# Reconstruction of Canonical *hv*-Convex Discrete Sets from Horizontal and Vertical Projections

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**Abstract.** The problem of reconstructing some special *hv*-convex discrete sets from their two orthogonal projections is considered. In general, the problem is known to be NP-hard, but it is solvable in polynomial time if the discrete set to be reconstructed is also 8-connected. In this paper, we define an intermediate class – the class of *hv*-convex canonical discrete sets – and give a constructive proof that the above problem remains computationally tractable for this class, too. We also discuss some further theoretical consequences and present experimental results as well.

**Keywords:** discrete tomography; *hv*-convex discrete set; reconstruction from projections; polynomial-time reconstruction.

## 1 Introduction

One of the main tasks of *Discrete Tomography* (DT) is to reconstruct discrete sets (finite subsets of the two-dimensional integer lattice) from few projections. Opposite to methods of Computerized Tomography – like filtered backprojection and algebraic reconstruction – which use several hundreds of projections [10], in DT just a few (typically less than ten) projections are available. For the basic algorithms and the wide area of applications of DT the reader is referred to [8,9]. The main problem arising from the very limited number of projections is that the reconstruction task is usually extremely underdetermined, i.e. there may be many different discrete sets with the same projections.

One way to reduce the number of possible solutions is to restrict the reconstruction to a class of discrete sets which satisfy some geometrical properties, such as connectivity and convexity. In the past 20-25 years many reconstruction algorithms have been developed for different classes of discrete sets, and also some strong theoretical results concerning the complexity of the reconstruction in those classes have been presented. Concerning the reconstruction from the horizontal and vertical projections with additional geometrical constrains the first

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[⋆] This research was partially supported by the TÁMOP-4.2.2/08/1/2008-0008 program of the Hungarian National Development Agency and by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

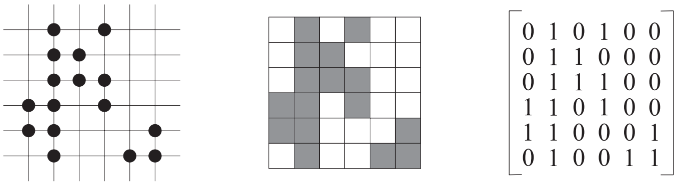
P. Wiederhold and R.P. Barneva (Eds.): IWCIA 2009, LNCS 5852, pp. 280–288, 2009.

approach was presented in [11] where the author gave a reconstruction heuristic for the class of horizontally and vertically convex (shortly, *hv*-convex) discrete sets using only two projections. Later, it was shown that this reconstruction task is NP-hard [13]. However, by this time it was known that assuming that the set to be reconstructed is also connected makes polynomial-time reconstruction possible [4,5,6,12]. The aim of this paper is to introduce an intermediate class between the classes of general and connected *hv*-convex discrete sets, and study the computational complexity of the reconstruction in that class.

The structure of the contribution is the following. First, the necessary definitions are introduced in Section 2. In Section 3 we supply a polynomial time algorithm to reconstruct canonical *hv*-convex discrete sets and do some further theoretical observations. In Section 4 we present experimental results concerning the average running time of the algorithm. Finally, Section 5 is for the conclusion.

## 2 Preliminaries

An arbitrary finite subset of the two-dimensional integer lattice defined up to translation is called *discrete set* and it can be represented by a binary image or binary matrix, too (see Fig. 1). In the following – depending on technical convenience – we will use both terms discrete set and binary matrix in the same sense. To avoid confusion, without loss of generality, we will assume that the vertical axis of the 2D integer lattice is directed top-down and the upper left corner of the smallest containing rectangle of a discrete set is the position $(1, 1)$.



$$\begin{bmatrix} 0&1&0&1&0&0\\ 0&1&1&0&0&0\\ 0&1&1&1&0&0\\ 1&1&0&1&0&0\\ 1&1&0&0&0&1\\ 0&1&0&0&1&1 \end{bmatrix}$$

**Fig. 1.** A discrete set represented by its elements (*left*), a binary picture (*center*), and a binary matrix (*right*)

A discrete set $F$ is *4-connected (8-connected)* if for any two distinct positions $P, Q \in F$ there exists a sequence of distinct positions $(i_0, j_0) = P, \ldots, (i_k, j_k) = Q$ such that $(i_l, j_l) \in F$ and $|i_l - i_{l+1}| + |j_l - j_{l+1}| = 1$ $(|i_l - i_{l+1}| + |j_l - j_{l+1}| \leq 2)$ for each $l = 0, \ldots, k - 1$. The 4-connected sets are also called *polyominoes.* If a discrete set is not 4-connected then it can be partitioned (in a uniquely determined way) into maximal 4-connected subsets which are called *components* of the discrete set. We say that the discrete set is *canonical (anticanonical)* if consists of just one component or the smallest containing rectangles of the components are connected to each other with their bottom-right and upper-left (bottom-left and upper-right) corners (see Fig. 2). A discrete set is called *hv-convex* if all the rows and columns of the set are 4-connected, i.e., the 1s of the corresponding representing matrix are consecutive in each row and column. For example, the discrete sets in Fig. 2 is *hv*-convex.

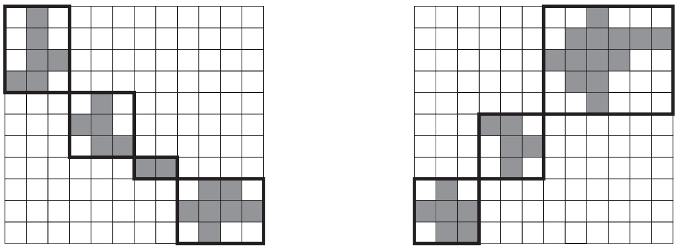


**Fig. 2.** A canonical *hv*-convex discrete set with 4 components (*left*) and an anticanonical *hv*-convex discrete set with 3 components (*right*). The smallest containing rectangles of the components are drawn bold.

The *size* of the discrete set is defined by the size of its minimal bounding rectangle. Given a discrete set $F$ of size $m \times n$, its *horizontal* and *vertical projections* are defined by the vectors $\mathcal{H}(F) = H = (h_1, \ldots, h_m)$, and $\mathcal{V}(F) = V = (v_1, \ldots, v_n)$, respectively, where

$$h_i = \sum_{j=1}^{n} f_{ij}, \qquad i = 1, \ldots, m \ , \tag{1}$$

and

$$v_j = \sum_{i=1}^{m} f_{ij}, \qquad j = 1, \ldots, n \ . \tag{2}$$

For example, the horizontal and vertical projections of the discrete set in Fig. 1 are $H = (2, 2, 3, 3, 3, 3)$ and $V = (2, 6, 2, 3, 1, 2)$, respectively. Discrete sets having zero row or column sums are not of interest in our work. In this paper we are going to study the reconstruction of certain discrete sets from two projections. Given a class $\mathcal{G}$ of discrete sets and two vectors $H \in \mathbb{N}^m$, $V \in \mathbb{N}^n$ (for arbitrary $m$ and $n$) this problem is the following

Reconstruction($\mathcal{G}, H, V$)
**Instance** Two non-negative vectors $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$.
**Task** Construct a discrete set $F \in \mathcal{G}$ such that $\mathcal{H}(F) = H$ and $\mathcal{V}(F) = V$.

## 3 Reconstruction of Canonical *hv*-Convex Discrete Sets

We first recall a result which describes the special structure of *hv*-convex 8-connected discrete sets.

**Lemma 1.** *An hv-convex 8-connected discrete set is either canonical or anti-canonical.*

*Proof.* See Theorem 2 of [2]. □

Now, let us denote the class of *hv*-convex 4-connected, *hv*-convex 8-connected canonical, *hv*-convex canonical, and general *hv*-convex discrete sets by $\mathcal{HV}_4$, $\mathcal{HV}_8^c$, $\mathcal{HV}^c$, and $\mathcal{HV}$, respectively. On the basis of Lemma 1 the class $\mathcal{HV}_8^c$ is non-empty. Moreover, based on the definitions it is obvious that $\mathcal{HV}_4 \subset \mathcal{HV}_8^c \subset \mathcal{HV}^c \subset \mathcal{HV}$ (see Fig. 3 for the proper inclusions).

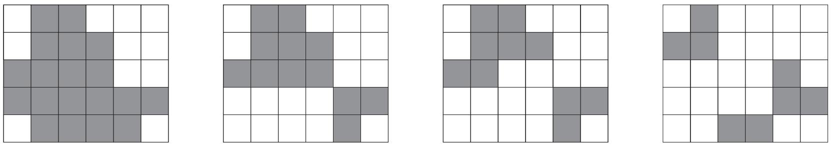


**Fig. 3.** An *hv*-convex 4-connected, an *hv*-convex 8-connected canonical, an *hv*-convex canonical, and a general *hv*-convex discrete set (from left to right, respectively)

Several algorithms have been proposed for solving the reconstruction problem in the class $\mathcal{HV}_4$ of *hv*-convex polyominoes in polynomial time using the horizontal and vertical projections, among them the fastest one has a worst case time complexity of $O(mn \cdot \min\{m^2, n^2\})$ [3,4,6]. Later, this result was extended to the class of (canonical) *hv*-convex 8-connected discrete sets, too [5,12]. Surprisingly, it also turned out that the reconstruction in the class of *hv*-convex 8- but not 4-connected discrete sets (with our notation in the class $\mathcal{HV}_8^c \setminus \mathcal{HV}_4$) can be performed faster, namely in $O(mn \cdot \min\{m, n\})$ time [2]. On the other hand, it was proven that the general problem $\textsc{Reconstruction}(\mathcal{H}V, H, V)$ is NP-hard [13]. Since the class $\mathcal{HV}^c$ of *hv*-convex canonical discrete sets forms an extension of the class $\mathcal{HV}_8^c$ and – in the same time – a restriction of the class $\mathcal{HV}$, it is an important and fascinating question whether the reconstruction complexity in this intermediate class is polynomial. In the followings we give a positive answer to this question.

Consider that the two vectors $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$ are given and we want to reconstruct a canonical *hv*-convex discrete set with those projections. For each $1 \le i_1 \le i_2 \le m$ and $1 \le j_1 \le j_2 \le n$ let us define a boolean variable $A_{i_1,i_2,j_1,j_2}$ such that $A_{i_1,i_2,j_1,j_2} = \mathit{true}$ if and only if there exists an *hv*-convex polyomino $P$ with the minimal bounding rectangle $[i_1, i_2] \times [j_1, j_2]$ such that $\mathcal{H}(P) = (h_{i_1}, \ldots, h_{i_2})$ and $\mathcal{V}(P) = (v_{j_1}, \ldots, v_{j_2})$. In addition, for each $1 \le i \le m$ and $1 \le j \le n$ let $B_{i,j}$ be a boolean variable such that $B_{i,j} = \mathit{true}$ if and only if there exists a canonical *hv*-convex discrete set $S$ with the minimal bounding rectangle $[1, i] \times [1, j]$ such that $\mathcal{H}(S) = (h_1, \ldots, h_i)$ and $\mathcal{V}(S) = (v_1, \ldots, v_j)$. In order to decide whether there exists a discrete set of $\mathcal{HV}^c$ with the given projections we have to calculate the value of $B_{m,n}$. For this we need the following

**Lemma 2.** *The variables* $B_{i,j}$ *satisfy the following recurrence*

*(i)* $B_{1,j} = A_{1,1,1,j}$ *for each* $1 \le j \le n$ ,

*(ii)* $B_{i,1} = A_{1,i,1,1}$ *for each* $1 \le i \le m$ , *and*
*(iii)* $B_{i,j} = A_{1,i,1,j} \vee \bigvee_{1<i'<i,1<j'<j} A_{i',i,j',j} \wedge B_{i'-1,j'-1}$ *(*$1 < i \le m$*;* $1 < j \le n$*).*

*Proof.* The components of a (canonical) $hv$-convex discrete set are $hv$-convex polyominoes. If the minimal bounding rectangle of an $hv$-convex discrete set $F$ consists of just one row or column then $F$ necessarily has just a single component, i.e. it is an $hv$-convex polyomino. Thus, (i) and (ii) hold. Now, suppose that $1 < i \le m$ and $1 < j \le n$ and we have to evaluate $B_{i,j}$. The structure of a canonical $hv$-convex discrete set $S$ in the minimal bounding rectangle $[1,i] \times [1,j]$ having projections $\mathcal{H}(S) = (h_1, \ldots, h_i)$ and $\mathcal{V}(S) = (v_1, \ldots, v_j)$ can only be of two types. It may happen that it consists of a single $hv$-convex component with the minimal bounding rectangle $[1,i] \times [1,j]$ and the proper projections (first term of the right hand side of (iii)). If not so, then it has an $hv$-convex polyomino in the bottom-right corner having a minimal bounding rectangle $[i',i] \times [j',j]$ for certain $1 < i' < i$ and $1 < j' < j$ and the rest of the set is a canonical $hv$-convex discrete set in the minimal bounding rectangle $[1, i'-1] \times [1, j'-1]$ with the corresponding projections, which proves the second term of the right hand side of (iii)) (see also Fig. 2, again). □

Based on this observation we now can outline the reconstruction algorithm for the class $\mathcal{HV}^c$. The algorithm uses an additional two-dimensional array $L$ for storing links to one of the *true* conjunction terms of formula (iii) (if exists) for each $B_{i,j}$ in order to make it possible to identify the smallest bounding rectangles of the components.

**Algorithm REC-HVC** for reconstructing canonical $hv$-convex discrete sets.
**Input:** Two vectors $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$.
**Output:** A discrete set $F \in \mathcal{HV}^c$ such that $\mathcal{H}(F) = H$ and $\mathcal{V}(F) = V$ or the message "NO SOLUTION".

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Step 1 for each $1 \le i_1 \le i_2 \le m$ and $1 \le j_1 \le j_2 \le n$ calculate $A_{i_1,i_2,j_1,j_2}$;
Step 2 for each $1 \le i \le m$ and $1 \le j \le n$
          { calculate $B_{i,j}$ ;
            if ($B_{i,j} = true$) then link $L_{i,j}$ to one of $B_{i,j}$'s true conjunction terms;
                                  else link $L_{i,j}$ to null; }
Step 3 if ($B_{m,n} = true$) then output the solution using the link array $L$;
                              else output "NO SOLUTION";
```

The main result of the paper is the following

**Theorem 1.** *Algorithm REC-HVC solves problem* Reconstruction $(\mathcal{H}V^c, H, V)$ *in* $O(m^3 n^3 \cdot \min\{m^2, n^2\})$ *time with* $O(m^2 n^2)$ *memory requirements.*

*Proof.* In Step 1 the algorithm checks for each pair $1 \le i_1 \le i_2 \le m$ and $1 \le j_1 \le j_2 \le n$ whether an $hv$-convex polyomino exists with the corresponding horizontal and vertical projections. The pairs $(i_i, i_2)$ and $(j_i, j_2)$ identify the smallest bounding rectangle of the polyomino. The number of such pairs is

$O(m^2n^2)$ and the values of all $A$'s can be stored on $O(m^2n^2)$ bytes. For a certain pair it takes $O(mn \cdot \min\{m^2, n^2\})$ time to reconstruct an *hv*-convex polyomino from the given projections (if exist) (see for example [3] for the reconstruction algorithm). Thus, performing Step 1 takes $O(m^3n^3 \cdot \min\{m^2, n^2\})$ time. The recursive formula given in Lemma 2 for all $B_{i,j}$ can be calculated by a dynamic programming in $O(m^2n^2)$ time and stored on $O(mn)$ bytes. So the total space complexity of the algorithm follows. Based on the definition of the boolean variables used and the recursive relations of Lemma 2, there exist a solution of the given task if and only if $B_{m,n} = \mathit{true}$, and in this case – using the link array $L$ – it is also possible to locate the smallest bounding rectangles of the components (otherwise the algorithm terminates with "NO SOLUTION"). Knowing those rectangles one can reconstruct each component independently. Since those components are *hv*-convex polyominoes this can be achieved in $O(mn \cdot \min\{m^2, n^2\})$ time (see again [3]). This does not affects the total execution time which is of $O(m^3n^3 \cdot \min\{m^2, n^2\})$. □

Some more restrictive forms of the above reconstruction problem can also be solved in polynomial time.

**Theorem 2.** *Given two vectors* $H \in \mathbb{N}^m$ *and* $V \in \mathbb{N}^n$ *it is possible to reconstruct a discrete set* $F \in \mathcal{HV}^c$ *(if exists) in polynomial time such that* $\mathcal{H}(F) = H$ *and* $\mathcal{V}(F) = V$ *and* $F$ *has exactly* $k$ *components for a prescribed* $k \in \mathbb{N}$.

*Proof.* We alter algorithm REC-HVC in the following way. For each $1 \le i \le m$, $1 \le j \le n$, and $1 \le c \le \min\{m, n\}$ let $B^c_{i,j}$ be a boolean variable such that $B^c_{i,j} = \mathit{true}$ if and only if there exists a canonical *hv*-convex discrete set $S$ with the minimal bounding rectangle $[1, i] \times [1, j]$ such that $\mathcal{H}(S) = (h_1, \ldots, h_i)$ and $\mathcal{V}(S) = (v_1, \ldots, v_j)$ and $S$ has exactly $c$ components. Note, that a (canonical) *hv*-convex discrete set of size $m \times n$ can have at most $\min\{m, n\}$ components. Then, the task is to calculate $B^k_{i,j}$ for which – in a similar way to Lemma 2 the following recursion can be applied

(i) $B^1_{i,j} = A_{1,i,1,j}$ for each $1 \le j \le n$ and $1 \le i \le m$ ,
(ii) for each $1 < i \le m$ , $1 < j \le n$ , and $1 < c \le \min\{m, n\}$

$$B^c_{i,j} = \begin{cases} \bigvee_{1<i'<i,1<j'<j} A_{i',i,j',j} \wedge B^{c-1}_{i'-1,j'-1}, & \text{if } c \le \min\{i, j\} \ , \\ \mathit{false}, & \text{otherwise} \ . \end{cases} \tag{3}$$

Those variables can be calculated by a dynamic programming in $O(m^2n^2 \cdot \min\{m^2, n^2\})$ time. □

**Corollary 1.** *Given two vectors* $H \in \mathbb{N}^m$ *and* $V \in \mathbb{N}^n$ *it is possible to reconstruct a discrete set* $F \in \mathcal{HV}^c$ *(if exists) in polynomial time such that* $\mathcal{H}(F) = H$ *and* $\mathcal{V}(F) = V$ *and* $F$ *has the maximal (minimal) number of components.*

*Proof.* Since the number of components can be at most $\min\{m, n\}$ one simply has to iterate Theorem 2 for $k = 1, \ldots, \min\{m, n\}$. □

We close with two further observations concerning the number of possible solutions. In [2] it was shown that the number of discrete sets in the class $\mathcal{HV}_8^c$ having the same horizontal and vertical projections can be at most $\min\{m, n\}$, i.e. the ambiguity of the reconstruction is always bounded by a polynomial. This no longer holds in the class $\mathcal{HV}^c$.

**Theorem 3.** *For some vectors $H$ and $V$ there can be exponentially many discrete sets in the class $\mathcal{HV}^c$ having $H$ and $V$ as horizontal and vertical projections, respectively.*

*Proof.* It follows from the fact that the number of *hv*-convex polyominoes having the same horizontal and vertical projections can be exponentially large [7] and any such polyomino can serve as a component of a canonical *hv*-convex discrete set. ☐

The following result complements Theorem 3 assuming that there is no ambiguity at component-level. Let $\mathcal{HV}^{c+}$ denote the class of canonical *hv*-convex discrete sets whose components are uniquely determined and consider the following problem

$\#\textsc{Consistency}(\mathcal{HV}^{c+}, H, V)$
**Instance** Two non-negative vectors $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$.
**Question** How many discrete sets of $\mathcal{HV}^{c+}$ exists with the horizontal and vertical projections $H$ and $V$, respectively?

**Theorem 4.** $\#\textsc{Consistency}(\mathcal{HV}^{c+}, H, V) \in \mathbf{P}$.

*Proof.* Algorithm REC-HVC can be modified in a straightforward way to solve this problem. Instead of storing one of the *true* conjunction terms (if exists) for each $B_{i,j}$, in the array $L$ we store the number of such terms. ☐

## 4 Experimental Results

The worst case time complexity of Algorithm REC-HVC is quite big. In order to investigate the average running time of this algorithm, we generated *hv*-convex canonical discrete sets (having more than one components) from uniform random distributions by using an altered version of the method presented in [1]. In this way we obtained 100-100 canonical *hv*-convex discrete sets of size $10 \times 10$, $20 \times 20$, $30 \times 30$, $40 \times 40$, and $50 \times 50$. The algorithm were implemented in C++ and the test run on a PC with Intel(R) Core(TM)2 Duo CPU of 2x2.4 GHz and 2 GB RAM under SuSE Linux. We must also note that for technical reasons the code was not optimized. The running times for each data set are given in Table 1. From the entries of this table we can deduce that, fortunately, the average running time of the algorithm does not grow as rapidly as one can expect from the formula of the worst case time complexity.

**Table 1.** The average running time in seconds (rounded to two digits) of Algorithm REC-HVC as it depends on the size of the discrete set to be reconstructed

| Size | Avg. time (s) |
|---|---|
| $10 \times 10$ | 1.35 |
| $20 \times 20$ | 5.92 |
| $30 \times 30$ | 21.75 |
| $40 \times 40$ | 39.61 |
| $50 \times 50$ | 55.06 |

## 5 Conclusion

We introduced an intermediate subclass of *hv*-convex discrete sets that lies in between the class of *hv*-convex 8-connected sets and the class of general *hv*-convex discrete sets. While in the former class the reconstruction from two projections can be solved in polynomial time, in the latter one this problem is known to be NP-hard. We showed that *hv*-convex canonical discrete sets can also be reconstructed from two projections in polynomial time, and they form one of the broadest known classes of discrete sets having this property. Thus, our work narrows the gap between problems of the complexity classes **P** and **NP**, from the viewpoint of discrete tomography. Beside giving a theoretical result for the worst case time complexity of the presented problem, we also conducted experiments to investigate the average running time, too. The results can be adapted to *hv*-convex anticanonical discrete sets in a straightforward way but further generalization seems to be difficult. However, we think that with a more effective implementation of the algorithm and a deeper analysis of its behaviour we can gain some further important theoretical and experimental observations.

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