Equational description of pseudovarieties of homomorphisms

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Eilenberg correspondence (1976)

Varieties of regular languages:

- closed under: 1. Boolean operations
 - 2. quotients
 - 3. preimages under (non-erasing) homomorphisms

Pseudovarieties of finite monoids (semigroups):

closed under: 1. homomorphic images

- 2. submonoids (subsemigroups)
- 3. finite direct products

Generalizations:

Pin (1995): not closed under complementation.

Straubing (2002): closed under preimages only for a certain class of homomorphisms.

Objects = Homomorphisms onto Finite Monoids

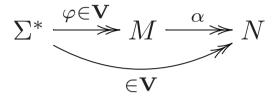
 $\mathbf{M}\dots$ the class of all onto homomorphisms $\varphi\colon \Sigma^*\twoheadrightarrow M$, Σ finite alphabet, M finite monoid

 \mathcal{C} ... category of homomorphisms between free monoids over finite alphabets:

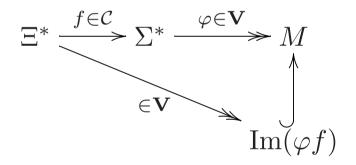
$$\begin{split} &\Sigma, \Xi \text{ finite alphabets} \\ &\mathcal{C}(\Sigma^*, \Xi^*) \dots \text{a set of monoid homomorphisms } \Sigma^* \to \Xi^* \\ &f \in \mathcal{C}(\Sigma^*, \Xi^*) \& g \in \mathcal{C}(\Xi^*, \Gamma^*) \implies gf \in \mathcal{C}(\Sigma^*, \Gamma^*) \\ &\text{id}_{\Sigma^*} \in \mathcal{C}(\Sigma^*, \Sigma^*) \end{split}$$

\mathcal{C} -pseudovariety of Homomorphisms

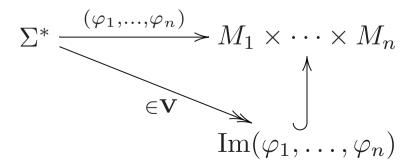
- ${\bf V}$ subclass of ${\bf M}$ closed under
- 1. homomorphic images: $\varphi \colon \Sigma^* \twoheadrightarrow M$, $\varphi \in \mathbf{V}$, $\alpha \colon M \twoheadrightarrow N$ homo $\implies \alpha \varphi \in \mathbf{V}$



2. substructures: $\varphi \colon \Sigma^* \twoheadrightarrow M$, $\varphi \in \mathbf{V}$, $f \in \mathcal{C}(\Xi^*, \Sigma^*) \implies \varphi f \in \mathbf{V}$



3. finite products: $\varphi_i \colon \Sigma^* \twoheadrightarrow M_i, \varphi_i \in \mathbf{V}$, for $i = 1, \ldots, n \implies$



Eilenberg-Type Correspondence

C-varieties of regular languages:

closed under: 1. Boolean operations

- 2. quotients
- 3. preimages under homomorphisms belonging to ${\cal C}$

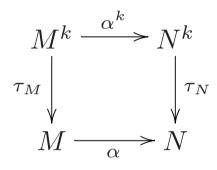
Particular cases of C:

all homomorphisms . . . pseudovarieties of monoids non-erasing homomorphisms . . . pseudovarieties of semigroups literal (length-preserving) homomorphisms length-multiplying homomorphisms

Implicit Operations

standard k-ary implicit operation:

- system $\tau = (\tau_M \colon M^k \to M)_M$ finite monoid
- $\bullet\,$ for every monoid homomorphism $\alpha\colon M\to N$,



 $\mathcal{I}_k \dots$ set of all standard k-ary implicit operations

Implicit C-operation:

- consists of partial operations defined on C-admissible tuples
- arity is an arbitrary finite alphabet Γ
- invariant only under C-morphisms

\mathcal{C} -admissible Tuples

 $k\text{-tuple} \text{ in } M^k \quad \sim \quad \text{mapping } k \to M \quad \sim \quad \text{homomorphism } k^* \to M$

 $\mathcal{C}\text{-admissible }\Gamma\text{-tuples for }\varphi\colon \Sigma^*\twoheadrightarrow M\colon$

$$\Gamma^* \xrightarrow{\iota \in \mathcal{C}} \Sigma^* \xrightarrow{\varphi} M \qquad \qquad M_{\varphi,\mathcal{C}}^{\Gamma} = \{ \varphi\iota \mid \iota \in \mathcal{C}(\Gamma^*, \Sigma^*) \}$$

Examples:

C = literal homomorphisms: tuples of distinguished generators

C = length-multiplying homomorphisms:

tuples of elements which are products of distinguished generators of equal length

Morphisms in $\mathbf{M} = \mathcal{C}$ -morphisms

 $\begin{array}{l} {\mathcal{C}}\text{-morphism from } \varphi \colon \Sigma^* \twoheadrightarrow M \text{ to } \psi \colon \Xi^* \twoheadrightarrow N \text{ in } \mathbf{M} \colon \\ \\ \text{monoid homomorphism } \alpha \colon M \to N \text{ such that} \end{array}$

$$\begin{array}{c|c} \Sigma^* \xrightarrow{\exists f \in \mathcal{C}} \Xi^* \\ \varphi \\ \psi \\ M \xrightarrow{} \alpha \end{array} \xrightarrow{} N \end{array}$$

Extension to $\mathcal C\text{-admissible}\ \Gamma\text{-tuples:}$

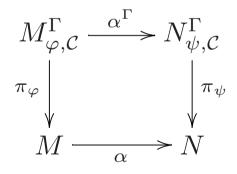
$$\alpha^{\Gamma} \colon M^{\Gamma}_{\varphi,\mathcal{C}} \to N^{\Gamma}_{\psi,\mathcal{C}} \qquad \alpha^{\Gamma}(\varphi\iota) = \alpha \varphi\iota$$

$$\begin{array}{c|c}
\Gamma^* \\
\iota \in \mathcal{C} \\
\downarrow \\
\Sigma^* \xrightarrow{\exists f \in \mathcal{C}} \\
\Sigma^* \xrightarrow{\neg \rightarrow} \Xi^* \\
\varphi \\
\downarrow \\
M \xrightarrow{\varphi} \\
M \xrightarrow{\alpha} N
\end{array}$$

Implicit C-operations

 Γ -ary implicit C-operation:

- system $\pi = (\pi_{\varphi} \colon M_{\varphi, \mathcal{C}}^{\Gamma} \to M)_{\varphi \in \mathbf{M}, \varphi \colon \Sigma^* \twoheadrightarrow M}$
- for every $\mathcal C\text{-morphism}\;\alpha$ from $\varphi\colon \Sigma^*\twoheadrightarrow M$ to $\psi\colon \Xi^*\twoheadrightarrow N$,



 $\mathcal{I}_{\Gamma}^{\mathcal{C}}$... set of all Γ -ary implicit \mathcal{C} -operations

Metric Monoids of Implicit C-operations

Operation on $\mathcal{I}_{\Gamma}^{\mathcal{C}}$: $(\pi \cdot \rho)_{\varphi}(\varphi \iota) = \pi_{\varphi}(\varphi \iota) \cdot \rho_{\varphi}(\varphi \iota)$

Metrics on $\mathcal{I}_{\Gamma}^{\mathcal{C}}$: $d(\pi, \rho) = 2^{-\min\{|M|; \exists \varphi \in \mathbf{M}, \varphi \colon \Sigma^* \twoheadrightarrow M, \pi_{\varphi} \neq \rho_{\varphi}\}}$

Proposition: The metric monoids $\mathcal{I}_{\Gamma}^{\mathcal{C}}$ and $\mathcal{I}_{|\Gamma|}$ are isomorphic.

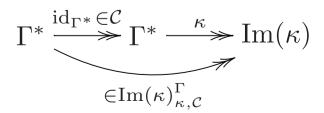
Proof:

au standard $|\Gamma|$ -ary implicit operation ($au_M \colon M^{\Gamma} \to M$)

 \rightsquigarrow implicit C-operation $\pi_{\varphi} = \tau_M |_{M_{\varphi,C}^{\Gamma}}$

 π implicit $\mathcal C$ -operation

 \rightsquigarrow standard implicit operation $\tau_M \colon M^{\Gamma} \to M$ defined by $\tau_M(\kappa) = \pi_{\kappa}(\kappa)$, where $\kappa \in M^{\Gamma}$ understood as $\kappa \colon \Gamma^* \twoheadrightarrow \operatorname{Im}(\kappa)$ in \mathbf{M}



Pseudoidentities

Pseudoidentity: $\sigma \doteq \tau$, where σ, τ are k-ary implicit operations $M \models \sigma \doteq \tau \iff \sigma_M = \tau_M$

Reiterman (1982):

Pseudovarieties = classes of finite monoids definable by a set of pseudoidentities.

Γ-ary *C*-pseudoidentity: $\pi \doteq \rho$, where π , ρ are Γ-ary implicit *C*-operations $\varphi \models_{\mathcal{C}} \pi \doteq \rho \iff \pi_{\varphi} = \rho_{\varphi}$ For a set *T* of *C*-pseudoidentities, $Mod_{\mathcal{C}}(T) = \{ \varphi \in \mathbf{M} \mid \varphi \models_{\mathcal{C}} \pi \doteq \rho \ \forall \pi \doteq \rho \in T \}.$

Theorem: A subclass V of M is a C-pseudovariety if and only if $V = Mod_{\mathcal{C}}(T)$ for some set of C-pseudoidentities T.

$\mathcal C\text{-}pseudoidentities$ for Everyday Use

Examples:

C = literal homomorphisms: pseudoidentities verified on distinguished generators C = injective homomorphisms: pseudoidentities verified on images of codes under φ

${\mathcal C}$ contains all isomorphisms \implies

for alphabets of the same cardinality, C-pseudoidentities are the same C contains all literal homomorphisms \implies usual relationship between k-ary and k + 1-ary C-pseudoidentities

Stable subsemigroups

Stable subsemigroup of $\varphi \colon \Sigma^* \twoheadrightarrow M$: subsemigroup $\operatorname{Stab} \varphi = (\varphi(\Sigma))^{\omega}$ of MFor a pseudovariety V of finite semigroups, $\operatorname{SV} = \{ \varphi \in \mathbf{M} \mid \operatorname{Stab} \varphi \in \mathbf{V} \}.$

C = length-multiplying homomorphisms $(\forall a, b \in \Sigma : |f(a)| = |f(b)|)$ Straubing (2002): SV is a C-pseudovariety.

$$\mathbf{V} = \operatorname{Mod}(\pi_i \doteq \rho_i \mid i \in I) \Longrightarrow$$

$$\mathbf{SV} = \operatorname{Mod}_{\mathcal{C}}\left(\pi_i(x_1 y_1^{\omega^{-2}} z_1, \dots, x_k y_k^{\omega^{-2}} z_k) \doteq \rho_i(x_1 y_1^{\omega^{-2}} z_1, \dots, x_k y_k^{\omega^{-2}} z_k) \mid i \in I\right)$$

Example:

Barrington, Compton, Straubing, Thérien (1992): $AC^0 \cap \text{Reg corresponds to } SA = Mod_{\mathcal{C}} \left((xy^{\omega-2}z)^{\omega} \doteq (xy^{\omega-2}z)^{\omega+1} \right)$