Trace languages and the variety **DA**

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University of Stuttgart

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Introduction

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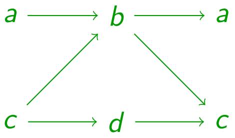
word

trace



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- Linear temporal logics (Ebinger 1994, Thiagarajan/Walukiewicz 1997, Diekert/Gastin 2005)

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- Traces have a graphical interpretation

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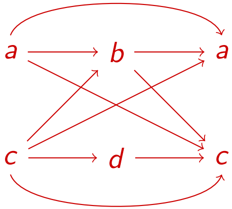

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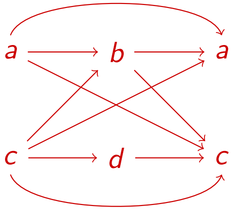

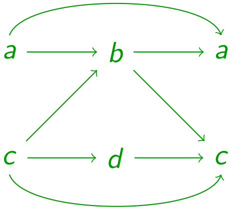

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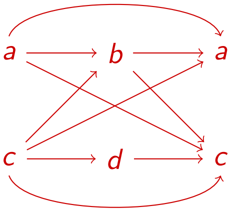

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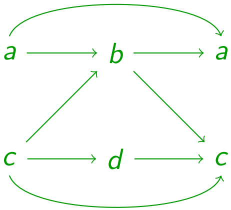
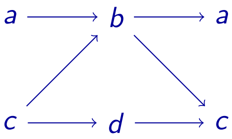


Hasse diagram:



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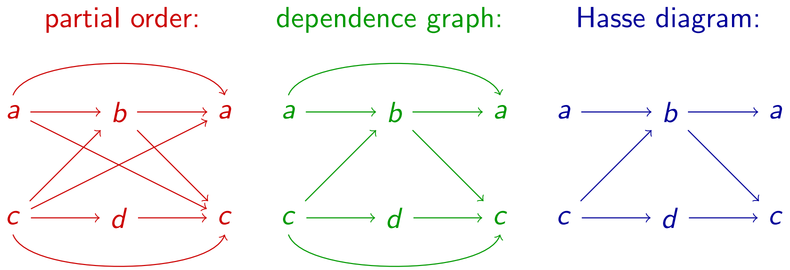
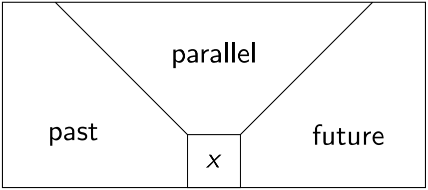


In t, the node labeled with d is parallel to all a's and b's.

Why are things more complicated for traces?

For each node x of a trace we can define three disjoint regions:



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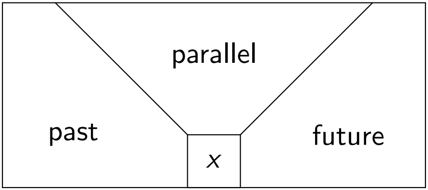

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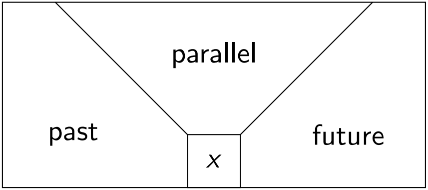

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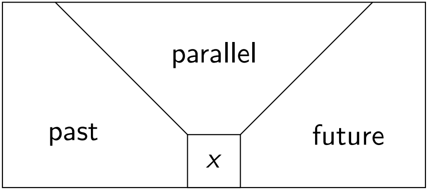

- Combinatorically: more involved due to concurrency.
- Algebraically: more involved due to lack of univeral property.
- M recognizes $L \subseteq \mathbb{M}(\Sigma, I) \Rightarrow M(L)$ divides M. *(but not $\Leftarrow$)*

Temporal logic (syntax)

In this context, a *temporal formula* is a term of the form

$$\varphi \ ::= a \mid \neg\varphi \mid (\varphi_1 \vee \varphi_2) \mid (\varphi_1 \wedge \varphi_2) \mid \\ \mathsf{XF}\varphi \mid \mathsf{YP}\varphi \mid \mathsf{PAR}\varphi \mid \mathsf{M}\varphi$$

where $a \in \Sigma$.

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t, x \models \mathsf{M}\varphi &\iff \exists y \text{ somewhere}\colon\ t, y \models \varphi
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Since membership is decidable for the variety **DA**, the above characterizations are decidable.

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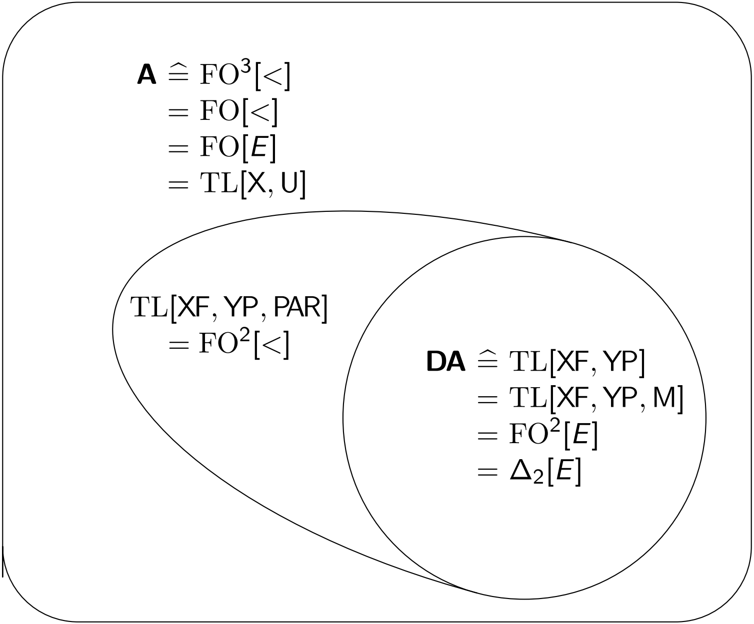

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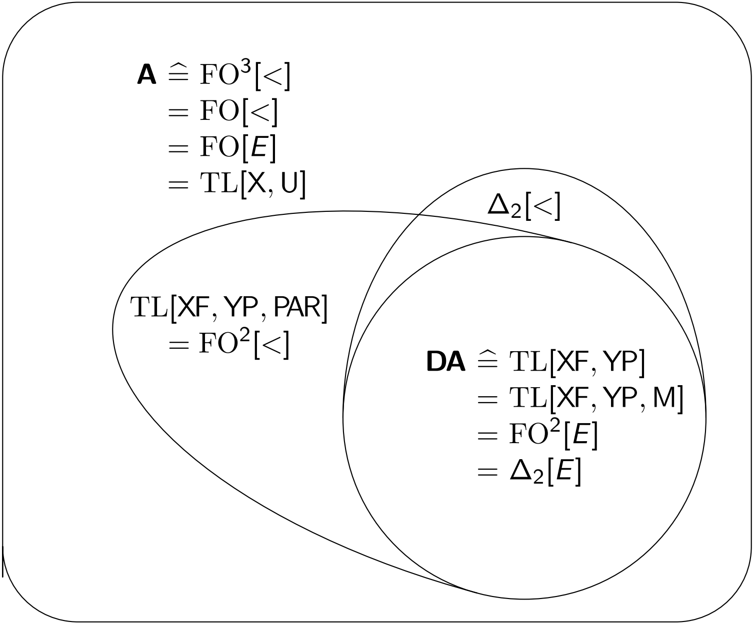

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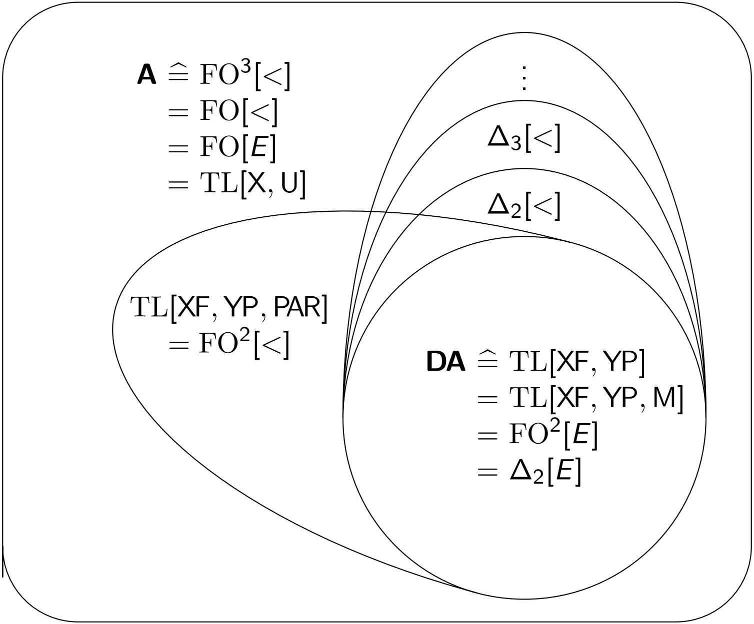

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- $\Sigma^* ab \Sigma^* \cup \Sigma^* ba \Sigma^* \subseteq \Sigma^*$ is NOT expressible in $\mathrm{FO}^2[<]$ over words.

Complexity of the satisfiability problem

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Theorem (Diekert,Horsch,K.)

The following problem is **NP***-complete:*

Input: $\varphi \in \mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{M}]$ *and* (Σ, I).
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Complexity of the satisfiability problem

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The following problem is **PSPACE***-complete:*

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Proof idea for **PSPACE**-hardness of SAT for $\mathrm{TL}[\mathsf{F}, \mathsf{PAR}]$

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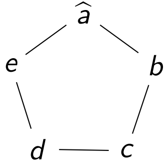
- Let $\widehat{a}$ be a shorthand for either a or $\overline{a}$ and consider the following trace monoid.

$I =$



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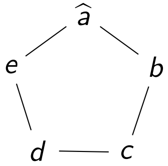
$I =$ 

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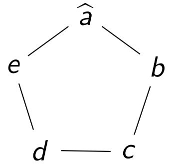
$I =$ 

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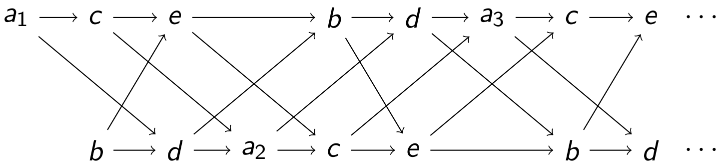

Outlook and open problems

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Outlook and open problems

- Is expressibility in $\mathrm{FO}^2[<]$ decidable?
- Is expressibility in $\Delta_2[<]$ decidable?
- How difficult is the satisfiability problem for $FO^2[E]$ and $FO^2[<]$?
- What is the largest class $\mathcal{C}$ such that $\mathrm{FO}^2[<]$ is a $\mathcal{C}$-variety?

Thank you!