Topology Preserving Parallel Smoothing for 3D Binary Images

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Abstract. This paper presents a new algorithm for smoothing 3D binary images in a topology preserving way. Our algorithm is a reduction operator: some border points that are considered as extremities are removed. The proposed method is composed of two parallel reduction operators. We are to apply our smoothing algorithm as an iteration-by-iteration pruning for reducing the noise sensitivity of 3D parallel surface-thinning algorithms. An efficient implementation of our algorithm is sketched and its topological correctness for (26,6) pictures is proved.

1 Introduction

Contour smoothing is a frequently required pre-processing step in image processing, image understanding, pattern recognition, and visualization. There exist numerous approaches for smoothing curves and surfaces [1,2,10,11].

Yu and Yan developed a 2D sequential boundary smoothing algorithm that uses operations on chain codes [11]. It removes some noisy pixels along a contour, decomposes the contour into a set of straight lines, and detects structural feature points which correspond to convex and concave segments along the contour. Based on this work, Hu and Yan proposed an improved algorithm [2]. The method that is introduced by Taubin is suitable for smoothing piecewise linear shapes of arbitrary dimensions [10]. This method is a linear low-pass filter that removes high curvature variations. In [1], Couprie and Bertrand assumed binary images that are composed of several objects. They introduced the homotopic alternating sequence filter (HASF), a topology preserving operator which is controlled by a constraint set. Their HASF is a composition of homotopic cuttings and fillings by spheres of various radii. In these works (with the exception of [1]), a single object was considered (i.e., just one closed curve or surface was smoothed).

This paper presents a new 3D smoothing algorithm for 3D binary images that may contain arbitrary number of objects. Our algorithm is a reduction operator: some border points that are considered as extremities are removed from the input image. It is composed of two topology preserving parallel reduction operators, hence the entire algorithm is topology preserving too. The support (i.e., the

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minimal set of points whose values determine whether a point is deletable) of these two operators is $3 \times 3 \times 3$ that makes an efficient implementation possible.

It is well-known that skeletonization algorithms are rather sensitive to coarse object boundaries and surfaces, hence the produced false skeletal segments must be removed by a pruning process as a post-processing step [9]. We are to apply our smoothing algorithm for reducing the noise sensitivity of 3D parallel surface-thinning algorithms [7].

2 Basic Notions and Results

Let p be a point in the 3D digital space denoted by $\mathbb{Z}^3$. Let us denote $N_j(p)$ (for $j = 6, 18, 26$) the set of points that are *j-adjacent* to point p (see Figure 1a).

The sequence of distinct points $\langle x_0, x_1, \ldots, x_n \rangle$ is called a *j-path* (for $j = 6, 18, 26$) of length n from point x_0 to point x_n in a non-empty set of points X if each point of the sequence is in X and x_i is j-adjacent to x_{i-1} for each $1 \leq i \leq n$ (see Figure 1a). Note that a single point is a j-path of length 0. Two points are said to be *j-connected* in the set X if there is a j-path in X between them.

The *3D binary* $(26, 6)$ *digital picture* $\mathcal{P}$ is a quadruple $\mathcal{P} = (\mathbb{Z}^3, 26, 6, B)$ [4]. Each element of $\mathbb{Z}^3$ is called a *point* of $\mathcal{P}$. Each point in $B \subseteq \mathbb{Z}^3$ is called a *black point* and has a value of 1. Each point in $\mathbb{Z}^3 \setminus B$ is called a *white point* and has a value of 0. 26-adjacency is associated with the black points and 6-adjacency is assigned to the white ones. A *black component* is a maximal 26-connected set of points in B, while a *white component* is a maximal 6-connected set of points in $\mathbb{Z}^3 \setminus B$. A black point is called a *border point* in a $(26, 6)$ picture if it is 6-adjacent to at least one white point.

A *reduction operator* transforms a binary picture only by changing some black points to white ones (which is referred to as the deletion of 1's). A *parallel reduction operator* deletes all points satisfying its condition simultaneously. A

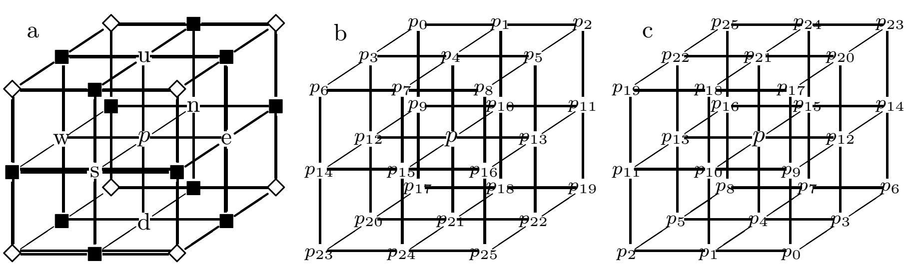

Fig. 1. Frequently used adjacencies in $\mathbb{Z}^3$ (a). The set $N_6(p)$ of the central point $p \in \mathbb{Z}^3$ contains p and the 6 points marked u=$u(p)$, n=$n(p)$, e=$e(p)$, s=$s(p)$, w=$w(p)$, and d=$d(p)$. The set $N_{18}(p)$ contains the set $N_6(p)$ and the 12 points marked "■". The set $N_{26}(p)$ contains the set $N_{18}(p)$ and the 8 points marked "◇".

Indexing schemes to encode all possible $3 \times 3 \times 3$ configurations (b-c). They are assigned to the first (b) and the second (c) parallel reduction operators of the proposed method.

3D reduction operator does *not* preserve topology [3] if any black component is split or is completely deleted, any white component is merged with another white component, a new white component is created, or a hole (that donuts have) is eliminated or created.

A *simple* point is a black point whose deletion is a topology preserving reduction [4]. Now we will make use the following result:

Theorem 1. [5] *A black point p is simple in picture* $(\mathbb{Z}^3, 26, 6, B)$ *if and only if all of the following conditions hold:*

1. *The set* $(B \setminus \{p\}) \cap N_{26}(p)$ *contains exactly one* 26*–component.*
2. *The set* $(\mathbb{Z}^3 \setminus B) \cap N_6(p)$ *is not empty.*
3. *Any two points in* $(\mathbb{Z}^3 \setminus B) \cap N_6(p)$ *are 6–connected in the set* $(\mathbb{Z}^3 \setminus B) \cap N_{18}(p)$*.*

Based on Theorem 1, simple points can be locally characterized; the support of an operator which deletes $(26, 6)$–simple points is $3 \times 3 \times 3$.

Parallel reduction operators delete a set of black points and not just a single simple point. Hence we need to consider what is meant by topology preservation when a number of black points are deleted simultaneously. The following theorem provides *sufficient conditions* for 3D parallel reduction operators to preserve topology.

Definition 1. *A unit square is a* $2 \times 2 \times 1$*, a* $2 \times 1 \times 2$*, or a* $1 \times 2 \times 2$ *subset of* $\mathbb{Z}^3$*.*

Theorem 2. [6] *Let* $\mathcal{O}$ *be a parallel reduction operator. Let p be any black point in any picture* $\mathcal{P} = (\mathbb{Z}^3, 26, 6, B)$ *such that p is deleted by* $\mathcal{O}$*. Let* $\mathcal{Q}$ *be the family of all the sets of* $Q \subseteq (N_{18}(p) \setminus \{p\}) \cap B$ *contained in a unit square. The operator* $\mathcal{O}$ *is* topology preserving *if all of the following conditions hold:*

1. *p is simple in the picture* $(\mathbb{Z}^3, 26, 6, B \setminus Q)$ *for any* Q *in* $\mathcal{Q}$*.*
2. *No black component contained in a* $2 \times 2 \times 2$ *cube can be deleted completely by* $\mathcal{O}$*.*

3 The New Smoothing Algorithm

The proposed algorithm for smoothing 3D binary pictures are composed of two parallel reduction operators denoted by R_1 and R_2. Deletable points (i.e., black points to be deleted simultaneously by R_1 and R_2) are given by a set of matching templates. A point is deletable by R_1 if at least one template in the set of 13 templates

$$\mathcal{T}_{R_1} = \{U, N, W, UN, UE, US, UW, NW, NE, UNW, UNE, USE, USW\}$$

shown in Figures 2-4 matches it. In these figures, we use the following notations: each element marked "p" or "●" matches a black point; each element marked "○" matches a white point; each "·" (don't care) matches either a black or a white point. Deletable points by operator R_2 are defined by matching templates too. Templates in Figures 2-4 reflected to the point p are taken into consideration.

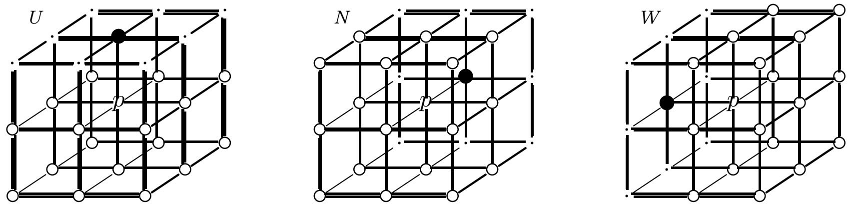

Fig. 2. Templates assigned to the first three faces

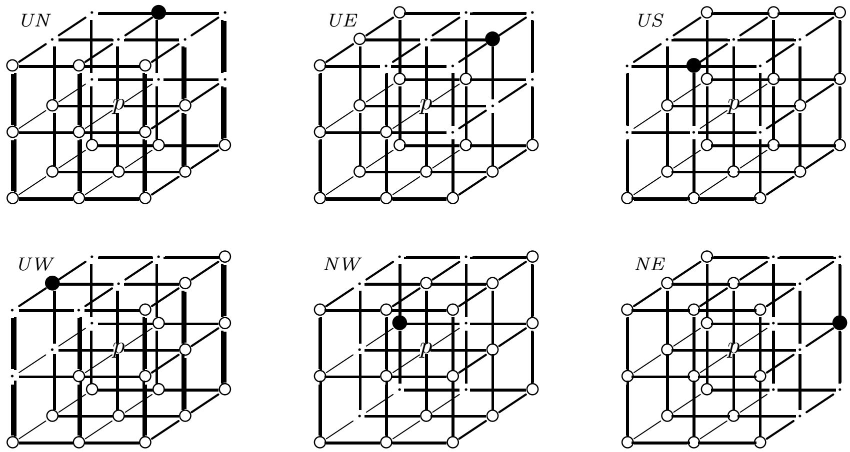

Fig. 3. Templates assigned to the first six edges

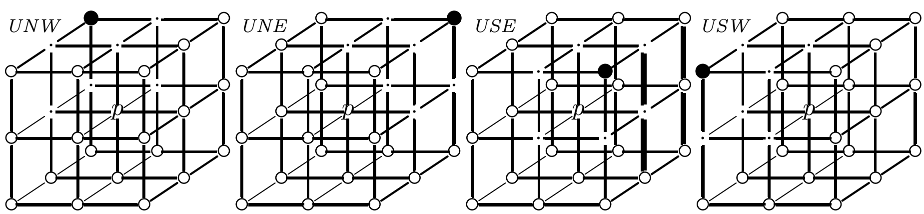

Fig. 4. Templates assigned to the first four nodes

Our algorithm is sketched by the following program:

Input: picture $(\mathbb{Z}^3, 26, 6, X)$
Output: picture $(\mathbb{Z}^3, 26, 6, Y)$
begin
$Y \ = \ X$;
$Y \ = \ Y \setminus \{ \ p \mid p$ is deletable by R_1 in $(\mathbb{Z}^3, 26, 6, Y) \ \}$;
$Y \ = \ Y \setminus \{ \ p \mid p$ is deletable by R_2 in $(\mathbb{Z}^3, 26, 6, Y) \ \}$;
end

If the 13+13 templates of operators R_1 and R_2 are considered, then one may think that the proposed algorithm is time consuming and it is rather difficult to implement it on conventional sequential computers. Thus we sketch here an efficient and fairly general implementation method. It can be used for various reduction operators (e.g., parallel thinning algorithms) as well [7,8].

The proposed implementation uses just one pre-calculated look-up-table to encode deletable points. Since the $3 \times 3 \times 3$ support of our operators contains 26 points with the exception of the central point p in question (see Figures 2-3), the look-up-table has 2^{26} entries of 1 bit in size. It is not hard to see that it requires just 8 MB of storage space in memory.

An integer in $[0, 2^{26})$ can be assigned to each $3\times3\times3$ configuration. This index is calculated as $\sum_{k=0}^{25} 2^k p_k$. Operator R_1 uses the indexing scheme depicted in Figure 1b, and R_2 takes the reflected scheme (see Figure 1c) into consideration. Then the calculated index is to address the look-up-table.

In addition, two lists are used to speed up the process: one for storing the border points in the current picture (since operators R_1 and R_2 can only delete border points, thus the repeated scans of the entire image array are avoided); the second list is to store all deletable points in the current phase of the process.

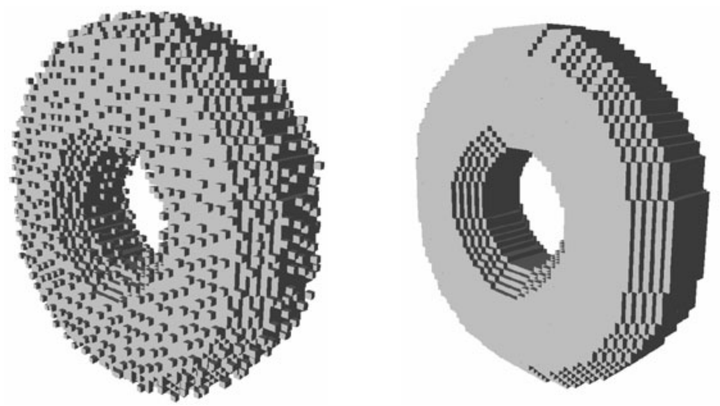

Fig. 5. A $64\times64\times19$ 3D image of a noisy torus (left) and the smoothed image produced by the proposed algorithm

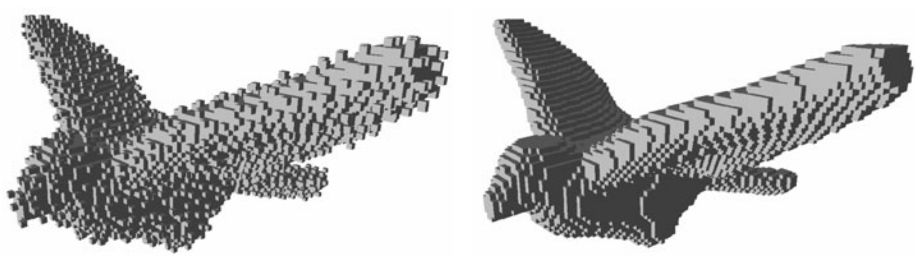

Fig. 6. A $103\times47\times75$ 3D image of a noisy bird (left) and the smoothed image produced by our algorithm

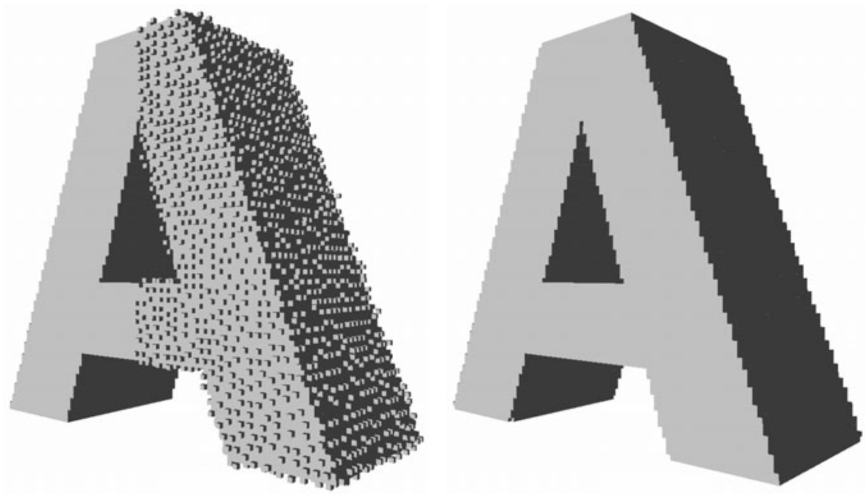

Fig. 7. A 100 × 100 × 40 3D image of a noisy character "A" (left) and its smoothed version. Note that the smooth boundary segments are not altered.

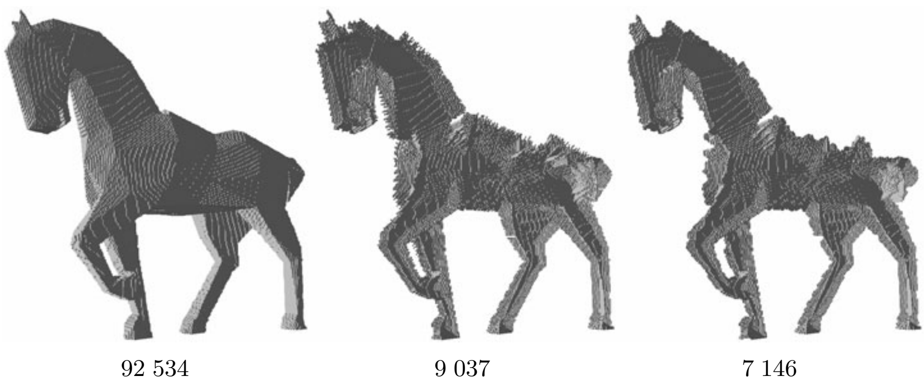

Fig. 8. A 300 × 239 × 83 3D image of a horse (left); its surface-skeleton produced by a parallel surface-thinning algorithm (middle); and the produced skeleton with iteration-by-iteration smoothing (right). Numbers mean the count of black points. We can state that the thinning with smoothing produces much less skeletal points without overshrinking.

In experiments the proposed smoothing algorithm was tested on objects of various images. Here we present three illustrative examples below (Figures 5-7). It is illustrated in Figure 7 that our algorithm is a proper smoothing one: it does not alter the smooth boundary segments of the original image.

We are to apply our smoothing algorithm for reducing the noise sensitivity of 3D parallel surface-thinning algorithms. In Figure 8 we present an example in which the proposed smoothing is applied before each iteration step of the 3D parallel surface-thinning algorithm described in [8].

4 Verification

Now we will show that the proposed smoothing algorithm is topology preserving for (26,6) pictures. We are to prove that the first operator R_1 given by the set of matching templates $\mathcal{T}_{R_1}$ fulfills both conditions of Theorem 2. It can be proved for the second operator R_2 in a similar way. Hence the entire smoothing algorithm is topology preserving, since it is composed of topology preserving reductions.

Let us classify the elements of the templates in the set of templates $\mathcal{T}_{R_1}$ (see Figures 2-4). The element in the centre of a template (marked "p") is called *central*. A noncentral template element is called *black* if it is marked "●". A noncentral template element is called *white* if it is marked "○". Any other noncentral template element which is neither white nor black, is called *potentially black* (marked "·"). A black or a potentially black noncentral template element is called *nonwhite*.

A black point p is *deletable* if at least one template in the set of 13 templates in $\mathcal{T}_{R_1}$ matches it (i.e., if it is deletable by R_1).

Now let us state some properties of the set of templates $\mathcal{T}_{R_1}$.

Proposition 1. *Element $d(p)$ is white in each template in $\mathcal{T}_{R_1}$.*

Proposition 2. *Element $s(d(p))$ is white in each template in $\mathcal{T}_{R_1}$.*

Proposition 3. *If element $s(p)$ is nonwhite in a template in $\mathcal{T}_{R_1}$, then $u(s(p))$, $e(u(s(p)))$, or $w(u(s(p)))$ is black.*

Lemma 1. *Each deletable point is simple.*

Proof. The first thing we need to verify is that there exists a 26-path between any two potentially black elements (condition 1 of Theorem 1). Here it is sufficient to show that any potentially black element is 26-adjacent to a black element and any black element is 26-adjacent to another black element. This is really apparent from a careful examination of the templates in $\mathcal{T}_{R_1}$.

To prove that conditions 2 and 3 of Theorem 1 hold, it is sufficient to show that, for each template,

- there exists a white element 6-adjacent to the central element,
- for any potentially black or white element 6-adjacent to the central element p, there exists a 6-adjacent white 18-neighbor which is 6-adjacent to a white element 6-adjacent to p.

These two points are obvious by Proposition 1 and a careful examination of the set of templates $\mathcal{T}_{R_1}$. □

Lemma 2. *The simplicity of a deletable point does not depend on any black point coinciding with a potentially black template element. (In other words, a deletable point remains simple after the deletion of any (sub)set of points coinciding with template elements marked "·".)*

It can be proved similarly as Lemma 1.

Lemma 3. *Let p and q be any two black points in a picture* $(\mathbb{Z}^3, 26, 6, B)$ *such that* $q \in N_{18}(p)$. *If both points p and q are deletable, then p is simple in picture* $(\mathbb{Z}^3, 26, 6, B \setminus \{q\})$.

Proof. Since point p is deletable, by Lemma 1 it is simple. To prove this lemma, we must show that p remains simple after the deletion of q.

If q coincides with a potentially black template element, then this lemma holds by Lemma 2. Hence it is sufficient to deal with the deletable points coinciding with template elements marked "●" in templates U, N, W, UN, UE, US, UW, NW, and NE (see Figures 2-3). We do not have to take templates UNW, UNE, USE, and USW into consideration since elements marked "●" in these four templates are not 18-adjacent to their central elements marked "p" (see Figure 4).

Let us see the 9 templates in question:

- If p is deleted by U, then $q = u(p)$ is not deletable by Proposition 1.
- If p is deleted by N, then $q = n(p)$ may be deleted by templates US, USE, or USW since element $s(p)$ is nonwhite in these three templates. The three possible configurations are depicted in Figure 9. Consequently, point q is not deletable.
- If p is deleted by W, then $q = w(p)$ may be deleted by templates UE, NE, UNE, or USE since element $e(p)$ is nonwhite in these four templates. The four possible configurations are depicted in Figure 10. Consequently, point q is not deletable.
- If p is deleted by UN, then $q = n(u(p))$ is not deletable by Proposition 2.
- If p is deleted by UE, then $q = e(u(p))$ may be deleted by templates W and NW since element $w(d(p))$ is nonwhite in these two templates. The two possible configurations are depicted in Figure 11. Black point r (see Figure 11a) is not deletable by Proposition 1, and the set of black points $\{p, q, s\}$ is not contained in a unit square (see Figure 11b). It is easy to see that p remains simple after the deletion of q in both configurations.
- If p is deleted by US, then point $q = s(u(p))$ may be deleted by templates N, NW, and NE since element $n(d(p))$ is nonwhite in these three templates. The three possible configurations are depicted in Figure 12. Black point r (see Figure 12a) is not deletable by Proposition 1, and the sets of black points $\{p, q, s\}$ and $\{p, q, t\}$ are not contained in a unit square (see Figures 12b-c). It is not hard to see that p remains simple after the deletion of q in all the three configurations.
- If p is deleted by UW, then point $q = w(u(p))$ may only be deleted by template NE since it is the only template in which element $e(d(p))$ is nonwhite. The possible configuration is depicted in Figure 11c in which the set of black points $\{p, q, r\}$ is not contained in a unit square. It is not hard to see that p remains simple after the deletion of q.
- If p is deleted by NW, then point $q = w(n(p))$ may be deleted by templates UE, US, and USE since element $e(s(p))$ is nonwhite in these three templates. The three possible configurations are depicted in Figure 13. The sets of black points $\{p, q, r\}$ and $\{p, q, s\}$ are not contained in a unit square (see

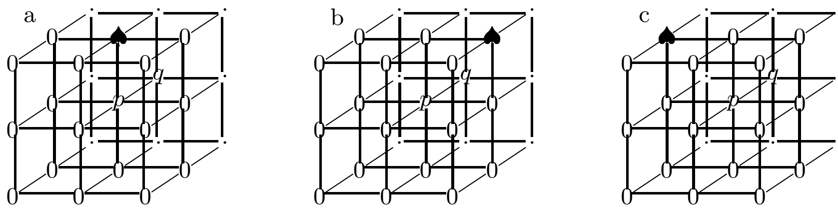

Fig. 9. Possible configurations in which point p is deleted by template N and q is to be deleted by templates US (a), USE (b), and USW (c). Each point marked "♠" coincides a white template element in N, but it coincides with a black element if q coincides with the central element of templates US, USE, and USW.

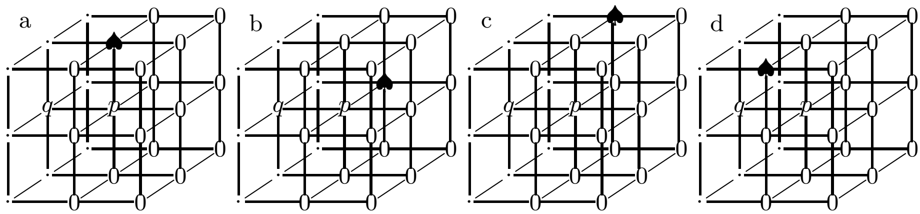

Fig. 10. Possible configurations in which point p is deleted by template W and q is to be deleted by templates UE (a), NE (b), UNE (c), and USE (d). Each point marked "♠" coincides a white template element in W, but it coincides with a black element if q coincides with the central element of templates UE, NE, UNE, and USE.

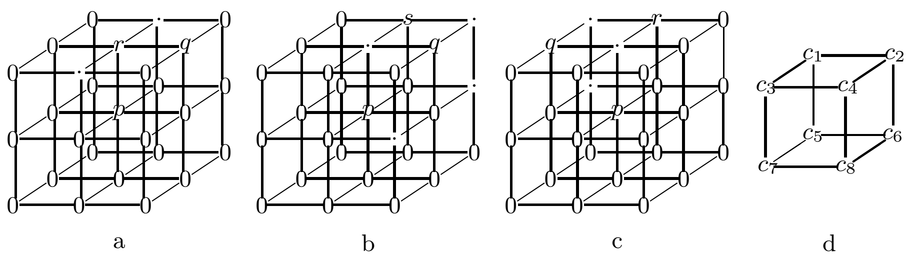

Fig. 11. Possible configurations in which point p is deleted by template UE and q is to be deleted by templates W (a) and NW (b), where r and s are black points. The possible configuration in which point p is deleted by template UW and q is to be deleted by template NE (c). The $2 \times 2 \times 2$ cube that contains a black component C (d).

Figures 13a-b). It is easy to see that p remains simple after the deletion of q in these configurations. Black point q in the last configuration (see Figure 13c) is not deletable.

– If p is deleted by NE, then $q = e(n(p))$ may be deleted by templates W, US, UW, or USW since element $w(s(p))$ is nonwhite in these four templates. The

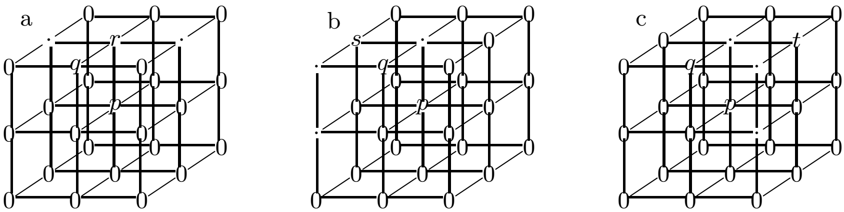

Fig. 12. Possible configurations in which point p is deleted by template US and q is to be deleted by templates N (a), NW (b), and NE (c), where r, s, and t are black points

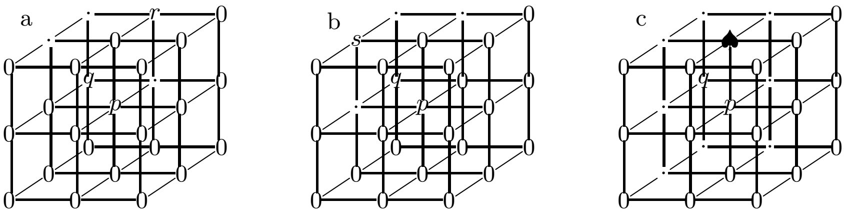

Fig. 13. Possible configurations in which point p is deleted by template NW and q is to be deleted by templates UE (a), US (b), and USE (c), where r and s are black points. The point marked "♠" coincides a white template element in NW, but it coincides with a black element if q coincides with the central element of USE.

four possible configurations are depicted in Figure 14. Black point r (see Figure 14a) is not deletable by Proposition 3, and the sets of black points $\{p, q, s\}$ and $\{p, q, t\}$ are not contained in a unit square (see Figure 14b-c). It is easy to see that p remains simple after the deletion of q in all these three configurations. Black point q in the last configuration (see Figure 14d) is not deletable. □

Lemma 4. *No black component C contained in a $2 \times 2 \times 2$ cube can be deleted completely by the operator R_1.*

Proof. Let us examine the $2 \times 2 \times 2$ cube depicted in Figure 11d.

It is easy to check that if $c_1 \in C$, then c_1 is not deletable by R_1, and if $c_k \in C$ $(k = 2, \ldots, 8)$, then there exists a $c_j \in C$ $(j = 1, \ldots, k-1)$ that is not deletable by R_1. Thus C cannot be deleted completely. □

We are now ready to state our main theorem.

Theorem 3. *Operator R_1 is topology preserving for $(26, 6)$ pictures.*

Proof. We need to show that both conditions of Theorem 2 are satisfied:

1. Let us examine the simplicity of a deletable point p in picture $(\mathbb{Z}^3, 26, 6, B \backslash Q)$, where the set of deletable points $Q \subseteq (N_{18}(p) \backslash \{p\}) \cap B$ is contained in a unit square. It is clear that the number of elements in Q (denoted by $\#(Q)$) is less than or equal to 3.

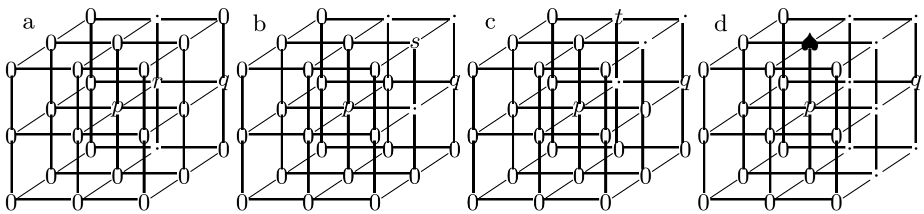

Fig. 14. Possible configurations in which point p is deleted by template NE and q is to be deleted by templates W (a), US (b), UW (c), or USW (d), where r, s, and t are black points. The point marked "♠" (d) coincides a white element in NE, but it coincides with a black element if q coincides with the central element of template USW.

The following points have to be checked:
- $\#(Q) = 0$ ($Q = \emptyset$):
  Condition 1 of Theorem 2 is satisfied by Lemma 1.
- $\#(Q) = 1$ ($Q = \{q\}$):
  Condition 1 of Theorem 2 is satisfied by Lemma 3.
- $\#(Q) = 2, 3$:
  Each template in $\mathcal{T}_{R_1}$ contains just one black element and the simplicity of p does not depend on any black point coinciding with a potentially black template element by Lemma 2. Hence these cases are to be ignored.

2. Condition 2 of Theorem 2 (i.e., no black component contained in a $2 \times 2 \times 2$ cube can be deleted completely) is satisfied by Lemma 4. □

Acknowledgements

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