Equational description of pseudovarieties of homomorphisms

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Eilenberg correspondence (1976)

Varieties of regular languages:

closed under:
1. Boolean operations
2. quotients
3. preimages under (non-erasing) homomorphisms

Pseudovarieties of finite monoids (semigroups):

closed under:
1. homomorphic images
2. submonoids (subsemigroups)
3. finite direct products

Generalizations:

Pin (1995): not closed under complementation.

Straubing (2002): closed under preimages only for a certain class of homomorphisms.

Objects = Homomorphisms onto Finite Monoids

$\mathbf{M}$ ... the class of all onto homomorphisms $\varphi\colon \Sigma^* \twoheadrightarrow M$, Σ finite alphabet, M finite monoid

$\mathcal{C}$ ... category of homomorphisms between free monoids over finite alphabets:

Σ, Ξ finite alphabets

$\mathcal{C}(\Sigma^*, \Xi^*)$ ... a set of monoid homomorphisms $\Sigma^* \to \Xi^*$

$f \in \mathcal{C}(\Sigma^*, \Xi^*)$ & $g \in \mathcal{C}(\Xi^*, \Gamma^*) \implies gf \in \mathcal{C}(\Sigma^*, \Gamma^*)$

$\mathrm{id}_{\Sigma^*} \in \mathcal{C}(\Sigma^*, \Sigma^*)$

$\mathcal{C}$-pseudovariety of Homomorphisms

$\mathbf{V}$ subclass of $\mathbf{M}$ closed under

1. homomorphic images: $\varphi\colon \Sigma^* \twoheadrightarrow M$, $\varphi \in \mathbf{V}$, $\alpha\colon M \twoheadrightarrow N$ homo $\Longrightarrow$ $\alpha\varphi \in \mathbf{V}$

$$\begin{tikzcd} \Sigma^* \arrow[r, two heads, "\varphi\in\mathbf{V}"] \arrow[rr, two heads, bend right, "\in\mathbf{V}"'] & M \arrow[r, two heads, "\alpha"] & N \end{tikzcd}$$

2. substructures: $\varphi\colon \Sigma^* \twoheadrightarrow M$, $\varphi \in \mathbf{V}$, $f \in \mathcal{C}(\Xi^*, \Sigma^*)$ $\Longrightarrow$ $\varphi f \in \mathbf{V}$

$$\begin{tikzcd} \Xi^* \arrow[r, "f\in\mathcal{C}"] \arrow[drr, two heads, "\in\mathbf{V}"'] & \Sigma^* \arrow[r, two heads, "\varphi\in\mathbf{V}"] & M \\ & & \mathrm{Im}(\varphi f) \arrow[u, hook] \end{tikzcd}$$

3. finite products: $\varphi_i\colon \Sigma^* \twoheadrightarrow M_i$, $\varphi_i \in \mathbf{V}$, for $i = 1, \ldots, n$ $\Longrightarrow$

$$\begin{tikzcd} \Sigma^* \arrow[r, "(\varphi_1,\ldots,\varphi_n)"] \arrow[dr, two heads, "\in\mathbf{V}"'] & M_1 \times \cdots \times M_n \\ & \mathrm{Im}(\varphi_1, \ldots, \varphi_n) \arrow[u, hook] \end{tikzcd}$$

Eilenberg-Type Correspondence

$\mathcal{C}$-varieties of regular languages:

closed under:

1. Boolean operations
2. quotients
3. preimages under homomorphisms belonging to $\mathcal{C}$

Particular cases of C:

all homomorphisms . . . pseudovarieties of monoids

non-erasing homomorphisms . . . pseudovarieties of semigroups

literal (length-preserving) homomorphisms

length-multiplying homomorphisms

Implicit Operations

standard k-ary implicit operation:

- system $\tau = (\tau_M \colon M^k \to M)_{M \text{ finite monoid}}$
- for every monoid homomorphism $\alpha\colon M \to N$,

$$\begin{array}{ccc} M^k & \xrightarrow{\alpha^k} & N^k \\ {\scriptstyle \tau_M}\downarrow & & \downarrow{\scriptstyle \tau_N} \\ M & \xrightarrow[\alpha]{} & N \end{array}$$

$\mathcal{I}_k$ . . . set of all standard k-ary implicit operations

Implicit $\mathcal{C}$-operation:

- consists of partial operations defined on $\mathcal{C}$-admissible tuples
- arity is an arbitrary finite alphabet Γ
- invariant only under $\mathcal{C}$-morphisms

$\mathcal{C}$-admissible Tuples

k-tuple in M^k $\quad\sim\quad$ mapping $k \longrightarrow M$ $\quad\sim\quad$ homomorphism $k^* \longrightarrow M$

$\mathcal{C}$-admissible Γ-tuples for $\varphi\colon\ \Sigma^* \twoheadrightarrow M$:

$$\Gamma^* \xrightarrow{\ \iota\in\mathcal{C}\ } \Sigma^* \xrightarrow{\ \varphi\ } \!\!\!\!\!\rightarrow M \qquad\qquad M^{\Gamma}_{\varphi,\mathcal{C}} = \{\, \varphi\iota \mid \iota \in \mathcal{C}(\Gamma^*, \Sigma^*)\,\}$$

Examples:

$\mathcal{C}$ = literal homomorphisms: tuples of distinguished generators

$\mathcal{C}$ = length-multiplying homomorphisms:

tuples of elements which are products of distinguished generators of equal length

Morphisms in $\mathbf{M}$ = $\mathcal{C}$-morphisms

$\mathcal{C}$-morphism from $\varphi\colon \Sigma^* \twoheadrightarrow M$ to $\psi\colon \Xi^* \twoheadrightarrow N$ in $\mathbf{M}$:

monoid homomorphism $\alpha\colon M \longrightarrow N$ such that

$$\begin{tikzcd} \Sigma^* \arrow[r, dashed, "\exists f \in \mathcal{C}"] \arrow[d, two heads, "\varphi"'] & \Xi^* \arrow[d, two heads, "\psi"] \\ M \arrow[r, "\alpha"'] & N \end{tikzcd}$$

Extension to $\mathcal{C}$-admissible Γ-tuples: $\alpha^\Gamma\colon M^\Gamma_{\varphi,\mathcal{C}} \longrightarrow N^\Gamma_{\psi,\mathcal{C}}$ $\qquad \alpha^\Gamma(\varphi\iota) = \alpha\varphi\iota$

$$\begin{tikzcd} \Gamma^* \arrow[d, "\iota\in\mathcal{C}"'] & \\ \Sigma^* \arrow[r, dashed, "\exists f \in \mathcal{C}"] \arrow[d, two heads, "\varphi"'] & \Xi^* \arrow[d, two heads, "\psi"] \\ M \arrow[r, "\alpha"'] & N \end{tikzcd}$$

Implicit $\mathcal{C}$-operations

Γ-ary implicit $\mathcal{C}$-operation:

- system $\pi = (\pi_\varphi \colon M^\Gamma_{\varphi,\mathcal{C}} \to M)_{\varphi \in \mathbf{M},\ \varphi\colon \Sigma^* \twoheadrightarrow M}$
- for every $\mathcal{C}$-morphism α from $\varphi\colon \Sigma^* \twoheadrightarrow M$ to $\psi\colon \Xi^* \twoheadrightarrow N$,

$$\begin{tikzcd} M^\Gamma_{\varphi,\mathcal{C}} \arrow[r, "\alpha^\Gamma"] \arrow[d, "\pi_\varphi"'] & N^\Gamma_{\psi,\mathcal{C}} \arrow[d, "\pi_\psi"] \\ M \arrow[r, "\alpha"'] & N \end{tikzcd}$$

$\mathcal{I}^{\mathcal{C}}_\Gamma$ . . . set of all Γ-ary implicit $\mathcal{C}$-operations

Metric Monoids of Implicit $\mathcal{C}$-operations

Operation on $\mathcal{I}_\Gamma^{\mathcal{C}}$: $\quad (\pi \cdot \rho)_\varphi(\varphi\iota) = \pi_\varphi(\varphi\iota) \cdot \rho_\varphi(\varphi\iota)$

Metrics on $\mathcal{I}_\Gamma^{\mathcal{C}}$: $\quad \mathrm{d}(\pi, \rho) = 2^{-\min\{\, |M|\,;\, \exists\varphi\in\mathbf{M},\, \varphi\colon\, \Sigma^* \twoheadrightarrow M,\ \pi_\varphi \neq \rho_\varphi\, \}}$

Proposition: The metric monoids $\mathcal{I}_\Gamma^{\mathcal{C}}$ and $\mathcal{I}_{|\Gamma|}$ are isomorphic.

Proof:

τ standard $|\Gamma|$-ary implicit operation ($\tau_M\colon\, M^\Gamma \to M$)

$\rightsquigarrow$ implicit $\mathcal{C}$-operation $\pi_\varphi = \tau_M|_{M^\Gamma_{\varphi,\mathcal{C}}}$

π implicit $\mathcal{C}$-operation

$\rightsquigarrow$ standard implicit operation $\tau_M\colon\, M^\Gamma \to M$ defined by $\tau_M(\kappa) = \pi_\kappa(\kappa)$,

where $\kappa \in M^\Gamma$ understood as $\kappa\colon\, \Gamma^* \twoheadrightarrow \mathrm{Im}(\kappa)$ in $\mathbf{M}$

$$\begin{tikzcd} \Gamma^* \arrow[r, two heads, "\mathrm{id}_{\Gamma^*} \in \mathcal{C}"] \arrow[rr, bend right, two heads, "\in \mathrm{Im}(\kappa)^\Gamma_{\kappa,\mathcal{C}}"'] & \Gamma^* \arrow[r, two heads, "\kappa"] & \mathrm{Im}(\kappa) \end{tikzcd}$$

Pseudoidentities

Pseudoidentity: $\sigma \doteq \tau$, where σ, τ are k-ary implicit operations

$$M \models \sigma \doteq \tau \iff \sigma_M = \tau_M$$

Reiterman (1982):

Pseudovarieties = classes of finite monoids definable by a set of pseudoidentities.

Γ-ary $\mathcal{C}$-pseudoidentity: $\pi \doteq \rho$, where π, ρ are Γ-ary implicit $\mathcal{C}$-operations

$$\varphi \models_{\mathcal{C}} \pi \doteq \rho \iff \pi_\varphi = \rho_\varphi$$

For a set T of $\mathcal{C}$-pseudoidentities, $\mathrm{Mod}_{\mathcal{C}}(T) = \{ \varphi \in \mathbf{M} \mid \varphi \models_{\mathcal{C}} \pi \doteq \rho \; \forall \pi \doteq \rho \in T \}$.

Theorem: A subclass $\mathbf{V}$ of $\mathbf{M}$ is a $\mathcal{C}$-pseudovariety if and only if $\mathbf{V} = \mathrm{Mod}_{\mathcal{C}}(T)$ for some set of $\mathcal{C}$-pseudoidentities T.

$\mathcal{C}$-pseudoidentities for Everyday Use

Examples:

$\mathcal{C}$ = literal homomorphisms: pseudoidentities verified on distinguished generators

$\mathcal{C}$ = injective homomorphisms: pseudoidentities verified on images of codes under φ

$\mathcal{C}$ contains all isomorphisms $\Longrightarrow$

for alphabets of the same cardinality, $\mathcal{C}$-pseudoidentities are the same

$\mathcal{C}$ contains all literal homomorphisms $\Longrightarrow$

usual relationship between k-ary and $k+1$-ary $\mathcal{C}$-pseudoidentities

Stable subsemigroups

Stable subsemigroup of $\varphi\colon \Sigma^* \twoheadrightarrow M$: subsemigroup $\operatorname{Stab}\varphi = (\varphi(\Sigma))^\omega$ of M

For a pseudovariety $\mathbf{V}$ of finite semigroups, $\mathrm{S}\mathbf{V} = \{\, \varphi \in \mathbf{M} \mid \operatorname{Stab}\varphi \in \mathbf{V} \,\}$.

$\mathcal{C}$ = length-multiplying homomorphisms $\big(\forall a, b \in \Sigma\colon |f(a)| = |f(b)|\big)$

Straubing (2002): $\mathrm{S}\mathbf{V}$ is a $\mathcal{C}$-pseudovariety.

$$\mathbf{V} = \operatorname{Mod}(\pi_i \doteq \rho_i \mid i \in I) \implies$$
$$\mathrm{S}\mathbf{V} = \operatorname{Mod}_{\mathcal{C}}\big(\pi_i(x_1 y_1^{\omega-2} z_1, \ldots, x_k y_k^{\omega-2} z_k) \doteq \rho_i(x_1 y_1^{\omega-2} z_1, \ldots, x_k y_k^{\omega-2} z_k) \mid i \in I\big)$$

Example:

Barrington, Compton, Straubing, Thérien (1992):

$AC^0 \cap \mathrm{Reg}$ corresponds to $\mathrm{S}\mathbf{A} = \operatorname{Mod}_{\mathcal{C}}\big((xy^{\omega-2}z)^\omega \doteq (xy^{\omega-2}z)^{\omega+1}\big)$