

Reconstruction of hv -convex Binary Matrices from Their Absorbed Projections

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Abstract

The reconstruction of hv -convex binary matrices from their absorbed projections is considered. Although this problem is NP-complete if the non-absorbed row and column sums are available, it is proved that such a reconstruction problem can be solved in polynomial time from absorbed projections when the absorption is represented by $\beta = (1+\sqrt{5})/2$. Also a reconstruction algorithm is given to determine the whole structure of hv -convex binary matrices from such projections.

1 Introduction

The reconstruction of binary matrices from their row and column sums is a basic problem in discrete tomography (DT). There are several theories, algorithms, and applications connected with this problem. As a collection of related papers see [1]. One of the most intensively studied classes of DT is the class of hv -convex binary matrices, in which there is no 0 between two 1s in their rows and columns (in other words, the rows and columns have consecutive-1 property). This problem was posed and a reconstruction algorithm was given by Kuba [2]. As it was proved later by Woeginger the complexity of this reconstruction problem is NP-complete [7].

Recently a new kind of discrete tomography problems have been introduced [3]. These new type of problems can be considered as the topics of the *emission discrete tomography*, shortly *EDT*, connected to a kind of emission model: The function to be reconstructed represents a radioactive object emitting activity into the surrounding space, where the space is filled with some

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absorbing material. Accordingly, the measurements in EDT are so-called *absorbed projections* depending on not the emitting object only but also on the absorption itself. It is known that the problem of uniqueness in EDT (in the case of certain absorption) is more complicated [3] than the same problem with non-absorbed projections.

In this paper we are going to show that there is at least one problem which is easier in the case of absorbed projections, where the absorption is represented with a special value of β . This is the problem of reconstructing *hv-convex* binary matrices from their absorbed row and column sums. We are going to show that this problem can be solved in polynomial time and a reconstruction algorithm is also given.

The organisation of this paper as follows. First the necessary definitions and notation are introduced. Section 3 contains the concept of β -representation which can be used very well in the case of absorbed row and column sums. It turns out that there is a very limited way to create binary rows and columns having the consecutive-1 property with given absorbed row and column sums. From this limitation it follows in Section 4 that many 0s and 1s of the binary matrix can be recognised simply from the row or column sums. Finally, in Section 5 we give an algorithm with polynomial time complexity, which is able to reconstruct all *hv-convex* binary matrices. The whole theory to be presented in this paper can be extended in higher dimensions as well.

2 Definitions and Notation

Let $A = (a_{ij})_{m \times n}$ be a (0,1)-matrix (in another words: binary matrix) with size $m \times n$, i.e., $a_{ij} \in \{0, 1\}$ for $i = 1, \dots, m, j = 1, \dots, n$. The pair (i, j) will be called *position*. The *row* and *column sum vectors* of A , $R(A) = (r_1, \dots, r_m)$ and $S(A) = (s_1, \dots, s_n)$, respectively, are defined as

$$r_i = \sum_{j=1}^n a_{ij}, \quad i = 1, \dots, m ,$$

$$s_j = \sum_{i=1}^m a_{ij}, \quad j = 1, \dots, n .$$

Then the reconstruction problem for binary matrices can be defined as follows.

Problem 2.1 RECONSTRUCTION M

Given: $m, n \in \mathbb{N}$ and $R \in \mathbb{N}_0^m, S \in \mathbb{N}_0^n$ (\mathbb{N}_0 denotes the set of nonnegative integers).

Task: Construct a binary matrix A with size $m \times n$ such that

$$R(A) = R \quad \text{and} \quad S(A) = S .$$

This problem was studied, for example, by Ryser [6], who gave also a reconstruction algorithm with time complexity $O(mn)$.

The reconstruction problem M is too general for many applications because of the high number of solutions. It is interesting to study similar reconstruction problems in different classes of binary matrices, where binary matrices with some special properties are to be reconstructed. Such a property can be the consecutive-1 property.

Definition 2.2 Let $a_1 \cdots a_k$ be a word of 0s and 1s, i.e., $a_i \in \{0, 1\}$ for $i = 1, \dots, k$. We say that $a_1 \cdots a_k$ has the *consecutive-1 property* if there is no 0 between two 1s in it. Accordingly, the consecutive-1 property can be defined for the words constructed from the rows and columns of binary matrices (it is called *horizontal* and *vertical convexity*, or shortly, *h-* and *v-convexity*). If all rows and columns of a binary matrix have this property then we say that the binary matrix is *hv-convex*.

Problem 2.3 RECONSTRUCTION hvM

Given: $m, n \in \mathbb{N}$ and $R \in \mathbb{N}_0^m, S \in \mathbb{N}_0^n$.

Task: Construct an *hv-convex* binary matrix A with size $m \times n$ such that

$$R(A) = R \quad \text{and} \quad S(A) = S .$$

This problem was posed and a reconstruction algorithm was given by Kuba [2]. As it turned out later the complexity of this reconstruction problem is NP-complete [7].

We are going to study a similar reconstruction problem in the case of absorbed projections (see [3]). The absorbed projections are defined here in a special case when the absorption is characterised by the constant

$$\beta = \frac{1 + \sqrt{5}}{2} .$$

It is easy to see that constant β has the following property.

$$(1) \quad \beta^{-1} = \beta^{-2} + \beta^{-3} .$$

Then the absorbed projections can be defined as follows.

Definition 2.4 Let A be a binary matrix with size $m \times n$. Its *absorbed row* and *column sum vectors*, $R_\beta(A) = (r_1, \dots, r_m)$ and $S_\beta(A) = (s_1, \dots, s_n)$, respectively, are defined as

$$(2) \quad \begin{aligned} r_i &= \sum_{j=1}^n a_{ij} \beta^{-j}, & i &= 1, \dots, m , \\ s_j &= \sum_{i=1}^m a_{ij} \beta^{-i}, & j &= 1, \dots, n . \end{aligned}$$

Then consider the following reconstruction problem for *hv-convex* binary matrices from their absorbed row and column sums.

Problem 2.5 RECONSTRUCTION *hvMA*

Given: $m, n \in \mathbb{N}$ and $R \in \mathbb{R}_0^m, S \in \mathbb{R}_0^n$ (\mathbb{R}_0 denotes the set of non-negative real numbers).

Task: Construct an *hv-convex* binary matrix A with size $m \times n$ such that

$$R_\beta(A) = R \quad \text{and} \quad S_\beta(A) = S .$$

3 β -representations

Let R and S be the absorbed row and column sums of the binary matrix $A = (a_{ij})_{m \times n}$. Then, using the terminology of numeration system [5], we can say on the base of (2) that $a_{i1} \cdots a_{im}$ is a (*finite*) representation in base β of r_i or it is a (*finite*) β -representation of r_i for $i = 1, \dots, m$. Similarly, $a_{1j} \cdots a_{mj}$ is a β -representation of s_j for $j = 1, \dots, n$. It is quite easy to see that in general the β -representation is not unique. As an example, consider the following two finite β -representations of the number $1/\beta$:

$$(3) \quad 100 = 011 ,$$

because $1 \cdot \beta^{-1} + 0 \cdot \beta^{-2} + 0 \cdot \beta^{-3} = 0 \cdot \beta^{-1} + 1 \cdot \beta^{-2} + 1 \cdot \beta^{-3}$ on the base of (1).

Even more, if there is one of the sub-words 011 and 100 in a β_0 -representation then it can be replaced by the other one without changing the value of the representation. This operation is called *1D elementary switching*. For example, consider the word 01000 having length 5. A 1D elementary switching can be done in the positions 2, 3, and 4 getting the word 00110 still representing the same number. The words 011 and 100 are called *0-type* and *1-type 1D elementary switching words*, respectively, also the *switching pair* expression can be used. In [4] it is proved that the β -representations of the same number can be got from each other by such switchings.

Generally, the following Lemma is true, see Section 2.1 in [3].

Lemma 3.1 *Let $a_1 \cdots a_k$ and $b_1 \cdots b_k$ be different, k -digit-length β_0 -representations of the same number. Then $b_1 \cdots b_k$ can be get from $a_1 \cdots a_k$ by a finite number of 1D switchings having the form*

$$(4) \quad 01x_21x_41 \cdots x_{2l}11 \longleftrightarrow 10x_20x_40 \cdots x_{2l}00 \quad (l \geq 0) ,$$

where x_2, x_4, \dots, x_{2l} denotes positions in the corresponding sub-words where the two representations have the same binary digit.

A simple consequence of this lemma is that if A and A' are different binary matrices with the same absorbed row and column sums then the elements where the matrices are different constitute subsequences $01x_21x_41 \cdots x_{2l}11$ and $10x_20x_40 \cdots x_{2l}00$ ($l \geq 0$) in the rows and columns of the matrices.

Let $r \in \mathbb{R}$ and take the greatest β -representation of r with respect to the lexicographic order, it is called the β -*expansion* and it is denoted by $\langle r \rangle$. Furthermore, let the class of k -digit-length β -representations with the

consecutive-1 property be denoted by $r_k^{(c)}$. For example, if $r = 1/\beta$ then $r_5^{(c)} = \{10000, 01100\}$ and $\langle r \rangle = 10000$.

Let C_k denote the set of non-negative real numbers having a k -digit-length β -representation with consecutive-1 property, formally

$$(5) \quad C_k = \{r \mid r_k^{(c)} \neq \emptyset\} .$$

Lemma 3.2 *For any real number $r \in C_k$ ($k \in \mathbb{N}$) there are at most two k -digit-length β -representations with the consecutive-1 property.*

Proof. Let $r \neq 0$. $r \in C_k$ if and only if r has the form

$$(6) \quad r = 00 \cdots 0011 \cdots 1100 \cdots 00 ,$$

where the sub-sequence of 1s starts in position j_1 and ends in position j_2 ($1 \leq j_1 \leq j_2 \leq k$). According to Lemma 3.1 if there is a different β -representation of r then it can be generated from (6) by switchings

$$(7) \quad 01x_21x_41 \cdots x_{2l}11 \longleftrightarrow 10x_20x_40 \cdots x_{2l}00 \quad (l \geq 0) .$$

It is easy to check that only the switching

$$(8) \quad 011 \longleftrightarrow 100$$

is possible between two β -representations in $r_k^{(c)}$ and this switching can be done if and only if

$$(9) \quad 1 \leq j_1 = j_2, \quad j_2 + 2 \leq k$$

or

$$(10) \quad 1 < j_1, \quad j_1 + 1 = j_2 \leq k$$

giving

$$(11) \quad r_k^{(c)} = \{00 \cdots 010000 \cdots 0, 00 \cdots 001100 \cdots 0\} .$$

In every other cases $r_k^{(c)}$ contains only one representation (6). □

Let $r \in C_k$. The positions of $r_k^{(c)}$ can be classified as variant and invariant positions as follows.

Definition 3.3 The position i ($1 \leq i \leq k$) is *variant* in the class $r_k^{(c)}$ if there are two β -representations in $r_k^{(c)}$ such that they have different (binary) digits in position i . The position i is *invariant 0* if all of the β -representations in $r_k^{(c)}$ has digit 0 in position i . Finally, the position i is *invariant 1* in the class $r_k^{(c)}$ if all of the β -representations in $r_k^{(c)}$ has digit 1 in position i .

For example, let $r = 1/\beta$ again and consider the class $r_5^{(c)} = \{10000, 01100\}$. Then the positions 1,2, and 3 are variant, and positions 4 and 5 are invariant 0 in $r_5^{(c)}$.

From the viewpoint of variant and invariant positions Lemma 3.2 has the following consequence.

Corollary 3.4 *Let $r \in C_k$ ($r \neq 0$ and $k \in \mathbb{N}$). There are at most three variant positions in the class $r_k^{(c)}$ as it can be seen from the following cases.*

Case 1. If there is exactly one 1 in $\langle r \rangle$, say in position j , and $j < k - 1$ then the positions j , $j + 1$, and $j + 2$ are variant, and every other position in $r_k^{(c)}$ are invariant 0.

Case 2. Otherwise $r_k^{(c)}$ has only one β -representation, and so all 0s in this representation indicate invariant-0 positions and all its 1s indicate invariant-1 positions in $r_k^{(c)}$.

4 Variant and Invariant Positions of hv -convex Binary Matrices

Let $R \in C_n^m$ and $S \in C_m^n$. Let $\mathcal{A}^{(hv)} = \mathcal{A}^{(hv)}(R, S)$ denote the class of hv -convex binary matrices having absorbed projections R and S .

Definition 4.1 The position (i, j) ($1 \leq i \leq m$, $1 \leq j \leq n$) is *variant* in the class $\mathcal{A}^{(hv)}$ if there are $A, A' \in \mathcal{A}^{(hv)}$ such that $a_{ij} \neq a'_{ij}$. The position (i, j) is *invariant 0* in the class $\mathcal{A}^{(hv)}$ if $a_{ij} = 0$ for all $A \in \mathcal{A}^{(hv)}$. Finally, the position (i, j) is *invariant 1* in the class $\mathcal{A}^{(hv)}$ if $a_{ij} = 1$ for all $A \in \mathcal{A}^{(hv)}$.

It is easy to see that if $\mathcal{A}^{(hv)}(R, S) \neq \emptyset$ then we have the following relation between the variant and invariant positions in $(r_i)_n^{(c)}/(s_j)_m^{(c)}$ and $\mathcal{A}^{(hv)}(R, S)$. If j is an invariant position in $(r_i)_n^{(c)}$ then (i, j) is the same type invariant position in $\mathcal{A}^{(hv)}(R, S)$. Similarly, if i is an invariant position in $(s_j)_m^{(c)}$ then (i, j) is the same type invariant position in $\mathcal{A}^{(hv)}(R, S)$.

As a consequence we get

Corollary 4.2 *There are at most three variant positions in each row and column in $\mathcal{A}^{(hv)}$.*

Definition 4.3 A binary matrix is *unique* among the hv -convex binary matrices with respect to its absorbed row and column sums if there is no other hv -convex binary matrix with the same absorbed row and column sums. Otherwise the matrix is called *non-unique*.

As the simplest examples of non-unique hv -convex binary matrices consider

$$(12) \quad E^{(0)} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad E^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} .$$

On the base of (3) it is easy to check that $E^{(0)}$ and $E^{(1)}$ have the same absorbed row and column sums, therefore they are non-unique hv -convex binary matrices. These matrices play important role also in the theory of (not nec-

essarily hv -convex) unique binary matrices (see [3]), $E^{(0)}$ and $E^{(1)}$ are called the 0 -type and 1 -type 2D elementary switching patterns, respectively.

The 2D elementary switching patterns play important role also in the class of hv -convex binary matrices as it can be seen from the following theorem. Let $E_{(i,j)}^{(0)}$ and $E_{(i,j)}^{(1)}$ denote the corresponding elementary switching patterns if they are 3×3 sub-matrices located in the position $\{i, i+1, i+2\} \times \{j, j+1, j+2\}$ for some $i \in \{1, \dots, m-2\}$ and $j \in \{1, \dots, n-2\}$.

Theorem 4.4 *A binary matrix is non-unique among the hv -convex binary matrices with respect to its absorbed row and column sums if and only if it contains an elementary switching pattern $E_{(i,j)}^{(0)}$ or $E_{(i,j)}^{(1)}$ for some $i \in \{1, \dots, m-2\}$ and $j \in \{1, \dots, n-2\}$ such that every other matrix element in rows $i, i+1, i+2$ and columns $j, j+1, j+2$ are 0.*

Proof. One direction is trivial: If there is an elementary switching pattern as a sub-matrix of the hv -convex binary matrix A then by changing it to the other type of 2D elementary switching pattern we get a new A' hv -convex binary matrix with the same absorbed row and column sums, i.e., A is non-unique.

In order to prove the other direction let us suppose that there are two hv -convex binary matrices, A and $A' (\neq A)$, with the same absorbed row and column sums. Let i be the first row where A is different from A' ($1 \leq i \leq m$) and let j be the first column ($1 \leq j \leq n$) in this row where A is different from A' , that is $a_{ij} \neq a'_{ij}$. Without the lack of generality we can suppose that $a_{ij} = 0$ and $a'_{ij} = 1$. Then according to Lemma 3.1 a_{ij} and a'_{ij} are the first elements of the “difference” subsequences $01x_21x_41 \cdots x_{2k}11$ and $10x_20x_40 \cdots x_{2k}00$ ($k \geq 0$) in row i and column j (x_2, x_4, \dots, x_{2k} denotes the positions where both subsequences has the same elements). Because of hv -convexity, $k = 0$ in this case for any such subsequence. That is, $a_{ij}a_{i,j+1}a_{i,j+2} = 011$ and $a'_{ij}a'_{i,j+1}a'_{i,j+2} = 100$. Applying the same idea to the columns, we get that $a_{i+1,j}a_{i+1,j+1}a_{i+1,j+2} = 100$ and $a'_{i+1,j}a'_{i+1,j+1}a'_{i+1,j+2} = 011$, and $a_{i+2,j}a_{i+2,j+1}a_{i+2,j+2} = 100$ and $a'_{i+2,j}a'_{i+2,j+1}a'_{i+2,j+2} = 011$. That is, there is a 0-type/1-type 2D elementary switching pattern in A/A' , respectively, in the positions $\{i, i+1, i+2\} \times \{j, j+1, j+2\}$. Because of hv -convexity, there is no other 1 in these rows and columns of A and A' . \square

5 An Algorithm to Determine Variant and Invariant Positions

Instead of reconstructing an hv -convex binary matrix A from their absorbed row and column sums R and S directly, we determine the variant and invariant positions of the class $\mathcal{A}^{(hv)}(R, S)$, called the *structure* of $\mathcal{A}^{(hv)}(R, S)$. As we know from Theorem 4.4, the knowledge of the variant and invariant positions is equivalent to the knowledge of the positions of the (eventual) 2D elementary switching patterns in any element of $\mathcal{A}^{(hv)}(R, S)$.

This algorithm starts to fill a matrix X with the initial values *free*, indicating that the variability of none of the positions is decided yet. Then, on the base of Corollary 3.4, we write 1s and 0s in the rows and columns of X indicating the invariant 0s and 1s, respectively. At most 3 free positions remain in each row and column after this step. The remaining free positions that are in a 3×3 free sub-matrix are variant positions of the class, the others can be determined from the 0s and 1s in their 3×3 neighbourhood. Formally, the algorithm is as follows.

Algorithm 1 *for determining the variant and invariant positions of the class of hv-convex binary matrices from absorbed row and column sums*

Input: $m, n \in \mathbb{N}$, $R \in C_n^m$, $S \in C_m^n$.

Output: A matrix $X_{m \times n}$ indicating the variant and invariant positions or the algorithm terminates with contradiction.

Step 1: $X := (\text{free})_{m \times n}$

Step 2: If (i, j) is an invariant position of $(r_i)_n^{(c)}$ then let $x_{ij} = 0/1$ accordingly for $i = 1, \dots, m$ (see Corollary 3.4).

Step 3: If (i, j) is an invariant position of $(s_j)_m^{(c)}$ then let $x_{ij} = 0/1$ accordingly for $j = 1, \dots, n$ (see Corollary 3.4). If a position gets different values in Steps 2 and 3 then it is a contradiction and the algorithm terminates without giving any indication of variant/invariant positions.

Step 4: For each free position (i, j) if it is not in a free 3×3 sub-matrix then set (i, j) to 0 or 1 depending on its 3×3 neighbourhood.

As an example of using Algorithm 1 see Fig. 1. Consider the following reconstruction problem: $R = (r_1, \dots, r_9)$ and $S = (s_1, \dots, s_{10})$ where $\langle r_1 \rangle = \langle r_9 \rangle = 0000001000$, $\langle r_2 \rangle = 0000000100$, $\langle r_3 \rangle = \langle r_4 \rangle = \langle r_5 \rangle = 1000000000$, $\langle r_6 \rangle = \langle r_7 \rangle = \langle r_8 \rangle = 0001000000$, $\langle s_1 \rangle = \langle s_2 \rangle = \langle s_3 \rangle = 0010000000$, $\langle s_4 \rangle = \langle s_5 \rangle = \langle s_6 \rangle = 0000010000$, $\langle s_7 \rangle = 0000000001$, $\langle s_8 \rangle = 1100000000$, $\langle s_9 \rangle = 1000000000$, $\langle s_{10} \rangle = 0000000000$. In Fig. 1 the Steps of Algorithm 1 can be followed. The solutions of this reconstruction problem are in Fig. 2.

Theorem 5.1 *Algorithm 1 determines the variant and invariant positions of any class $\mathcal{A}^{(hv)}(R, S) \neq \emptyset$ in $O(mn)$ steps.*

It is easy to see that if $\mathcal{A}^{(hv)}(R, S) \neq \emptyset$ then any element of this class can be generated from the output of Algorithm 1 by replacing the 3×3 free sub-matrices with a suitable 2D elementary switching pattern ($E^{(0)}$ or $E^{(1)}$).

As a final remark we can say that the same method can be used to prove corresponding theorems and algorithms for reconstructing binary matrices in n -dimension from their n ($n \geq 2$) orthogonal absorbed projections when the absorption is characterised by the constant β .

0 0 0 0 0 0 . . . 0	0 0 0 0 0 0 0 1 . 0
0 0 0 0 0 0 0 . . .	0 0 0 0 0 0 0 1 . 0
. . . 0 0 0 0 0 0 0	. . . 0 0 0 0 0 0 0
. . . 0 0 0 0 0 0 0	. . . 0 0 0 0 0 0 0
. . . 0 0 0 0 0 0 0	. . . 0 0 0 0 0 0 0
0 0 0 . . . 0 0 0 0	0 0 0 . . . 0 0 0 0
0 0 0 . . . 0 0 0 0	0 0 0 . . . 0 0 0 0
0 0 0 . . . 0 0 0 0	0 0 0 . . . 0 0 0 0
0 0 0 0 0 0 . . . 0	0 0 0 0 0 0 1 0 0 0
a)	b)

0 0 0 0 0 0 0 1 1 0
 0 0 0 0 0 0 0 1 0 0
 . . . 0 0 0 0 0 0 0
 . . . 0 0 0 0 0 0 0
 . . . 0 0 0 0 0 0 0
 0 0 0 . . . 0 0 0 0
 0 0 0 . . . 0 0 0 0
 0 0 0 . . . 0 0 0 0
 0 0 0 0 0 0 1 0 0 0
 c)

Fig. 1. Determination of variant and invariant positions by Algorithm 1. a) Result of Step 2. b) Result of Step 3. c) Result of Step 4. (Positions indicated by “.” are free.)

References

- [1] Herman, G. T., and A. Kuba, “Discrete Tomography. Foundations, Algorithms, and Applications,” Birkhäuser, Boston, 1999.
- [2] Kuba, A., *Reconstruction of two-directionally connected binary patterns from their two orthogonal projections*, Comp. Vision Graph. and Image Proc. **27** (1984), 249–265.
- [3] Kuba, A., and M. Nivat, *Reconstruction of discrete sets with absorption*, Lecture Notes in Computer Sciences, vol. **1953**, Springer-Verlag, Berlin (2000), pp. 137–

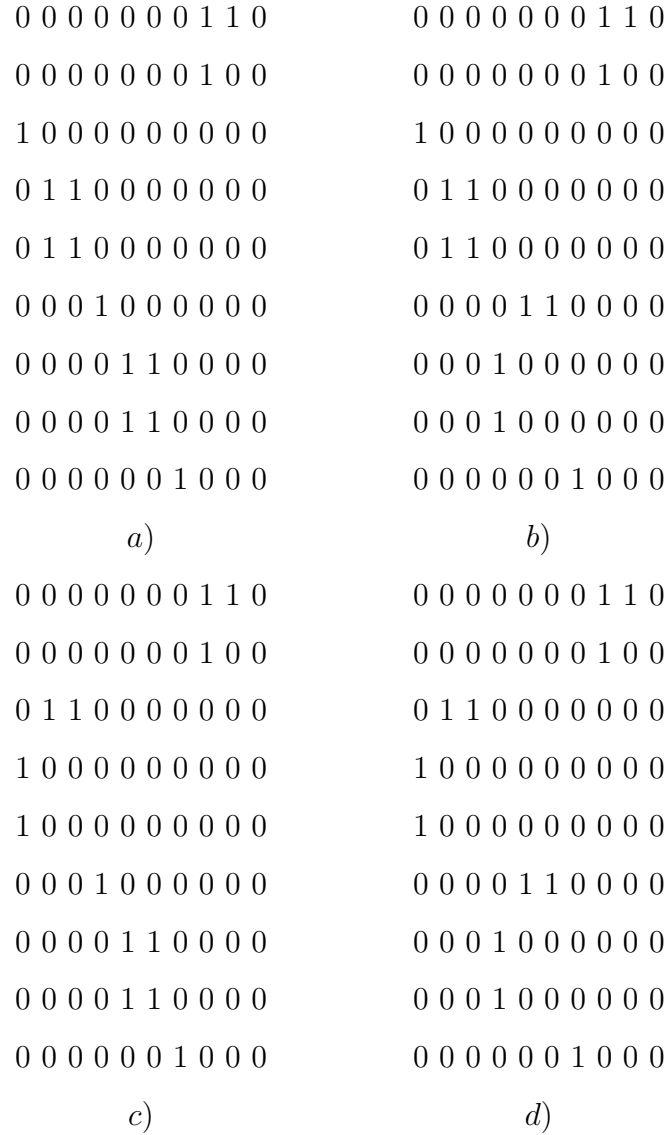


Fig. 2. The four solutions of the given reconstruction problem.

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- [4] Kuba, A., and M. Nivat, *A sufficient condition for non-uniqueness in binary tomography with absorption*, submitted for publication (2001).
- [5] Lothaire, M., “Combinatorics on Words,” Cambridge University Press, Cambridge, 1997.
- [6] Ryser, H. J., *Combinatorial properties of matrices of zeros and ones*, *Canad. J. Math.* **9** (1957), 371–377.
- [7] Woeginger, G. W., *The reconstruction of polyominoes from their orthogonal projections*, *Inform. Proc. Letters* **77** (2001), 225–229.