Proceedings of GO 2005, pp. 1 – 4.

A global optimization model for locating chaos*

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Abstract We present a global optimization model to find chaotic regions of certain dynamic systems. The technique has two innovations: first an interval arithmetic based guaranteed reliability checking routine to decide whether an inclusion relation holds, and a penalty function based nonlinear optimization problem that enables us to automatize the search for fitting problem instances. We provide the theoretical results proving correctness and convergence properties for the new algorithm. A companion talk discusses the results achieved by the presented method.

Keywords: Chaos, Computer-aided proof, Global optimization.

An important question is while studying approximations of the solutions of differential equations, whether the given problem has a chaotic solution. The problem is usually solved by careful studying of the given problem with much human interaction, followed by an estimation of the Lipschitz constant, bounding the rounding errors to be committed, and finally a number of grid points are checked one by one by a proper computer program [6].

We study verified computational methods to check and locate regions the points of which fulfill the conditions of chaotic behaviour. The investigated Hénon mapping is $\mathcal{H}(x, y) = (1 + y - Ax^2, Bx)$. The paper [6] considered the $A = 1.4$ and $B = 0.3$ values and some regions of the two dimensional Euclidean space: $E = E_1 \cup E_2 = \{(x, y) \mid x \geq 0.4,\ y \geq 0.28\} \cup \{(x, y) \mid x \leq 0.64,\ |y| \leq 0.01\}$, $\mathcal{O}_1 = \{(x, y) \mid x < 0.4,\ y > 0.01\}$, $\mathcal{O}_2 = \{(x, y) \mid y < 0\}$.

According to [6] Theorem 1 below ensures the chaotic behaviour for the points of the parallelograms Q_0 and Q_1 with parallel sides with the x axis (for $y_0 = 0.01$ and $y_1 = 0.28$, respectively), with the common tangent of 2, and x coordinates of the lower vertices are $x_a = 0.460$, $x_b = 0.556$; and $x_c = 0.558$, $x_d = 0.620$, respectively. The mapping and the problem details (such as the transformed sides of the parallelograms, $H^7(a)$, $H^7(b)$, $H^7(c)$, and $H^7(d)$) are illustrated on Figure 1.

Theorem 1. *Assume that the following relations hold for the given particular Hénon mapping:*

$$\mathcal{H}^7(a \cup d) \subset \mathcal{O}_2, \tag{1}$$

$$\mathcal{H}^7(b \cup c) \subset \mathcal{O}_1, \tag{2}$$

$$\mathcal{H}^7(Q_0 \cup Q_1) \subset \mathbb{R}^2 \setminus E, \tag{3}$$

then chaotic trajectories belong to the starting points of the regions Q_0 and Q_1.

To check the inclusion relations required in Theorem 1 we have set up an adaptive subdivision algorithm based on interval arithmetic:

*This work has been partially supported by the Bilateral Austrian-Hungarian project öu56011 as well as by the Hungarian National Science Foundation Grants OTKA No. T 037491, T 032118, T 034350, T 048377, and T 046822.

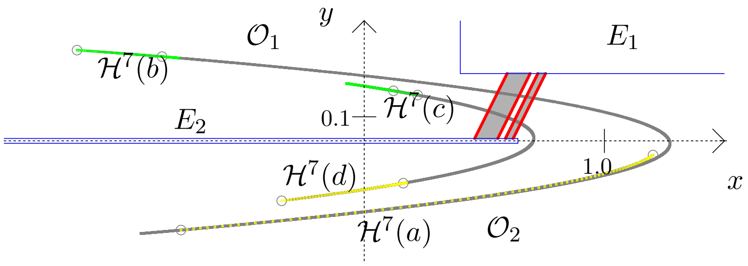

Figure 1. Illustration of the H^7 transformation for the classic Hénon parameters $A = 1.4$ and $B = 0.3$ together with the chaotic region of two parallelograms. The a, b, c, and d sides of the parallelograms are depicted on the upper left picture of Figure 2.

Algorithm 1 : The Checking Routine

Inputs: – ε: the user set limit size of subintervals,
– Q: the argument set to be proved,
– $\mathcal{O}$: the aimed set for which $\mathcal{T}(Q) \subset \mathcal{O}$ is to be checked.

Calculate the initial interval I, that contains the regions of interest
Push the initial interval into the stack
while (the stack is nonempty)
  Pop an interval v out of the stack
  Calculate the width of v
  Determine the widest coordinate direction
  Calculate the transformed interval $w = T(v)$
  if $v \cap Q \neq \emptyset$, and the condition $w \subset \mathcal{O}$ does not hold, **then**
    if the width of interval v is less than ε **then**
      print that the condition is hurt by v and **stop**
      else bisect v along the widest side: $v = v_1 \cup v_2$
        push the subintervals into the stack
    endif
  endif
end while
print that the condition is proven and **stop**

We have proven that this algorithm is capable to provide the positive answer after a finite number of steps, and also that the given answer is rigorous in the mathematical sense. Once we have a reliable computer procedure to check the conditions of chaotic behavior of a mapping it is straightforward to set up an optimization model that transforms the original chaos location problem to a global optimization problem.

The key question for the successful application of a global optimization algorithm was how to compose the penalty functions. On the basis of earlier experiences collected with similar constrained problems, we have decided to add a nonnegative value proportional to how much the given condition was hurt, plus a fixed penalty term in case at least one of the constraints was not satisfied.

As an example, consider the case when one of the conditions for the transformed region was hurt, e.g. when (2), i.e. the relation

$$\mathcal{H}^k(b \cup c) \subset \mathcal{O}_1$$

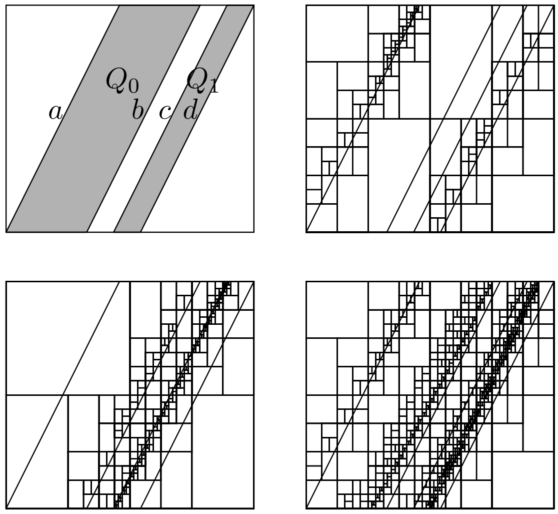

Figure 2. The parallelograms and the starting interval covered by the verified subintervals for which either the given condition holds (in the order of mentioning in Theorem 1), or they do not contain a point of the argument set.

does not hold for a given kth iterate, and for a region of two parallelograms. In such a case the checking routine will provide a subinterval that contains at least one point of the investigated region, and which contradicts the given condition. Then we have calculated the Hausdorff distance of the transformed subinterval $H^k(I)$ to the set $\mathcal{O}_1$ of the right side of the condition,

$$\max_{z \in H^k(I)} \inf_{y \in \mathcal{O}_1} d(z, y),$$

where $d(z, y)$ is a given metric, a distance between a two dimensional interval and a point. Notice that the use of maximum in the expression is crucial, with minimization instead our optimization approach could provide (and has provided) result regions that do not fulfill the given conditions of chaotic behaviour. On the other hand, the minimal distance according to points of the aimed set (this time $\mathcal{O}_1$) is satisfactory, since it enables the technique to push the search into proper directions. In cases when the checking routine answered that the investigated subinterval has fulfilled the given condition, we have not changed the objective function.

Summing it up, we have considered the following bound constrained problem for the T inclusion function of the mapping $\mathcal{T}$:

$$\min_{x \in X} g(x), \tag{4}$$

where

$$g(x) = f(x) + p\left(\sum_{i=1}^{m} \max_{z \in T(I)} \inf_{y \in S_i} d(z, y)\right),$$

X is the n-dimensional interval of admissible values for the parameters x to be optimized, $f(x)$ is the original, nonnegative objective function, and $p(y) = y + C$ if y is positive, and $p(y) = 0$ otherwise. C is a positive constant, larger than $f(x)$ for all the feasible x points, m is the number of conditions to be fulfilled, and S_i is the aimed set for the i-th condition.

The emerging global optimization problem has been solved by the clustering optimization method described in citecst. We have proven the correctness of this global optimization model:

Theorem 2. *For the bound constrained global optimization problem defined in (4) the following properties hold:*

1. In case a global optimization algorithm finds a point for which the objective function g has a value below C, i.e. when each penalty term $\max_{z\in T(I)} \inf_{y\in S_i} d(z,y)$ is zero, then all the conditions of chaos are fulfilled by the found region represented by the corresponding optimal parameters found. At the same time, the checking routine provided a guaranteed reliability computational proof of the respective subset relations.

2. In case the given problem does not have a parameter set within the bounds of the parameters to be optimized such that the corresponding region would fulfill the criteria of chaos, then the optimization cannot result in an approximate optimizer point with an objective function value below C.

Talk will provide an insight into the theoretical statements and their proofs. On this basis we have checked chaos for an earlier investigated 7th iterate Hénon mapping and also other problem instances, some of them have involved tolerance optimization too [3]. The numerical results (see also in [1],[4]) will be covered by an other talk.

Acknowledgments

This work has been partially supported by the Bilateral Austrian-Hungarian project öu56011 as well as by the Hungarian National Science Foundation Grants OTKA No. T 037491, T 032118, T 034350, T 048377, and T 046822.

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