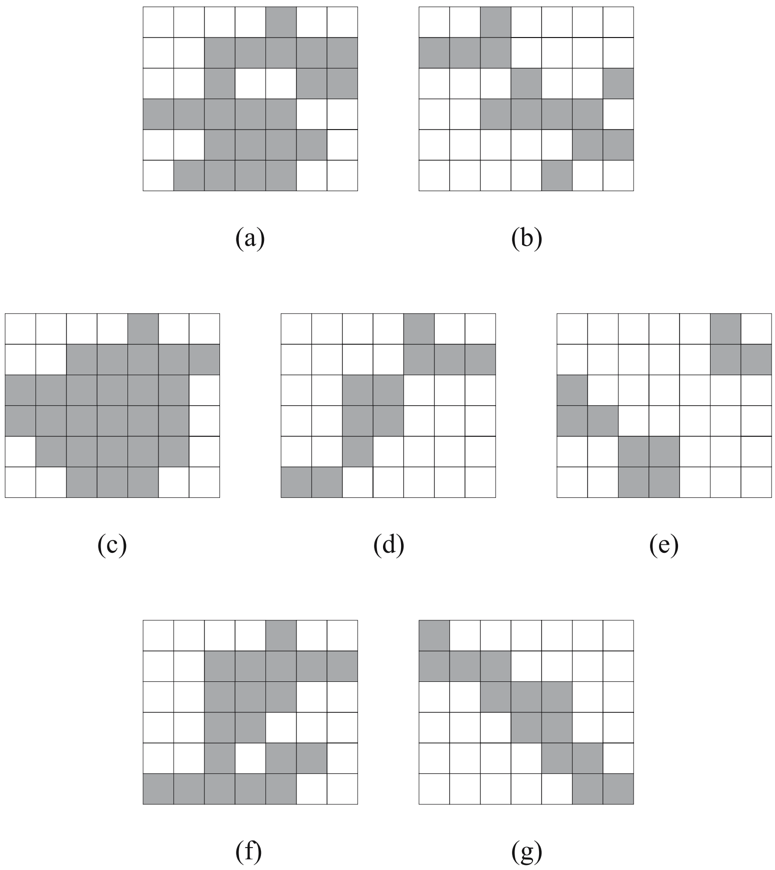

Fig. 2. (a) a polyomino, (b) an 8-connected discrete set, (c) an *hv*-convex polyomino, (d) an *hv*-convex 8-connected discrete set, (e) a general *hv*-convex discrete set, (f) an NE-directed polyomino, and (g) an NW-parallelogram polyomino

this question we have to identify the cardinality of the class of *hv*-convex discrete sets and its subclasses.

Regarding the class $\mathcal{P}$ we already have nice closed formulas for describing the number of *hv*-convex polyominoes according to several parameters. In [12] it was proved that the number P_{n+4} of *hv*-convex polyominoes with a semiperimeter value of $n+4$ is

$$P_{n+4} = (2n+11)4^n - 4(2n+1)\binom{2n}{n} . \quad (2)$$

Later, based on the above result in [13] it was shown that the number $P_{m+1,n+1}$ of *hv*-convex polyominoes of size $(m+1) \times (n+1)$ is

$$P_{m+1,n+1} = \frac{m+n+mn}{m+n}\binom{2m+2n}{2m} - \frac{2mn}{m+n}\binom{m+n}{m}^2 . \quad (3)$$

For the number of elements of the classes $\mathcal{HV}'$ and $\mathcal{HV}$ we obtain recursive formulas from [1]. Let $HV'^{(t)}_{m,n}$ denote the number of arbitrary *hv*-convex discrete sets of size $m \times n$ with nonempty rows and columns which have exactly t components. Then $HV'^{(t)}_{i,j} = 0$ if $i < t$ or $j < t$, and $HV'^{(1)}_{i,j} = P_{i,j}$ for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Finally, for every $t > 1$ and $m, n \geq 1$ the following recursion holds

$$HV'^{(t)}_{m,n} = \sum_{k<m,\ l<n} P_{k,l} \cdot HV'^{(t-1)}_{m-k,n-l} \cdot t \ . \tag{4}$$

With a simple calculation we find that the total number $HV'_{m,n}$ of arbitrary *hv*-convex discrete sets of size $m \times n$ with nonempty rows and columns is

$$HV'_{m,n} = \sum_{t=1}^{\min\{m,n\}} HV'^{(t)}_{m,n} \ . \tag{5}$$

In an analogous way, we also can describe a recursive formula for counting arbitrary *hv*-convex sets – perhaps with empty rows or/and columns – as well.

$$HV^{(t)}_{m,n} = \sum_{k<m,\ l<n} \ \sum_{i \leq m-k,\ j \leq n-l} P_{k,l} \cdot HV^{(t-1)}_{i,j} \cdot t \ . \tag{6}$$

Then, the total number of *hv*-convex discrete sets can be calculated by a formula similar to (5).

Now, let us investigate the number of *hv*-convex 8-connected discrete sets of size $m \times n$. For the sake of technical simplicity we recall some concepts of [1].

Let F be a discrete set with $k \geq 2$ components such that $I_l \times J_l = \{i_l, \ldots, i'_l\} \times \{j_l, \ldots j'_l\}$ is the minimal bounding rectangle of the l-th component of F. We say that the components of F are *disjoint* if for any $1 \leq l, l' \leq k$ $l \neq l'$ implies that $I_l \cap I_{l'} = \emptyset$ and $J_l \cap J_{l'} = \emptyset$. Now, without loss of generality we can assume that $i_l < i_{l+1}$ for each $l = 1, \ldots, k-1$. F is called *canonical* if $j_l < j_{l+1}$ for each $l = 1, \ldots, k-1$. F is called *anticanonical* if $j_l > j_{l+1}$ for each $l = 1, \ldots, k-1$. That is, the discrete set is canonical (anticanonical) if - omitting empty rows and columns - the minimal bounding rectangles of the components are connected to each other with their bottom right hand and upper left hand (bottom left hand and upper right hand) corners (see Fig. 3).

Let us introduce the notations $D_{m,n}$, $L_{m,n}$, and $Q_{m,n}$ for the number of NW-directed polyominoes, NW-parallelogram polyominoes, and *hv*-convex 8-connected discrete sets of size $m \times n$, respectively. Moreover let $T_{m,n}$ denote the number of canonical 8-connected discrete sets whose components are all NW-parallelogram polyominoes. With these notations we obtain

Theorem 1. *For each* $m, n > 1$

$$T_{m,n} = L_{m,n} + \sum_{k<m,\ l<n} L_{k,l} \cdot T_{m-k,n-l} \tag{7}$$

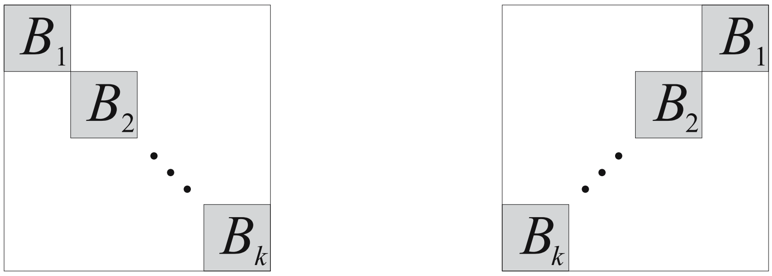

Fig. 3. The relative position of the minimal bounding rectangles of the components $B_1, \ldots, B_k$ of a canonical (*left*) and an anticanonical (*right*) discrete set

and

$$Q_{m,n} = P_{m,n} + 2 \sum_{i<m,\ j<n} \ \sum_{k \leq m-i,\ l \leq n-j} D_{i,j} \cdot D_{k,l} \cdot T_{m-k-i,n-j-l} \ . \qquad (8)$$

Proof. Let $\mathcal{T}$ denote the class of canonical 8-connected discrete sets which components are all NW-parallelogram polyominoes. With this notation Equation (7) can be proven in the following way. A discrete set $F \in \mathcal{T}$ of size $m \times n$ is either a NW-parallelogram polyomino (i.e. it has just one component) or it contains a NW-parallelogram polyomino of size $k \times l$ (where $k < m$ and $l < n$) as a subset in the upper left hand corner and the remaining part of F is a discrete set of size $(m-k) \times (n-l)$ which also belongs to the $\mathcal{T}$ class (see Fig. 3 again). This observation can be concisely expressed by the recursive formula (7).

To prove Equation (8) we recall the following observations from [3]. A set of $\mathcal{Q}$ is either an hv-convex polyomino or it consists of several hv-convex components. Let $F \in \mathcal{Q}$ be a discrete set having components $F_1, \ldots, F_k$ such that $\{i_l, \ldots, i'_l\} \times \{j_l, \ldots, j'_l\}$ is the minimal bounding rectangle of the l-th ($l = 1, \ldots, k$) component of F. Without loss of generality we can assume that $1 = i_1 \leq i'_1 < i_2 \leq i'_2 < \ldots \leq i'_k = m$. Then, either $1 = j_1 \leq j'_1 < j_2 \leq j'_2 < \ldots \leq j'_k = n$, or $n = j_1 \geq j'_1 > j_2 \geq j'_2 > \ldots \geq j'_k = 1$. Consequently, such a set of $\mathcal{Q}$ is always canonical or anticanonical.

Due to symmetry the number of canonical and anticanonical sets of $\mathcal{Q}$ which are not polyominoes are the same. Therefore, it is sufficient to calculate the number of canonical discrete sets of $\mathcal{Q}$ and multiply the result by 2. For a canonical set of $\mathcal{Q}$ it is always true that $F_1, \ldots, F_{k-1}$ are NW-directed and $F_2, \ldots, F_k$ are SE-directed (that is, $F_2, \ldots, F_{k-1}$ are NW-parallelogram polyominoes). In particular, we also get that there are hv-convex 8-connected sets which have just two components and with no parallelogram polyominoes between them. Additionally, the structure of a canonical set of $\mathcal{Q}$ is the following. It contains an NW-directed polyomino of size $i \times j$ in the upper-left corner (where $i < m$ and $j < n$), an SE-directed polyomino of size $k \times l$ in the bottom right corner (where $k \leq m-i$ and $l \leq n-j$) and the remaining part (if exists) is a canonical discrete set of size $(m-i-k) \times (n-j-l)$ which components are all NW-parallelogram polyominoes. Thus, we get the formula (8). $\square$

For the number of NW-directed (parallelogram) polyominoes we obtain the formulas from [7]. Namely,

$$D_{m,n} = \binom{m+n-2}{m-1}\binom{m+n-2}{n-1} \tag{9}$$

and

$$L_{m,n} = \frac{1}{m+n-1}\binom{m+n-1}{m-1}\binom{m+n-1}{n-1} . \tag{10}$$

Setting $D_{1,j} = D_{i,1} = L_{0,j} = L_{1,j} = L_{i,0} = L_{i,1} = T_{1,j} = T_{i,1} = 1$ and $T_{0,j} = T_{i,0} = 0$ for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$ (for technical reasons we set $T_{0,0} = 1$) we are now able to determine the number of hv-convex 8-connected discrete sets of size $m \times n$ for an arbitrary m and n.

4 Statistics on *hv*-Convex Discrete Sets

The recursive formulas of Section 3 allow us to examine some important properties of hv-convex discrete sets that can affect the reconstruction complexity. In order to get such statistics we first calculated the number of hv-convex discrete sets in the classes studied. Table 1 shows the number of elements in the classes $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{HV}'$, and $\mathcal{HV}$ with minimal bounding rectangles of semi-perimeter n for the first 15 values of n – represented by P_n, Q_n, HV'_n, and HV_n, respectively (the first column can also be calculated via formula (2) and it enumerates the first 15 elements of Sequence A005436 in [19]).

For $n = 5$ the corresponding hv-convex binary images are shown in Fig. 4.

Knowing the relations of (1) and with the aid of the statistics presented in Table 1, we can describe the relative cardinality of the classes examined. With this information we can, for example, address questions concerning the relative occurrence of certain hv-convex discrete sets and calculate the probability that an hv-convex discrete set chosen from a uniform random distribution has some special properties which can facilitate the reconstruction task.

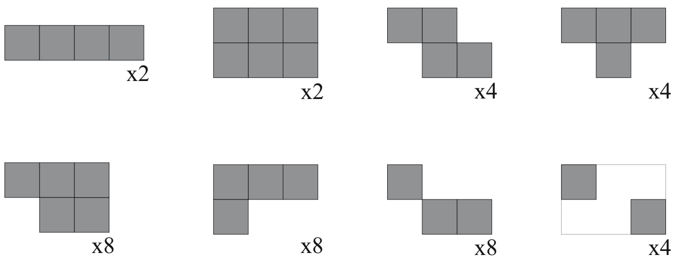

Fig. 4. Some hv-convex binary pictures with a perimeter value of 10. The numbers tell us that there are other solutions that can be obtained by mirroring or/and rotating the given polyomino.

Table 1. The values of P_n, Q_n, HV'_n, and HV_n

n	P_n	Q_n	HV'_n	HV_n
2	1	1	1	1
3	2	2	2	2
4	7	9	9	9
5	28	36	36	40
6	120	154	162	184
7	528	668	732	860
8	2344	2916	3368	4058
9	10416	12741	15520	19240
10	46160	55570	71618	91440
11	203680	241692	329988	435136
12	894312	1047604	1518090	2072672
13	3907056	4524464	6971112	9883264
14	16986352	19470660	3196392	47193776
15	73512288	83500968	146390016	225779728

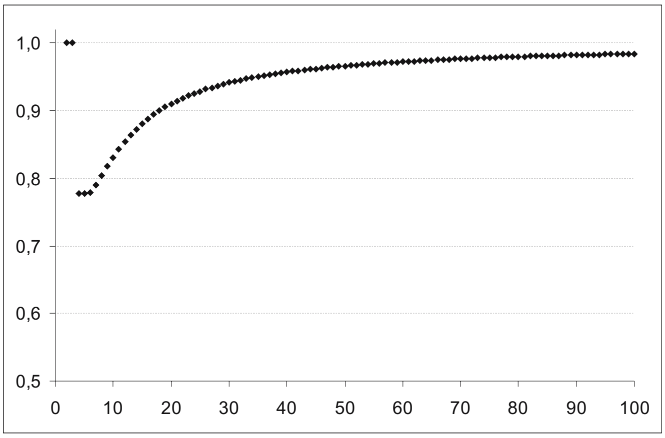

Fig. 5. The ratio P_n/Q_n (*vertical axis*) depending on the semiperimeter value n (*horizontal axis*)

Example 1. Using the entries of Table 1 we can calculate the probability that an hv-convex discrete set with semi-perimeter value of 6 chosen from a uniform random distribution is an hv-convex polyomino (i.e. it consists of one component), which turns out to be $120/184 \approx 0.65$. If we increase the semi-perimeter value to 10, say, then this probability decreases to $46160/91440 \approx 0.50$. Such information is especially useful in the reconstruction task as hv-convex polyominoes can be reconstructed from their horizontal and vertical projections in polynomial time. In contrast, if the hv-convex set has at least two components then the reconstruction is NP-hard (see the introduction here). Hence with this method we can calculate the probability that the reconstruction of the randomly chosen

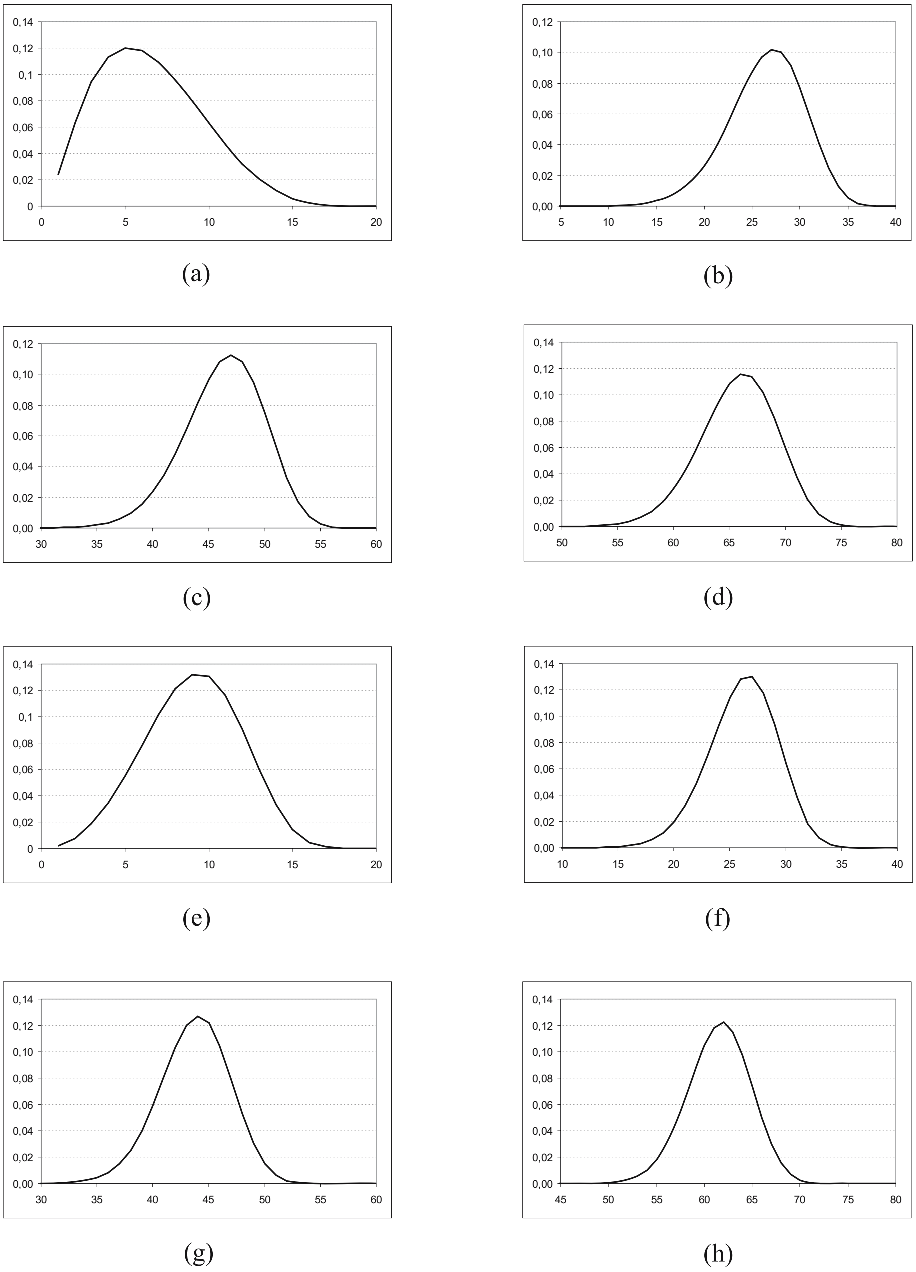

Fig. 6. The distributions of the number of components – which depend on the size of the test data – in the $\mathcal{HV}'$ ((a)-(d)) and $\mathcal{HV}$ ((e)-(h)) classes

Table 2. The expectation value $E_{\mathcal{HV}'}(n)$ ($E_{\mathcal{HV}}(n)$) and the variance $D^2_{\mathcal{HV}'}(n)$ ($D^2_{\mathcal{HV}}(n)$) of the components of a set with a minimal bounding rectangle of size $n \times n$ in the $\mathcal{HV}'$ ($\mathcal{HV}$) class. The values have been rounded to 5 digits

n	$E_{HV'}(n)$	$D^2_{HV'}(n)$	$E_{HV}(n)$	$D^2_{HV}(n)$
20	6.53981	9.84446	9.03570	8.12406
40	26.33821	16.00766	26.11090	9.54114
60	46.30283	12.92260	43.68220	10.00145
80	65.70631	12.05665	61.49588	10.72577

hv-convex set can be performed using a polynomial-time algorithm to reconstruct an *hv*-convex polyomino.

Example 2. In [3] the authors presented a very fast algorithm for the reconstruction of *hv*-convex 8-connected but not 4-connected discrete sets. From the first few entries of Table 1 we have the suggestion that the number of such kind of sets rapidly decreases as the semiperimeter value increases. To verify this, we calculated the first 100 values of P_n/Q_n (see Fig. 5). From this figure it is evident, that – unfortunately – even for sets of relatively small sizes there is almost no chance to apply this fast reconstruction algorithm in practice (assuming that the sets to be reconstructed are from a uniform random distribution), and things get worse if we want to reconstruct sets of bigger sizes.

With the aid of the formulas (4) and (6) it is also possible to describe the true distribution of the number of components of the generated *hv*-convex discrete set of the $\mathcal{HV}'$ class since, in this case, we can enumerate the discrete sets of a given class that have k components. This piece of information is also very useful when reconstructing images like these. For example, as was discussed in the introduction of Section 3, if the *hv*-convex set consists of a single component then the reconstruction from two projections can be solved in polynomial time, otherwise it is NP-hard. Furthermore, the number of components of an *hv*-convex set also affects the accuracy of the reconstruction heuristic that was presented in [2]. Namely, the more components the *hv*-convex discrete set has, it is more likely that ambiguity will occur in the reconstruction.

Table 2 lists the expectation values and the variances of the variables which represent the number of components of a discrete set generated using a uniform random distribution from the $\mathcal{HV}'$ and $\mathcal{HV}$ classes when the size of the minimal bounding rectangle is $n \times n$ for some fixed positive integer n. In addition, the corresponding distributions are depicted in Fig. 6.

Statistics about the expected number of components can be especially useful in the reconstruction task. It tells us something about the discrete set to be reconstructed before we attempt to reconstruct it. Thus, such statistics opens the way to the design of reconstruction algorithms that exploit information known beforehand about the expected number of components. The author believes that such algorithms could be more effective in practice than the previously developed ones which do not make use of such prior knowledge.

5 Conclusions

In this paper we have presented recursive formulas to count *hv*-convex discrete sets (which possibly have certain connectedness properties as well). With the aid of these formulas we have collected some statistics on several subclasses of *hv*-convex discrete sets. We can use these statistics to analyze the performance of certain reconstruction algorithms developed for the classes studied. In addition, it turned out that it is also possible to say something about the number of the components of an *hv*-convex discrete set before we attempt to reconstruct it (if the set arises from a uniform random distribution). Incorporating this prior knowledge into the reconstruction process can hopefully yield more effective reconstruction algorithms in the future.

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