

## Linearizing the product of two binary variables

Let  $y_1, y_2 \in \{0, 1\}$  two binary variables, and assume that its product,  $y_1 y_2$ , which is a nonlinear expression, appears in a given formulation. We can linearize the product as follows:

$$\begin{aligned}\delta &\leq y_1 \\ \delta &\leq y_2 \\ \delta &\geq y_1 + y_2 - 1 \\ \delta &\in \{0, 1\}\end{aligned}$$

Notice that  $\delta = y_1 y_2$ .



## Linearizing the product of a binary and a continuous variable

Let  $z$  be a continuous variable such that  $L \leq z \leq U$ , and  $x \in \{0, 1\}$  be a binary variable. Assume that its product,  $zx$ , which is a nonlinear expression, appears in a given formulation. We can linearize the product as follows:

$$\begin{aligned}y &\leq Ux \\y &\geq Lx \\z - y &\leq U(1 - x) \\z - y &\geq L(1 - x)\end{aligned}$$

Notice that  $y = zx$



# MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

- A chain wants to enter in a given area by opening  $p$  facilities.
- Those facilities are to be open in  $p$  of the  $s$  potential sites pre-selected by the chain.
- There already exists  $m$  competing facilities operating in the area.
- Customers follow a probabilistic choice rule (they patronize all the facilities, and the amount spent at each facility is proportional to its attraction).
- The objective is to maximize the market share captured by the locating chain.



# MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

## Indices

- $i$  index for demand points (or customers),  $i = \{1, \dots, n\}$ .
- $j$  index for the facilities,
  - $j = 1, \dots, s$ , for the potential new facilities,
  - $j = s + 1, \dots, s + m$ , for the existing competing facilities.

## Data

- $w_i$  demand (or buying power) of demand point  $i$ .
- $d_{ij}$  distance between demand point  $i$  and location  $j$ .
- $a_{ij}$  quality of facility  $j$  as perceived by demand point  $i$ .
- $\beta$  modulator of the distance



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## Computed data

$$u_{ij} = \frac{a_{ij}}{(d_{ij} + 1)^\beta} \quad \text{attraction that demand point } i \text{ feels towards facility } j.$$

## Variables

$$x_j = \begin{cases} 1 & \text{if a facility is open at } j \\ 0 & \text{otherwise} \end{cases}, j = 1 \dots, s$$



# MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$\begin{aligned} \max \quad & \sum_{i=1}^n w_i \frac{\sum_{j=1}^s u_{ij} x_j}{\sum_{j=1}^s u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}} \\ \text{s.t.} \quad & \sum_{j=1}^s x_j = p \\ & x_j \in \{0, 1\}, j = 1, \dots, s \end{aligned}$$



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If we denote

$$z_i = \frac{1}{\sum_{j=1}^s u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}, i = 1, \dots, n$$

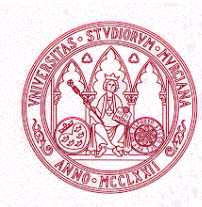
then the problem becomes



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Example: A discrete competitive location problem under the probabilistic choice rule.

$$\begin{aligned} \max \quad & \sum_{i=1}^n w_i z_i \sum_{j=1}^s u_{ij} x_j \\ \text{s.t.} \quad & z_i = \frac{1}{\sum_{j=1}^s u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}, i = 1, \dots, n \\ & \sum_{j=1}^s x_j = p \\ & x_j \in \{0, 1\}, j = 1, \dots, s \\ & z_i \geq 0, i = 1, \dots, n \end{aligned}$$





# MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

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# MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$\begin{aligned} \max \quad & \sum_{i=1}^n \sum_{j=1}^s (w_i z_i u_{ij}) x_j \\ \text{s.t.} \quad & z_i = \frac{1}{\sum_{j=1}^s u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}, i = 1, \dots, n \\ & \sum_{j=1}^s x_j = p \\ & x_j \in \{0, 1\}, j = 1, \dots, s \\ & z_i \geq 0, i = 1, \dots, n \end{aligned}$$



# MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

If we denote

$$y_{ij} = (w_i z_i u_{ij}) x_j, i = 1, \dots, n, j = 1, \dots, s$$

and taking into account that the product  $y = zx$ , where  $L \leq z \leq U$  is continuous and  $x$  binary can be linearized as

$$\begin{aligned} y &\leq Ux \\ y &\geq Lx \\ z - y &\leq U(1 - x) \\ z - y &\geq L(1 - x) \end{aligned}$$

we have that the product  $y_{ij} = (w_i z_i u_{ij}) x_j$  can be linearized as follows

$$\left. \begin{aligned} y_{ij} &\leq w_i x_j, \\ y_{ij} &\geq 0 x_j \Leftrightarrow y_{ij} \geq 0, \\ w_i z_i u_{ij} - y_{ij} &\leq w_i (1 - x_j), \\ w_i z_i u_{ij} - y_{ij} &\geq 0 (1 - x_j) \Leftrightarrow w_i z_i u_{ij} - y_{ij} \geq 0, \end{aligned} \right\} i = 1, \dots, n, j = 1, \dots, s$$



# MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$\max \sum_{i=1}^n \sum_{j=1}^s y_{ij}$$

$$\text{s.t.} \quad z_i = \frac{1}{\sum_{j=1}^s u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}, \quad i = 1, \dots, n$$

$$y_{ij} \leq w_i x_j, \quad i = 1, \dots, n, j = 1, \dots, s$$

$$y_{ij} \geq 0, \quad i = 1, \dots, n, j = 1, \dots, s$$

$$w_i z_i u_{ij} - y_{ij} \leq w_i (1 - x_j), \quad i = 1, \dots, n, j = 1, \dots, s$$

$$w_i z_i u_{ij} - y_{ij} \geq 0, \quad i = 1, \dots, n, j = 1, \dots, s$$

$$\sum_{j=1}^s x_j = p$$

$$x_j \in \{0, 1\}, \quad j = 1, \dots, s$$

$$z_i \geq 0, \quad i = 1, \dots, n$$

$$y_{ij} \geq 0, \quad i = 1, \dots, n, j = 1, \dots, s$$



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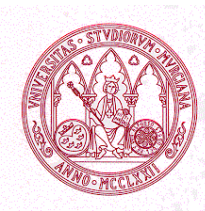
$$y_{ij} \geq 0, \quad i = 1, \dots, n, j = 1, \dots, s$$



# MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$z_i = \frac{1}{\sum_{j=1}^s u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}}$$



# MILP: modeling tricks

Example: A discrete competitive location problem under the probabilistic choice rule.

$$z_i = \frac{1}{\sum_{j=1}^s u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij}} \Leftrightarrow$$

$$z_i \left( \sum_{j=1}^s u_{ij} x_j + \sum_{j=s+1}^{s+m} u_{ij} \right) = 1$$



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$$z_i \sum_{j=1}^s u_{ij}x_j + z_i \sum_{j=s+1}^{s+m} u_{ij} = 1$$





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$$w_i z_i \sum_{j=1}^s u_{ij}x_j + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} = w_i$$



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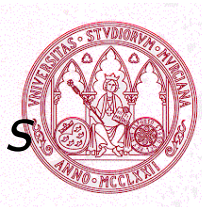
$$\sum_{j=1}^s y_{ij} + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} = w_i$$



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Example: A discrete competitive location problem under the probabilistic choice rule.

$$\begin{aligned} \max \quad & \sum_{i=1}^n \sum_{j=1}^s y_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^s y_{ij} + w_i z_i \sum_{j=s+1}^{s+m} u_{ij} = w_i, & i = 1, \dots, n \\ & y_{ij} \leq w_i x_j, & i = 1, \dots, n, j = 1, \dots, s \\ & y_{ij} \geq 0, & i = 1, \dots, n, j = 1, \dots, s \\ & w_i z_i u_{ij} - y_{ij} \leq w_i (1 - x_j), & i = 1, \dots, n, j = 1, \dots, s \\ & w_i z_i u_{ij} - y_{ij} \geq 0, & i = 1, \dots, n, j = 1, \dots, s \\ & \sum_{j=1}^s x_j = p \\ & x_j \in \{0, 1\}, & j = 1, \dots, s \\ & z_i \geq 0, & i = 1, \dots, n \\ & y_{ij} \geq 0, & i = 1, \dots, n, j = 1, \dots, s \end{aligned}$$



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