Products of tree automata with an application to temporal logic

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Joint work with Szabolcs Iván

Plan

- We associate a branching temporal logic FTL($\mathcal{L}$) with each class $\mathcal{L}$ of (regular) tree languages.

- Provide an algebraic characterization of the expressive power of the logics FTL($\mathcal{L}$) under some natural assumptions.

- We establish a formal connection between temporal logics and algebra.

- Applications: Szabolcs Iván

Ranked Alphabets and Trees

Definition A rank type R is a finite set of natural numbers *containing* 0. A ranked alphabet Σ of rank type R is a disjoint union of finite *nonempty* sets Σ_n, $n \in R$.

We fix a rank type R and only consider ranked alphabets of rank type R.

Notation When Σ is a ranked alphabet, the set of all ground Σ-terms or closed Σ-terms is denoted T_Σ. It is well-known that T_Σ is the carrier of the initial Σ-algebra.

We will keep calling terms as trees.

Tree Automata

Suppose that Σ is a ranked alphabet.

A Σ-tree automaton is a finite Σ-algebra that has no nontrivial subalgebra.

When $\mathbb{A}$ is a Σ-algebra each $t \in T_\Sigma$ evaluates to an element $t_\mathbb{A} \in A$. We say that a language $L \subseteq T_\Sigma$ is recognizable by a Σ-tree automaton $\mathbb{A}$ if there is a set $F \subseteq A$ with

$$L = \{t \in T_\Sigma : t_\mathbb{A} \in F\}.$$

A tree language $L \subseteq T_\Sigma$ is called regular if it is recognizable by a tree automaton.

Formulas

Our logic contains formulas over each ranked set Σ.

- For each letter $\sigma \in \Sigma$, p_σ is a formula.
- If φ and ψ are formulas, then $\neg\varphi$, $\varphi \vee \psi$, $\varphi \wedge \psi$, ... are formulas.
- If Δ is a ranked set, $(\varphi_\delta)_{\delta \in \Delta}$ is a family of formulas, $K \subseteq T_\Delta$ is a pattern language, then

$$K(\delta \mapsto \varphi_\delta)_{\delta \in \Delta}$$

is a formula.

$\mathcal{L}$ a class of regular languages – FTL($\mathcal{L}$)
$\mathbf{K}$ a class of finite tree automata – FTL($\mathbf{K}$)

Semantics 1

Suppose that φ is a formula over Σ and $t \in T_\Sigma$. Then $t \models \varphi$ if

1. $\varphi = p_\sigma$ and the root of t is labeled σ, or

2. $\varphi = \neg\psi$ and $t \models \psi$ does not hold, or $\varphi = \psi \vee \psi'$ and $t \models \psi$ or $t \models \psi'$ holds, or, ..., or

3. $\varphi = K(\delta \mapsto \varphi_\delta)_{\delta \in \Delta}$ and the characteristic tree $\hat{t} \in T_\Delta$ determined by t and $(\varphi_\delta)_{\delta \in \Delta}$ is in K.

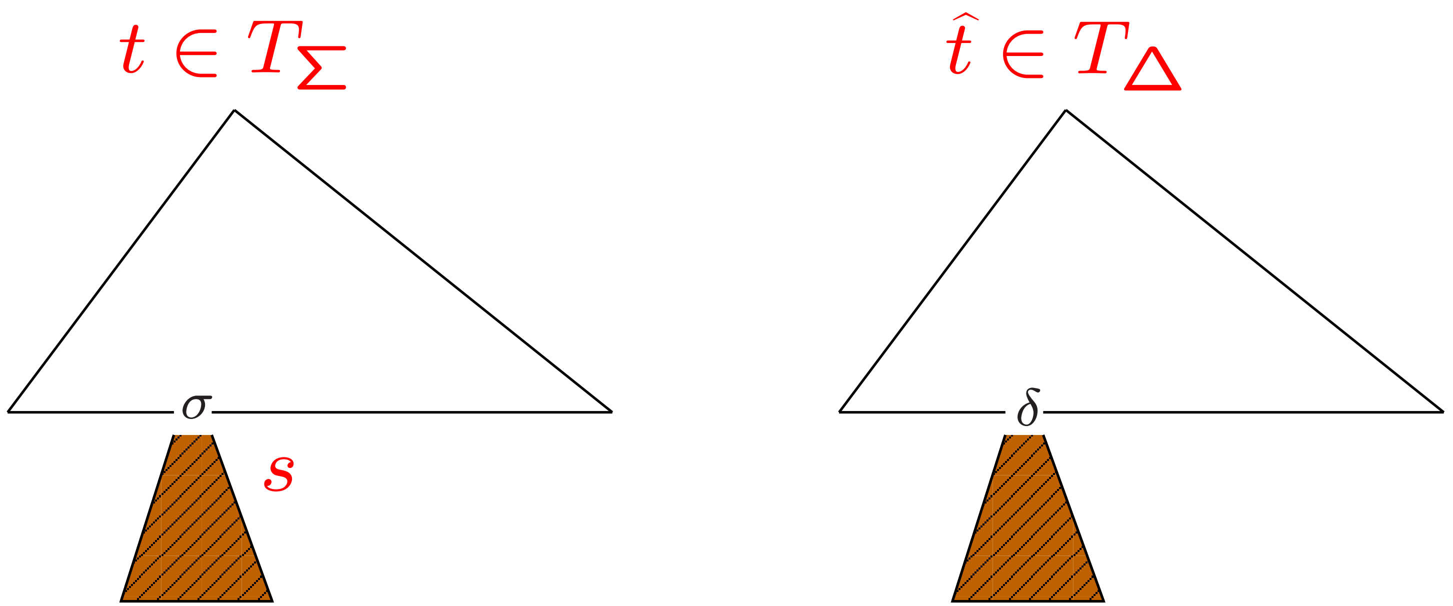
Semantics 2

$\widehat{t}$ differs from t only in the labeling of the vertices.

Suppose that v is a vertex labeled in t by a letter $\sigma \in \Sigma_n$. Then the label of v in $\widehat{t}$ is the first letter $\delta \in \Delta_n$ such that the subtree $t|_v$ of t rooted at v satisfies φ_δ, $t|_v \models \varphi_\delta$, if there is such a letter. Otherwise the label is the last letter in Δ_n.

We assume a fixed lexicographic order on each ranked alphabet. (But the ordering is not important ...)

The Characteristic Tree



$$s \models \varphi_{\delta}$$

Example

$\Delta_n = \{\uparrow_n, \downarrow_n\}$, $n \in R$.

$$
\begin{aligned}
L_{\mathsf{EF}^*} &= \{s \in T_\Delta : \exists v\, \exists n\ \ s(v) = \uparrow_n\} \\
L_{\mathsf{EG}^*} &= \{s \in T_\Delta : \exists p\, \forall v \in p\, \exists n\ \ s(v) = \uparrow_n\}
\end{aligned}
$$

Then for any $t \in T_\Sigma$, $t \models L_{\mathsf{EF}^*}(\uparrow_n \mapsto \varphi, \downarrow_n \mapsto \neg\varphi)_{n \in R}$ iff for some vertex v, $t|_v \models \varphi$: EF modality.

And $t \models L_{\mathsf{EG}^*}(\uparrow_n \mapsto \varphi, \downarrow_n \mapsto \neg\varphi)_{n \in R}$ iff there is a path p such that for each vertex $v \in p$, $t|_v \models \varphi$: EG modality.

Earlier Result

Theorem Suppose that $\mathcal{L}$ is a class of regular tree languages with:

1. Each quotient of any language in $\mathcal{L}$ is definable in FTL($\mathcal{L}$).
2. The “next modalities are expressible in FTL($\mathcal{L}$)”.

Then a language $L \subseteq T_\Sigma$ is definable in FTL($\mathcal{L}$) iff it is regular and its minimal automaton is in the least pseudovariety closed under the cascade product which contains the 2-state reset automaton $\mathbb{D}_0$ and the minimal automata of the languages in $\mathcal{L}$.

This Talk

Theorem Suppose that $\mathcal{L}$ is a class of regular tree languages such that each quotient of any language in $\mathcal{L}$ is definable in $\mathbf{FTL}(\mathcal{L})$. Then a language $L \subseteq T_\Sigma$ is definable in $\mathbf{FTL}(\mathcal{L})$ iff it is regular and its minimal automaton is in the least pseudovariety closed under the Moore product containing the 2-state reset automaton $\mathbb{D}_0$ and the minimal automata of the languages in $\mathcal{L}$.

Corollary Suppose that $\mathbf{K}$ is a class of finite tree automata. Then a language $L \subseteq T_\Sigma$ is definable in FTL$(\mathbf{K})$ iff it is regular and its minimal automaton is in the least pseudovariety closed under the Moore product containing $\mathbb{D}_0$ and $\mathbf{K}$.

Cascade Product (Ricci)

Let $\mathbb{A}$ be a Σ-tree automaton, $\mathbb{B}$ a Δ-tree automaton, and α a family of functions $\alpha_n : A^n \times \Sigma_n \to \Delta_n$, $n \in R$. The cascade product $\mathbb{A} \times_\alpha \mathbb{B}$ determined by α is the least subalgebra of the Σ-algebra with carrier $A \times B$ and operations

$$\sigma((a_1, b_1), \ldots, (a_n, b_n)) = (\sigma(a_1, \ldots, a_n), \delta(b_1, \ldots, b_n)),$$

where $\delta = \alpha_n(a_1, \ldots, a_n, \sigma)$, for all $((a_1, b_1), \ldots, (a_n, b_n)) \in A \times B$, $\sigma \in \Sigma_n$, $n \in R$.

The direct product is clearly a special case of the cascade product.

The cascade product is closely related to the wreath product of function clones (VanderWerf).

Moore Product

Let $\mathbb{A}$ be a Σ-tree automaton, $\mathbb{B}$ a Δ-tree automaton. We call a cascade product $\mathbb{A} \times_\alpha \mathbb{B}$ a Moore product if there exists a family of functions $\beta_n : A \times \Sigma_n \to \Delta_n$, $n \in R$ such that

$$\alpha_n(a_1, \ldots, a_n, \sigma) \;=\; \beta_n(\sigma(a_1, \ldots, a_n), \sigma)$$

for all $a_1, \ldots, a_n \in A$ and $\sigma \in \Sigma_n$, $n \in R$.

CTL

$$\text{CTL} = \text{FTL}(\{\mathbb{E}_U\}) \quad \text{where}$$

$\mathbb{E}_U$ is defined on the set $\{0,1\}$ by

$$\begin{aligned} \uparrow_n (b_1,\ldots,b_n) &= 1 \\ \mu_n(b_1,\ldots,b_n) &= b_1 \vee \ldots \vee b_n \\ \downarrow_n (b_1,\ldots,b_n) &= 0, \end{aligned}$$

for all $b_1,\ldots,b_n \in \{0,1\}$, $n \in R$.

Theorem A tree language is definable in CTL iff its minimal tree automaton is in the least pseudovariety containing $\mathbb{E}_U$ which is closed under the cascade product.

CTL(EF*)

$$\text{CTL}(\text{EF}^*) = \text{FTL}(\{\mathbb{E}_{\text{EF}^*}\}) \quad \text{where}$$

$\mathbb{E}_{\text{EF}^*}$ is defined on the set $\{0,1\}$ by

$$\begin{aligned} \uparrow_n (b_1, \ldots, b_n) &= 1 \\ \downarrow_n (b_1, \ldots, b_n) &= b_1 \vee \ldots \vee b_n \end{aligned}$$

for all $b_1, \ldots, b_n \in \{0,1\}$, $n \in R$.

Theorem A tree language is definable in CTL(EF*) iff its minimal tree automaton is in the least pseudovariety containing $\mathbb{E}_{\text{EF}^*}$ and $\mathbb{D}_0$ which is closed under the Moore product.

Wreath Product of Clones

Suppose that (A, S) and (B, T) are function clones. We define a clone on the set $A \times B$, denoted $(A \times B, S\, w\, T)$. The n-ary functions in $S\, w\, T$ are the functions (s, f), where $s : A^n \to A$ in S and f maps A^n into the set of n-ary functions in T, defined by

$$(s, f)((a_1, b_1), \ldots, (a_n, b_n)) \;=\; (s(a_1, \ldots, a_n), f^{(a_1,\ldots,a_n)}(b_1, \ldots, b_n)),$$

for all $(a_i, b_i) \in A \times B$, $i \in [n]$.

Then $(s, f) = (s', f')$ iff $s = s'$ and $f = f'$. Moreover, the functions (s, f) form a clone: they are closed under composition and contain the projections.