MILP: modeling tricks

Linearizing the product of two binary variables

Let $y_1, y_2 \in \{0, 1\}$ two binary variables, and assume that its product, $y_1 y_2$, which is a nonlinear expression, appears in a given formulation. We can linearize the product as follows:

$$\begin{aligned} \delta &\leq y_1 \\ \delta &\leq y_2 \\ \delta &\geq y_1 + y_2 - 1 \\ \delta &\in \{0, 1\} \end{aligned}$$

Notice that $\delta = y_1 y_2$.

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Linearizing the product of a binary and a continous variable

Let z be a continuous variable such that $L \leq z \leq U$, and $x \in \{0, 1\}$ be a binary variable. Assume that its product, zx, which is a nonlinear expression, appears in a given formulation. We can linearize the product as follows:

$$
\begin{aligned}
y &\leq Ux \\
y &\geq Lx \\
z - y &\leq U(1 - x) \\
z - y &\geq L(1 - x)
\end{aligned}
$$

Notice that $y = zx$

- A chain wants to enter in a given area by opening p facilities.
- Those facilities are to be open in p of the s potential sites pre-selected by the chain.
- There already exists m competing facilities operating in the area.
- Customers follow a probabilistic choice rule (they patronize all the facilities, and the amount spent at each facility is proportional to its attraction).
- The objective is to maximize the market share captured by the locating chain.

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Example: A discrete competitive location problem under the probabilistic choice rule.

Indices

i	index for demand points (or customers), $i = \{1, \ldots, n\}$.
j	index for the facilities,
	$j = 1, \ldots, s$, for the potential new facilities,
	$j = s + 1, \ldots, s + m$, for the existing competing facilities.

Data

w_i	demand (or buying power) of demand point i.
d_{ij}	distance between demand point i and location j.
a_{ij}	quality of facility j as perceived by deman point i.
β	modulator of the distance

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Example: A discrete competitive location problem under the probabilistic choice rule.

Computed data

$$u_{ij} = \frac{a_{ij}}{(d_{ij}+1)^{\beta}}$$ attraction that demand point i feels towards facility j.

Variables

$$x_j = \begin{cases} 1 & \text{if a facility is open at } j \\ 0 & \text{otherwise} \end{cases}, j = 1\ldots, s$$

$$\max \quad \sum_{i=1}^{n} w_i \frac{\sum_{j=1}^{s} u_{ij} x_j}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}$$

$$\text{s.t.} \quad \sum_{j=1}^{s} x_j = p$$

$$x_j \in \{0, 1\}, j = 1, \ldots, s$$

$$\max \quad \sum_{i=1}^{n} w_i \frac{\sum_{j=1}^{s} u_{ij}x_j}{\sum_{j=1}^{s} u_{ij}x_j + \sum_{j=s+1}^{s+m} u_{ij}}$$

$$\text{s.t.} \quad \sum_{j=1}^{s} x_j = p$$

$$x_j \in \{0,1\}, j = 1, \ldots, s$$

If we denote

$$z_i = \frac{1}{\sum_{j=1}^{s} u_{ij}x_j + \sum_{j=s+1}^{s+m} u_{ij}}, i = 1, \ldots, n$$

then the problem becomes

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Example: A discrete competitive location problem under the probabilistic choice rule.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} w_i z_i \sum_{j=1}^{s} u_{ij} x_j \\
\text{s.t.} \quad & z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}, i = 1, \ldots, n \\
& \sum_{j=1}^{s} x_j = p \\
& x_j \in \{0, 1\}, j = 1, \ldots, s \\
& z_i \geq 0, i = 1, \ldots, n
\end{aligned}
$$

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Example: A discrete competitive location problem under the probabilistic choice rule.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} \sum_{j=1}^{s} w_i z_i u_{ij} x_j \\
\text{s.t.} \quad & z_i = \frac{1}{\displaystyle\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}, i = 1, \ldots, n \\
& \sum_{j=1}^{s} x_j = p \\
& x_j \in \{0, 1\}, j = 1, \ldots, s \\
& z_i \geq 0, i = 1, \ldots, n
\end{aligned}
$$

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Example: A discrete competitive location problem under the probabilistic choice rule.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} \sum_{j=1}^{s} (w_i z_i u_{ij}) x_j \\
\text{s.t.} \quad & z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}, i = 1, \ldots, n \\
& \sum_{j=1}^{s} x_j = p \\
& x_j \in \{0, 1\}, j = 1, \ldots, s \\
& z_i \geq 0, i = 1, \ldots, n
\end{aligned}
$$

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Example: A discrete competitive location problem under the probabilistic choice rule.

If we denote

$$y_{ij} = (w_i z_i u_{ij})x_j, i = 1, \ldots, n, j = 1, \ldots, s$$

and taking into account that the product $y = zx$, where $L \leq z \leq U$ is continuous and x binary can be linearized as

$$\begin{aligned} y &\leq Ux \\ y &\geq Lx \\ z - y &\leq U(1-x) \\ z - y &\geq L(1-x) \end{aligned}$$

we have that the product $y_{ij} = (w_i z_i u_{ij})x_j$ can be linearized as follows

$$\left.\begin{array}{l} y_{ij} \leq w_i x_j, \\ y_{ij} \geq 0 x_j \Leftrightarrow y_{ij} \geq 0, \\ w_i z_i u_{ij} - y_{ij} \leq w_i(1 - x_j), \\ w_i z_i u_{ij} - y_{ij} \geq 0(1 - x_j) \Leftrightarrow w_i z_i u_{ij} - y_{ij} \geq 0, \end{array}\right\} i = 1, \ldots, n, j = 1, \ldots, s$$

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Example: A discrete competitive location problem under the probabilistic choice rule.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} \sum_{j=1}^{s} y_{ij} \\
\text{s.t.} \quad & z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}, && i = 1, \ldots, n \\
& y_{ij} \leq w_i x_j, && i = 1, \ldots, n, j = 1, \ldots, s \\
& y_{ij} \geq 0, && i = 1, \ldots, n, j = 1, \ldots, s \\
& w_i z_i u_{ij} - y_{ij} \leq w_i (1 - x_j), && i = 1, \ldots, n, j = 1, \ldots, s \\
& w_i z_i u_{ij} - y_{ij} \geq 0, && i = 1, \ldots, n, j = 1, \ldots, s \\
& \sum_{j=1}^{s} x_j = p \\
& x_j \in \{0, 1\}, && j = 1, \ldots, s \\
& z_i \geq 0, && i = 1, \ldots, n \\
& y_{ij} \geq 0, && i = 1, \ldots, n, j = 1, \ldots, s
\end{aligned}
$$

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Example: A discrete competitive location problem under the probabilistic choice rule.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n}\sum_{j=1}^{s} y_{ij} \\
\text{s.t.} \quad & z_i = \frac{1}{\sum_{j=1}^{s} u_{ij}x_j + \sum_{j=s+1}^{s+m} u_{ij}}, && i = 1,\dots,n \\
& y_{ij} \le w_i x_j, && i = 1,\dots,n, j = 1,\dots,s \\
& y_{ij} \ge 0, && i = 1,\dots,n, j = 1,\dots,s \\
& w_i z_i u_{ij} - y_{ij} \le w_i(1 - x_j), && i = 1,\dots,n, j = 1,\dots,s \\
& w_i z_i u_{ij} - y_{ij} \ge 0, && i = 1,\dots,n, j = 1,\dots,s \\
& \sum_{j=1}^{s} x_j = p \\
& x_j \in \{0,1\}, && j = 1,\dots,s \\
& z_i \ge 0, && i = 1,\dots,n \\
& y_{ij} \ge 0, && i = 1,\dots,n, j = 1,\dots,s
\end{aligned}
$$

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Example: A discrete competitive location problem under the probabilistic choice rule.

$$z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}$$

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Example: A discrete competitive location problem under the probabilistic choice rule.

$$z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}} \Leftrightarrow$$

$$z_i \left(\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}\right) = 1$$

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Example: A discrete competitive location problem under the probabilistic choice rule.

$$z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}} \Leftrightarrow$$

$$z_i \left(\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}\right) = 1 \Leftrightarrow$$

$$z_i \sum_{j=1}^{s} u_{ij} x_j + z_i \sum_{j=s+1}^{s+m} u_{ij} = 1$$

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Example: A discrete competitive location problem under the probabilistic choice rule.

$$z_i = \frac{1}{\sum_{j=1}^{s} u_{ij}x_j + \sum_{j=s+1}^{s+m} u_{ij}} \Leftrightarrow$$

$$z_i(\sum_{j=1}^{s} u_{ij}x_j + \sum_{j=s+1}^{s+m} u_{ij}) = 1 \Leftrightarrow$$

$$z_i\sum_{j=1}^{s} u_{ij}x_j + z_i\sum_{j=s+1}^{s+m} u_{ij} = 1 \Leftrightarrow$$

$$w_i z_i\sum_{j=1}^{s} u_{ij}x_j + w_i z_i\sum_{j=s+1}^{s+m} u_{ij} = w_i$$

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Example: A discrete competitive location problem under the probabilistic choice rule.

$$z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}} \Leftrightarrow$$

$$z_i \left(\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}\right) = 1 \Leftrightarrow$$

$$z_i \sum_{j=1}^{s} u_{ij} x_j + z_i \sum_{j=s+1}^{s+m} u_{ij} = 1 \Leftrightarrow$$

$$w_i z_i \sum_{j=1}^{s} u_{ij} x_j + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} = w_i \Leftrightarrow$$

$$\sum_{j=1}^{s} w_i z_i u_{ij} x_j + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} = w_i$$

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Example: A discrete competitive location problem under the probabilistic choice rule.

$$z_i = \frac{1}{\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}} \Leftrightarrow$$

$$z_i\left(\sum_{j=1}^{s} u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}\right) = 1 \Leftrightarrow$$

$$z_i \sum_{j=1}^{s} u_{ij} x_j + z_i \sum_{j=s+1}^{s+m} u_{ij} = 1 \Leftrightarrow$$

$$w_i z_i \sum_{j=1}^{s} u_{ij} x_j + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} = w_i \Leftrightarrow$$

$$\sum_{j=1}^{s} w_i z_i u_{ij} x_j + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} = w_i \Leftrightarrow$$

$$\sum_{j=1}^{s} y_{ij} + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} = w_i$$

$$
\begin{array}{lll}
\max & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{s} y_{ij} & \\
\text{s.t.} & \sum_{j=1}^{s} y_{ij} + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} = w_i, & i = 1,\ldots,n \\
& y_{ij} \le w_i x_j, & i = 1,\ldots,n, j = 1,\ldots,s \\
& y_{ij} \ge 0, & i = 1,\ldots,n, j = 1,\ldots,s \\
& w_i z_i u_{ij} - y_{ij} \le w_i(1 - x_j), & i = 1,\ldots,n, j = 1,\ldots,s \\
& w_i z_i u_{ij} - y_{ij} \ge 0 & i = 1,\ldots,n, j = 1,\ldots,s \\
& \displaystyle\sum_{j=1}^{s} x_j = p & \\
& x_j \in \{0,1\}, & j = 1,\ldots,s \\
& z_i \ge 0, & i = 1,\ldots,n \\
& y_{ij} \ge 0, & i = 1,\ldots,n, j = 1,\ldots,s
\end{array}
$$

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Example: A discrete competitive location problem under the probabilistic choice rule.

$$
\begin{array}{lll}
\max & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{s} y_{ij} & \\
\text{s.t.} & \sum_{j=1}^{s} y_{ij} + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} \leq w_i, & i = 1,\dots,n \\
& y_{ij} \leq w_i x_j, & i = 1,\dots,n, j = 1,\dots,s \\
& y_{ij} \geq 0, & i = 1,\dots,n, j = 1,\dots,s \\
& w_i z_i u_{ij} - y_{ij} \leq w_i(1 - x_j), & i = 1,\dots,n, j = 1,\dots,s \\
& w_i z_i u_{ij} - y_{ij} \geq 0 & i = 1,\dots,n, j = 1,\dots,s \\
& \displaystyle\sum_{j=1}^{s} x_j = p & \\
& x_j \in \{0,1\}, & j = 1,\dots,s \\
& z_i \geq 0, & i = 1,\dots,n \\
& y_{ij} \geq 0, & i = 1,\dots,n, j = 1,\dots,s
\end{array}
$$