Available at www.ElsevierMathematics.com
POWERED BY SCIENCE @ DIRECT®
Discrete Applied Mathematics 139 (2004) 137 – 148
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Reconstruction of *hv*-convex binary matrices from their absorbed projections☆

Attila Kuba, Antal Nagy, Emese Balogh
Department of Applied Informatics, University of Szeged, Árpád tér 2, Szeged H-6720, Hungary

Received 26 October 2001; received in revised form 22 August 2002; accepted 23 November 2002

Abstract

The reconstruction of *hv*-convex binary matrices from their absorbed projections is considered. Although this problem is NP-hard if the non-absorbed row and column sums are available, it is proved that such a reconstruction problem can be solved in polynomial time from absorbed projections when the absorption is represented by $\beta=(1+\sqrt{5})/2$. Also a reconstruction algorithm is given to determine the whole structure of *hv*-convex binary matrices from such projections.

Keywords: Discrete tomography; Absorbed projections; *hv*-convex binary matrices; Numeration systems

1. Introduction

The reconstruction of binary matrices from their row and column sums is a basic problem in discrete tomography (DT). There are several theories, algorithms, and applications connected with this problem. As a collection of related papers see [1]. One of the most intensively studied classes of DT is the class of *hv*-convex binary matrices, in which there is no 0 between two 1's in their rows and columns (in other words, the rows and columns have consecutive-1 property). This problem was posed and a reconstruction algorithm was given by Kuba [2]. As it was proved later by Woeginger [7] the complexity of this reconstruction problem is NP-hard.

Recently, a new kind of discrete tomography problems has been introduced [3]. These types of problems can be considered as the topics of the *emission discrete*

☆ Supported by the Grants FKFP 0908/1997 and OTKA T032241, Hungary.
E-mail address: kuba@inf.u-szeged.hu (A. Kuba).

0166-218X/$ - see front matter
doi:10.1016/j.dam.2002.11.001

tomography, shortly *EDT*, connected to a kind of emission model. In this model, the whole space is filled with some homogeneous absorbing material and the function to be reconstructed represents an object emitting radioactive rays into the surrounding space. Quantitatively, the detected activity emitted from a point of the object can be described as

$$I = I_0 e^{-\mu x}, \tag{1}$$

where I_0 denotes the initial activity in the point of the object, I is the detected activity, $\mu \geqslant 0$ denotes the absorption coefficient of the homogeneous material and x is the length of the path between the point and the detector.

Accordingly, the measurements in EDT are so-called *absorbed projections*. They depend on both the emitting object and the absorption. It is known that the problem of uniqueness in EDT (in the case of certain absorption) is more complicated [3] than the same problem with non-absorbed projections.

In this paper, we are going to show that there is at least one problem which is easier in the case of absorbed projections, where the absorption is represented with a special absorption value. This is the problem of reconstructing *hv*-convex binary matrices from their absorbed row and column sums. We are going to show that this problem can be solved in polynomial time and a reconstruction algorithm is also given.

The organization of this paper is the following. First, the necessary definitions and notation are introduced. Section **??** contains the concept of β-representation which is an important tool when dealing with the case of absorbed row and column sums. It turns out that there is a very limited way to create binary rows and columns having the consecutive-1 property with given absorbed row and column sums. From this limitation it follows in Section 4 that many 0's and 1's of the binary matrix can be recognised simply from the row or column sums. Finally, in Section 5 we give an algorithm with polynomial time complexity, which is able to reconstruct all *hv*-convex binary matrices. The whole theory to be presented in this paper can be extended in higher dimensions as well.

2. Definitions and notation

Let $A = (a_{ij})_{m \times n}$ be a (0,1)-matrix (in other words: binary matrix) with size $m \times n$, i.e., $a_{ij} \in \{0, 1\}$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$. The pair (i, j) will be called *position*. The *row* and *column sum vectors* of A, $R(A) = (r_1, \ldots, r_m)$ and $S(A) = (s_1, \ldots, s_n)$, respectively, are defined as

$$r_i = \sum_{j=1}^{n} a_{ij}, \quad i = 1, \ldots, m,$$

$$s_j = \sum_{i=1}^{m} a_{ij}, \quad j = 1, \ldots, n.$$

Then the reconstruction problem for binary matrices can be defined in the following.

Reconstruction M.
Given: $m, n \in \mathbb{N}$ *and* $R \in \mathbb{N}_0^m$, $S \in \mathbb{N}_0^n$ ($\mathbb{N}_0$ *denotes the set of non-negative integers*).
Task: *Construct a binary matrix A with size $m \times n$ such that*

$$R(A) = R \quad \text{and} \quad S(A) = S.$$

This problem was studied, for example, by Ryser [6], who gave also a reconstruction algorithm with time complexity $O(mn)$.

The reconstruction problem M is too general for many applications because of the high number of solutions. It is interesting to study similar reconstruction problems in different classes of binary matrices, where binary matrices with some special properties are to be reconstructed. Such a property can be the consecutive-1 property.

Definition 1. Let $a_1 \cdots a_k$ be a word of 0's and 1's, i.e., $a_i \in \{0, 1\}$ for $i = 1, \ldots, k$. We say that the word $a_1 \cdots a_k$ has the consecutive-1 property if there is no 0 between two 1s in it. Accordingly, the consecutive-1 property can be defined for the words constructed from the rows and columns of binary matrices (it is called horizontal and vertical convexity, or shortly, *h*- and *v*-convexity, respectively). If all rows and columns of a binary matrix have this property then we say that the binary matrix is *hv*-convex.

Reconstruction hvM.
Given: $m, n \in \mathbb{N}$ *and* $R \in \mathbb{N}_0^m$, $S \in \mathbb{N}_0^n$.
Task: *Construct an hv-convex binary matrix A with size $m \times n$ such that*

$$R(A) = R \quad \text{and} \quad S(A) = S.$$

This problem was posed and a reconstruction algorithm was given by Kuba [2]. As it turned out later the complexity of this reconstruction problem is NP-hard [7].

We are going to study a similar reconstruction problem in the case of absorbed projections (see [3]). The absorbed projections are defined here in a special case when the absorption is characterized by the constant $\beta = e^{\mu}$, where

$$\beta = \frac{1 + \sqrt{5}}{2},$$

which is the golden ratio. It is easy to see that constant β has the following property:

$$\beta^{-1} = \beta^{-2} + \beta^{-3}. \tag{2}$$

Then the absorbed projections can be defined in the following.

Definition 2. Let A be a binary matrix with size $m \times n$. Its absorbed row and column sum vectors, $R_\beta(A) = (r_1, \ldots, r_m)$ and $S_\beta(A) = (s_1, \ldots, s_n)$, respectively, are defined as

$$r_i = \sum_{j=1}^{n} a_{ij} \beta^{-j}, \quad i = 1, \ldots, m,$$

$$s_j = \sum_{i=1}^{m} a_{ij} \beta^{-i}, \quad j = 1, \ldots, n. \tag{3}$$

Then consider the following reconstruction problem for *hv*-convex binary matrices from their absorbed row and column sums.

Reconstruction *hvMA*.
Given: $m, n \in \mathbb{N}$ *and* $R \in \mathbb{R}_0^m$, $S \in \mathbb{R}_0^n$ ($\mathbb{R}_0$ *denotes the set of non-negative real numbers*).
Task: *Construct an hv-convex binary matrix A with size* $m \times n$ *such that*

$$R_\beta(A) = R \quad \text{and} \quad S_\beta(A) = S.$$

3. β-Representations

Let R and S be the absorbed row and column sums of the binary matrix $A=(a_{ij})_{m\times n}$. Then, using the terminology of numeration system [5], we can say on the base of (3) that the word $a_{i1}\cdots a_{in}$ is a (*finite*) *representation in base* β *of* r_i or it is a (*finite*) β-*representation* of r_i for $i=1,\ldots,m$. Similarly, $a_{1j}\cdots a_{mj}$ is a β-representation of s_j for $j=1,\ldots,n$. It is quite easy to see that in general the β-representation is not unique. As an example, consider the following two finite β-representations of the number $1/\beta$:

$$100 = 011, \tag{4}$$

because $1\cdot\beta^{-1}+0\cdot\beta^{-2}+0\cdot\beta^{-3} = 0\cdot\beta^{-1}+1\cdot\beta^{-2}+1\cdot\beta^{-3}$ on the base of (2).

Even more, if there is one of the sub-words 011 and 100 in a β-representation then it can be replaced by the other one without changing the value of the representation. This operation is called 1*D elementary switching*. For example, consider the word 01000 having length 5. A 1D elementary switching can be done in the positions 2, 3, and 4 getting the word 00110 still representing the same number. The words 011 and 100 are called 0-*type* and 1-*type* 1*D elementary switching words*, respectively, also the *switching pair* expression can be used. In [4] it is proved that the β-representations of the same number can be obtained from each other by such switchings.

Generally, the following Lemma is true, see [3, Section 2.1].

Lemma 3. *Let* $a_1\cdots a_k$ *and* $b_1\cdots b_k$ *be different*, *k-digit-length* β*-representations of the same number. Then* $b_1\cdots b_k$ *can be obtained from* $a_1\cdots a_k$ *by a finite number of* 1*D switchings having the form*

$$011 \leftrightarrow 100 \tag{5}$$

or

$$01x_2 1x_4 1\cdots x_{2l}11 \leftrightarrow 10x_2 0x_4 0\cdots x_{2l}00 \quad (l \geqslant 1), \tag{6}$$

where $x_2, x_4, \ldots, x_{2l}$ *denotes positions in the corresponding sub-words where the two representations have the same binary digit.*

A simple consequence of this lemma is that if A and A' are different binary matrices with the same absorbed row and column sums then the elements where the matrices are different constitute subsequences $01x_2 1x_4 1\cdots x_{2l}11$ and $10x_2 0x_4 0\cdots x_{2l}00$ ($l \geqslant 0$) in the rows and columns of the matrices.

Let $r \in \mathbb{R}$, and take the greatest β-representation of r with respect to the lexicographic order, it is called the *β-expansion* of r and it is denoted by $\langle r \rangle$. Furthermore, let the class of k-digit-length β-representations with the consecutive-1 property be denoted by $r_k^{(c)}$. For example, if $r = 1/\beta$ then $r_5^{(c)} = \{10000, 01100\}$ and $\langle r \rangle = 10000$. It is easy to see that $\langle r \rangle$ (and $r_k^{(c)}$) can be determined from r in polynomial time. Let r be a real number having k-digit-length β-representation, then its β-expansion $a_1 \cdots a_k$ can be determined by the algorithm

$$r_0 := r,$$

$$a_i := \lfloor \beta \cdot r_{i-1} \rfloor, \quad r_i := \{\beta \cdot r_{i-1}\}, \quad i = 1, \ldots, k,$$

where $\lfloor . \rfloor$ and $\{.\}$ denote the integer and fractional, respectively, part of the argument.

Let C_k denote the set of non-negative real numbers having a k-digit-length β-representation with consecutive-1 property, formally

$$C_k = \{r \mid r_k^{(c)} \neq \emptyset\}. \tag{7}$$

Lemma 4. *For any real number $r \in C_k$ $(k \in \mathbb{N})$ there are at most two k-digit-length β-representations with the consecutive-1 property.*

Proof. Let $r \neq 0$. $r \in C_k$ if and only if r has the form

$$r = 00 \cdots 0011 \cdots 1100 \cdots 00, \tag{8}$$

where the sub-sequence of 1's starts in position j_1 and ends in position j_2 $(1 \leqslant j_1 \leqslant j_2 \leqslant k)$. According to Lemma 3, if there is a different β-representation of r then it can be generated from (8) by switchings

$$01x_2 1x_4 1 \cdots x_{2l} 11 \leftrightarrow 10x_2 0x_4 0 \cdots x_{2l} 00 \quad (l \geqslant 0). \tag{9}$$

It is easy to check that only the switching

$$011 \leftrightarrow 100$$

is possible between two β-representations in $r_k^{(c)}$ and this switching can be done if and only if

$$1 \leqslant j_1 = j_2, \quad j_2 + 2 \leqslant k \tag{10}$$

or

$$1 < j_1, \quad j_1 + 1 = j_2 \leqslant k \tag{11}$$

giving

$$r_k^{(c)} = \{00 \cdots 010000 \cdots 0, 00 \cdots 001100 \cdots 0\}. \tag{12}$$

In all other cases $r_k^{(c)}$ contains only one representation with the form of (8). □

Let $r \in C_k$. The positions of $r_k^{(c)}$ can be classified as variant and invariant positions in the following.

Definition 5. The position i $(1 \leqslant i \leqslant k)$ is variant in the class $r_k^{(c)}$ if there are two β-representations in $r_k^{(c)}$ such that they have different (binary) digits in position i. The position i is invariant 0 if all of the β-representations in $r_k^{(c)}$ has digit 0 in position i. Finally, the position i is invariant 1 in the class $r_k^{(c)}$ if all of the β-representations in $r_k^{(c)}$ has digit 1 in position i.

For example, let $r = 1/\beta$ again and consider the class $r_5^{(c)} = \{10000, 01100\}$. Then the positions 1, 2, and 3 are variant, and positions 4 and 5 are invariant 0 in $r_5^{(c)}$.

From the viewpoint of variant and invariant positions Lemma 4 has the following consequence.

Corollary 6. *Let* $r \in C_k$ $(r \neq 0$ *and* $k \in \mathbb{N})$. *There are at most three variant positions in the class* $r_k^{(c)}$ *as it can be seen from the following cases.*

Case 1: *If there is exactly one* 1 *in* $\langle r \rangle$, *say in position* j, *and* $j < k-1$ *then the positions* j, $j+1$, *and* $j+2$ *are variant*, *and every other position in* $r_k^{(c)}$ *are invariant* 0.

Case 2: *Otherwise* $r_k^{(c)}$ *has only one* β*-representation*, *and so all* 0*s in this representation indicate invariant*-0 *positions and all its* 1*'s indicate invariant*-1 *positions in* $r_k^{(c)}$.

The variant and invariant positions in $r_k^{(c)}$ *can be determined in polynomial time.*

4. Variant and invariant positions of *hv*-convex binary matrices

Let $R \in C_n^m$ and $S \in C_m^n$, i.e., the components of vectors R and S are from the sets of non-negative real numbers having an n-digit-length and an m-digit-length, respectively, β-representation with consecutive-1 property. Let $\mathscr{A}^{(hv)} = \mathscr{A}^{(hv)}(R,S)$ denote the class of *hv*-convex binary matrices having absorbed projections R and S.

Definition 7. The position (i,j) $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$ is variant in the class $\mathscr{A}^{(hv)}$ if there are $A, A' \in \mathscr{A}^{(hv)}$ such that $a_{ij} \neq a'_{ij}$. The position (i,j) is invariant 0 in the class $\mathscr{A}^{(hv)}$ if $a_{ij} = 0$ for all $A \in \mathscr{A}^{(hv)}$. Finally, the position (i,j) is invariant 1 in the class $\mathscr{A}^{(hv)}$ if $a_{ij} = 1$ for all $A \in \mathscr{A}^{(hv)}$.

It is easy to see that if $\mathscr{A}^{(hv)}(R,S) \neq \emptyset$ then we have the following relation between the variant and invariant positions in $(r_i)_n^{(c)}$, $(s_j)_m^{(c)}$, and $\mathscr{A}^{(hv)}(R,S)$. If j is an invariant position in $(r_i)_n^{(c)}$ then (i,j) is the same type invariant position in $\mathscr{A}^{(hv)}(R,S)$. Similarly, if i is an invariant position in $(s_j)_m^{(c)}$ then (i,j) is the same type invariant position in $\mathscr{A}^{(hv)}(R,S)$.

As a consequence we get

Corollary 8. *There are at most three variant positions in each row and column in* $\mathscr{A}^{(hv)}$.

Definition 9. A binary matrix is unique among the *hv*-convex binary matrices with respect to its absorbed row and column sums if there is no other *hv*-convex binary matrix with the same absorbed row and column sums. Otherwise the matrix is called non-unique.

As the simplest examples of non-unique *hv*-convex binary matrices consider

$$E^{(0)} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad E^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

On the base of (4) it is easy to check that $E^{(0)}$ and $E^{(1)}$ have the same absorbed row and column sums, therefore they are non-unique *hv*-convex binary matrices. These matrices play important role also in the theory of (not necessarily *hv*-convex) unique binary matrices (see [3]), $E^{(0)}$ and $E^{(1)}$ are called the 0-*type* and 1-*type* 2*D elementary switching patterns*, respectively.

The 2D elementary switching patterns play important role also in the class of *hv*-convex binary matrices as it can be seen from the following theorem. Let $E^{(0)}_{(i,j)}$ and $E^{(1)}_{(i,j)}$ denote the corresponding elementary switching patterns if they are 3×3 sub-matrices located in the position $\{i, i+1, i+2\} \times \{j, j+1, j+2\}$ for some $i \in \{1, \dots, m-2\}$ and $j \in \{1, \dots, n-2\}$.

Theorem 10. *A binary matrix is non-unique among the hv-convex binary matrices with respect to its absorbed row and column sums if and only if it contains an elementary switching pattern* $E^{(0)}_{(i,j)}$ *or* $E^{(1)}_{(i,j)}$ *for some* $i \in \{1, \dots, m-2\}$ *and* $j \in \{1, \dots, n-2\}$ *such that every other matrix element in rows* $i, i+1, i+2$ *and columns* $j, j+1, j+2$ *is* 0.

Proof. One direction is trivial: Let us suppose that the *hv*-convex binary matrix A has a 3×3 elementary switching pattern, say $E^{(0)}_{(i,j)}$, as its sub-matrix. Then by changing the sub-matrix to the other type of 2D elementary switching pattern, i.e. to $E^{(1)}_{(i,j)}$, we get a new A' *hv*-convex binary matrix with the same absorbed row and column sums as A. That is, A is non-unique.

In order to prove the other direction let us suppose that there are two *hv*-convex binary matrices, A and $A'(\neq A)$, with the same absorbed row and column sums. Let i be the first row where A is different from A' $(1 \leqslant i \leqslant m)$ and let j be the first column $(1 \leqslant j \leqslant n)$ in this row where A is different from A', that is $a_{ij} \neq a'_{ij}$. Without the lack of generality we can suppose that $a_{ij} = 0$ and $a'_{ij} = 1$. Then according to Lemma 3 a_{ij} and a'_{ij} are the first elements of the "difference" subsequences $01x_2 1x_4 1 \cdots x_{2k} 11$ and $10x_2 0x_4 0 \cdots x_{2k} 00$ $(k \geqslant 0)$ in row i and column j ($x_2, x_4, \dots x_{2k}$ denotes the positions where both subsequences has the same elements). Because of *hv*-convexity, $k = 0$ in this case for any such subsequence. That is, $a_{ij}a_{i,j+1}a_{i,j+2} = 011$ and $a'_{ij}a'_{i,j+1}a'_{i,j+2} = 100$. Applying the same idea to the columns, we get that $a_{i+1,j}a_{i+1,j+1}a_{i+1,j+2} = 100$ and

$a'_{i+1,j}a'_{i+1,j+1}a'_{i+1,j+2}=011$, and $a_{i+2,j}a_{i+2,j+1}a_{i+2,j+2}=100$ and $a'_{i+2,j}a'_{i+2,j+1}a'_{i+2,j+2}=011$. That is, there is a 0-type/1-type 2D elementary switching pattern in A/A', respectively, in the positions $\{i,i+1,i+2\}\times\{j,j+1,j+2\}$. Because of *hv*-convexity, there is no other 1 in these rows and columns of A and A'. □

Now, we have all of the tools necessary for the reconstruction of *hv*-convex binary matrices from their absorbed row and column sums. From the β-representation of the row and column sums, the invariant 0 and 1 positions can be determined in each row and column. The rest, i.e., at most three consecutive positions in each row and column, can be variant position in the matrix. Finally, Theorem 10 gives the information about the variant positions of the matrix. We can start to describe the reconstruction algorithm.

5. An algorithm to determine variant and invariant positions

Instead of reconstructing an *hv*-convex binary matrix A from their absorbed row and column sums R and S directly, we determine the variant and invariant positions of the class $\mathscr{A}^{(hv)}(R,S)$, called the *structure* of $\mathscr{A}^{(hv)}(R,S)$. As we know from Theorem 10, the knowledge of the variant and invariant positions is equivalent to the knowledge of the positions of the (eventual) 2D elementary switching patterns in any element of $\mathscr{A}^{(hv)}(R,S)$.

Algorithm 1 starts to fill a matrix X with the initial values $free$ (Step 1), indicating that the variability of none of the positions is decided yet. Then, on the base of Corollary 6, we write 0s and 1s in the rows and columns of X indicating the invariant 0 and 1 positions, respectively (Steps 2 and 3). At most 3 free positions remain in each row and column after Step 3. The remaining free positions that are in a 3×3 free sub-matrix are variant positions of the class, the others can be determined from the 0's and 1's in their 3×3 neighbourhood. Formally, the algorithm is the following.

Algorithm 1. *For determining the variant and invariant positions of the class of hv-convex binary matrices from absorbed row and column sums*

Input: $m,n\in\mathbb{N}$, $R\in C_n^m$, $S\in C_m^n$.

Output: *A matrix* $X_{m\times n}$ *indicating the variant and invariant positions or the algorithm terminates with contradiction.*

Step 1: $X:=(free)_{m\times n}$

Step 2: *For* $i=1,\ldots,m$ *if* (i,j) *is an invariant position of* $(r_i)_n^{(c)}$ *then let* $x_{ij}=0/1$ *accordingly* (*see Corollary* 6).

Step 3: *For* $j=1,\ldots,n$ *if* (i,j) *is an invariant position of* $(s_j)_m^{(c)}$ *then let* $x_{ij}=0/1$ *accordingly* (*see Corollary* 6). *If a position gets different values in Steps* 2 *and* 3 *then it is a contradiction and the algorithm terminates without giving any indication of variant/invariant positions.*

Step 4: *For each free position* (i,j) *if it is not in a free* 3×3 *sub-matrix then set* (i,j) *to* 0 *or* 1 *depending on its* 3×3 *neighbourhood.*

0 0 0 0 0 0 . . . 0

0 0 0 0 0 0 0 . . .

. . . 0 0 0 0 0 0 0

. . . 0 0 0 0 0 0 0

. . . 0 0 0 0 0 0 0

0 0 0 . . . 0 0 0 0

0 0 0 . . . 0 0 0 0

0 0 0 . . . 0 0 0 0

0 0 0 0 0 0 . . . 0

(a)

0 0 0 0 0 0 0 1 . 0

0 0 0 0 0 0 0 1 . 0

. . . 0 0 0 0 0 0 0

. . . 0 0 0 0 0 0 0

. . . 0 0 0 0 0 0 0

0 0 0 . . . 0 0 0 0

0 0 0 . . . 0 0 0 0

0 0 0 . . . 0 0 0 0

0 0 0 0 0 0 1 0 0 0

(b)

0 0 0 0 0 0 0 1 1 0

0 0 0 0 0 0 0 1 0 0

. . . 0 0 0 0 0 0 0

. . . 0 0 0 0 0 0 0

. . . 0 0 0 0 0 0 0

0 0 0 . . . 0 0 0 0

0 0 0 . . . 0 0 0 0

0 0 0 . . . 0 0 0 0

0 0 0 0 0 0 1 0 0 0

(c)

Fig. 1. Determination of variant and invariant positions by Algorithm 1: (a) Result of Step 2; (b) result of Step 3; (c) result of Step 4. (Positions indicated by "." are free.)

As an example of using Algorithm 1 see Fig. 1. Consider the following reconstruction problem: $R=(r_1,\ldots,r_9)$ and $S=(s_1,\ldots,s_{10})$ where $\langle r_1\rangle=\langle r_9\rangle=0000001000$, $\langle r_2\rangle=0000000100$, $\langle r_3\rangle=\langle r_4\rangle=\langle r_5\rangle=1000000000$, $\langle r_6\rangle=\langle r_7\rangle=\langle r_8\rangle=0001000000$, $\langle s_1\rangle=\langle s_2\rangle=\langle s_3\rangle=001000000$, $\langle s_4\rangle=\langle s_5\rangle=\langle s_6\rangle=000001000$, $\langle s_7\rangle=000000001$, $\langle s_8\rangle=110000000$, $\langle s_9\rangle=100000000$, $\langle s_{10}\rangle=000000000$. In Fig. 1 the Steps of Algorithm 1 can be followed. The solutions of this reconstruction problem are in Fig. 2.

Theorem 11. *Algorithm* 1 *determines the variant and invariant positions of any class* $\mathscr{A}^{(hv)}(R,S)\neq\emptyset$ *in* $\mathrm{O}(mn)$ *steps.*

Theorem 11 follows from the fact that each step of Algorithm 1 is performed in $\mathrm{O}(mn)$ steps.

(a)

$$\begin{array}{cccccccccc} 0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0 \end{array}$$

(b)

$$\begin{array}{cccccccccc} 0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0 \end{array}$$

(c)

$$\begin{array}{cccccccccc} 0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&1&1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0 \end{array}$$

(d)

$$\begin{array}{cccccccccc} 0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&1&1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0 \end{array}$$

Fig. 2. The four solutions of the given reconstruction problem.

It is easy to see that if $\mathscr{A}^{(hv)}(R,S) \neq \emptyset$ then any element of this class can be generated from the output of Algorithm 1 by replacing the 3×3 free sub-matrices with a suitable 2D elementary switching pattern ($E^{(0)}$ or $E^{(1)}$).

We implemented Algorithm 1. The output given by this program is visible in Fig. 3. Instead of the β-representations of the horizontal and vertical projections, the implemented algorithm prints two matrices indicating the invariant and variant positions of the horizontal and vertical projetions (i.e., the results of the Steps 2 and 3 of Algorithm 1). The β-expansions of the horizontal and vertical projection values, i.e., the input of Algorithm 1, can be recalculated easily from the rows and columns of these matrices, respectively. For example, the first row of the horizontal projections matrix gives that $\langle r_1\rangle = 0000000000001$ and the sixth column of the vertical projections matrix gives that $\langle s_6\rangle = 00100000000$ in the input data.

As a final remark we can say that the same method can be used to prove corresponding theorems and algorithms for reconstructing binary matrices in n dimension from

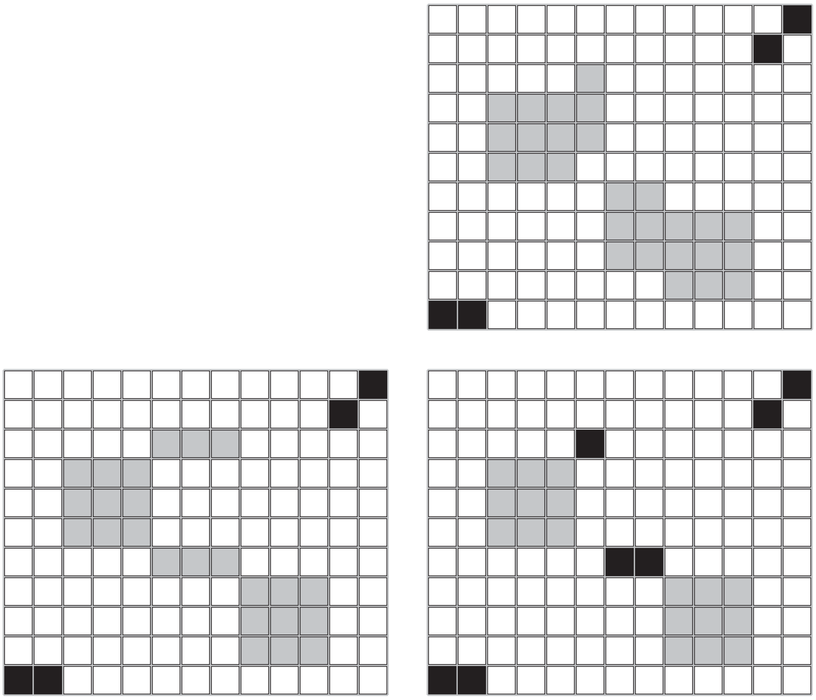

Fig. 3. The output of the implemented program of Algorithm 1. The vertical and horizontal projections are represented by matrices in the upper right and lower left quadrants, respectively. The meanings of the elementary squares are the followings: empty square: invariant 0, black square: invariant 1, gray square: variant. The matrix in the lower right quadrant represents the structure of the class.

their n ($n \geqslant 2$) orthogonal absorbed projections when the absorption is characterized by the constant β.

6. Discussion

It has been shown in this paper that the problem of reconstructing *hv*-convex binary matrices from their absorbed row and column sums can be solved in polynomial time when the absorption is represented by the constant $\beta = (1 + \sqrt{5})/2$. It is a natural question: What can we say about this reconstruction problem in the case of other absorption values? As it has been proven in [4], there are absorption values for which any binary matrix is uniquely determined by its absorbed row sums (e.g., if $\beta \geqslant 2$). Mathematically, the question is interesting only for those absorption values for which the row sum does not determine the sequence of 1's and 0's in the row uniquely, that is, when different rows may have equal absorbed sums. Therefore, the interesting β values are those for which

$$\beta^{-p_1} + \cdots + \beta^{-p_t} = \beta^{-q_1} + \cdots + \beta^{-q_z}, \tag{13}$$

where t, z and $1 \leqslant p_1 < \cdots < p_t \leqslant n$, $1 \leqslant q_1 < \cdots < q_z \leqslant n$ are positive integers such that $\{p_1, \ldots, p_t\} \neq \{q_1, \ldots, q_z\}$. One of the possible values satisfying this condition is $\beta = (1 + \sqrt{5}/2)$ studied in this paper (cf. (2)). Other possible values satisfying (13) can be, for example, the β's satisfying the equation

$$\beta^{-1} = \beta^{-2} + \cdots + \beta^{-z} \tag{14}$$

(for some $z > 3$). For the class of absorption values determined by (14), the generalization of the theories and the reconstruction method given here are straightforward. This kind of reconstruction problem for other absorption values needs some different idea probably.

References

[1] G.T. Herman, A. Kuba, Discrete Tomography, Foundations, Algorithms, and Applications, Birkhäuser, Boston, 1999.

[2] A. Kuba, Reconstruction of two-directionally connected binary patterns from their two orthogonal projections, Comput. Vision Graph. Image Process. 27 (1984) 249–265.

[3] A. Kuba, M. Nivat, Reconstruction of discrete sets from absorbed projections, in: G. Borgefors, I. Nyström, G. Sanniti di Baja (Eds.), Discrete Geometry for Computer Imagery, Proceedings of the Ninth International Conference, Lecture Notes in Computer Sciences, Vol. 1953, Springer, Berlin, 2000, pp. 137–148.

[4] A. Kuba, M. Nivat, Reconstruction of discrete sets with absorption, Linear Algebra Appl. 339 (2001) 171–194.

[5] M. Lothaire, Combinatorics on Words, Cambridge University Press, Cambridge, 1997.

[6] H.J. Ryser, Combinatorial properties of matrices of zeros and ones, Canad. J. Math. 9 (1957) 371–377.

[7] G.W. Woeginger, The reconstruction of polyominoes from their orthogonal projections, Inform. Process. Lett. 77 (2001) 225–229.