Myhill-Nerode theory
for fuzzy languages and automata

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The original Zadeh’s definition of a fuzzy set is:

A fuzzy subset of a set $A$ is any mapping $f : A \to [0, 1]$, where $[0, 1]$ is the real unit closed interval.

For $x \in A$, the value $f(x)$ is interpreted as

the degree of membership of $x$ to $f$

that is

the truth value of the proposition $'x \in f'$

Of course, if $f$ takes values only in the set $\{0, 1\}$, then it is treated as an ordinary crisp subset of $A$. 
Nowadays, various more general structures of truth values are used instead of $[0, 1]$.

- Gödel algebras (algebraic counterpart of the Gödel logic)
- MV-algebras or Wajsberg algebras (Łukasiewicz logic)
- Product algebras (Product logic)
- BL-algebras (Basic fuzzy logic)
- Heyting algebras (Intuitionistic logic)
- Complete residuated lattices (Residuated logic)
- Complete orthomodular lattices (Quantum logic), and others.

Here we work with **complete residuated lattices**, which include the first five kinds of the above mentioned algebras as special cases.
A residuated lattice is an algebra $\mathcal{L} = (L, \land, \lor, \otimes, \rightarrow, 0, 1)$ satisfying the following conditions

(L1) $(L, \land, \lor, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,
(L2) $(L, \otimes, 1)$ is a commutative monoid with the unit 1,
(L3) $\otimes$ and $\rightarrow$ form an adjoint pair, i.e., they satisfy the adjunction property: for all $x, y, z \in L$,

$$x \otimes y \leq z \iff x \leq y \rightarrow z.$$

If, in addition, $(L, \land, \lor, 0, 1)$ is a complete lattice, then $\mathcal{L}$ is called a complete residuated lattice.
The operation $\otimes$ is called a \textit{multiplication}, and $\rightarrow$ a \textit{residuum}.

They are intended for modeling the \textit{conjunction} and \textit{implication} of the corresponding logical calculus.

Supremum $\lor$ and infimum $\land$ are intended for modeling of the \textit{general} and \textit{existential quantifier}, respectively.

A \textit{biresiduum} or \textit{biimplication} in $\mathcal{L}$ is an operation $\leftrightarrow$ defined by

$$x \leftrightarrow y = (x \rightarrow y) \land (y \rightarrow x),$$

It is used for modeling the \textit{equivalence} of truth values.

A \textit{negation} in $\mathcal{L}$ is a unary operation $\neg$ defined by

$$\neg x = x \rightarrow 0.$$
The most studied and applied set of truth values is the real unit interval $[0, 1]$ with

$$x \land y = \min(x, y), \quad x \lor y = \max(x, y),$$

and three important pairs of adjoint operations:

- Łukasiewicz operations

$$x \otimes y = \max(x + y - 1, 0), \quad x \rightarrow y = \min(1 - x + y, 1),$$

$$x \leftrightarrow y = 1 - |x - y|, \quad \neg x = 1 - x;$$
Łukasiewicz, Product and Gödel Operations

**Product operations**

\[ x \otimes y = x \cdot y, \quad x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}, \]

\[ x \leftrightarrow y = \frac{\min(x, y)}{\max(x, y)}, \quad \neg x = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x > 0 \end{cases}; \]

**Gödel operations**

\[ x \otimes y = \min(x, y), \quad x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}, \]

\[ x \leftrightarrow y = \begin{cases} 1 & \text{for } x = y \\ \min(x, y) & \text{otherwise} \end{cases}, \quad \neg x = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x > 0 \end{cases}; \]
Another important set of truth values is
\[ \{a_0, a_1, \ldots, a_n\}, \quad 0 = a_0 < \cdots < a_n = 1, \]

with
\[ a_k \otimes a_l = a_{\max(k+l-n,0)}, \quad a_k \rightarrow a_l = a_{\min(n-k+l,n)}. \]

A special case of the latter algebras is the two-element Boolean algebra of classical logic with the support \( \{0, 1\} \).

The only adjoint pair on it consist of the classical conjunction and implication operations.
Let $\mathcal{L}$ will be a complete residuated lattice.

A **fuzzy subset** of a set $A$ over $\mathcal{L}$, or simply a fuzzy subset of $A$, is any mapping $f : A \rightarrow \mathcal{L}$.

The set $\mathcal{F}(A)$ of all fuzzy subsets of $A$ we call the **fuzzy power set** of $A$.

For $f, g \in \mathcal{F}(A)$ we define

**Equality:** $f = g$ if and only if $f(x) = g(x)$, for every $x \in A$

**Inclusion:** $f \leq g$ if and only if $f(x) \leq g(x)$, for every $x \in A$

The **meet** or **intersection** $\bigwedge_{i \in I} f_i$ and the **join** or **union** $\bigvee_{i \in I} f_i$ of a family $\{f_i\}_{i \in I} \subseteq \mathcal{F}(A)$ are mappings from $A$ into $\mathcal{L}$ defined by

$$\left( \bigwedge_{i \in I} f_i \right)(x) = \bigwedge_{i \in I} f_i(x), \quad \left( \bigvee_{i \in I} f_i \right)(x) = \bigvee_{i \in I} f_i(x).$$
The crisp part of a fuzzy subset $f \in \mathcal{F}(A)$ is a crisp set defined by

$$\hat{f} = \{ x \in A | f(x) = 1 \}.$$

A fuzzy relation on $A$ is any mapping $\mu : A \times A \rightarrow L$, i.e., any fuzzy subset of $A \times A$.

Hence, the equality, inclusion, joins, meets and ordering of fuzzy relations are defined as for fuzzy sets.
A fuzzy relation $\mu$ on $A$ is said to be

(R) reflexive or fuzzy reflexive if $\mu(x, x) = 1$, for every $x \in A$;

(S) symmetric or fuzzy symmetric if $\mu(x, y) = \mu(y, x)$, for all $x, y \in A$;

(T) transitive or fuzzy transitive if for all $x, a, y \in A$

$$
\mu(x, a) \otimes \mu(a, y) \leq \mu(x, y).
$$

A reflexive, symmetric and transitive fuzzy relation on $A$ is called a fuzzy equivalence relation, or just a fuzzy equivalence, on $A$.

With respect to the ordering of fuzzy relations, the set $E(A)$ of all fuzzy equivalence relations on a set $A$ is a complete lattice.
Let $\mu$ be a fuzzy equivalence relation on $A$.

For each $a \in A$ we define $\mu_a \in \mathcal{F}(A)$, i.e., $\mu_a : A \to L$, by:

$$\mu_a(x) = \mu(a, x), \quad \text{for every } x \in A.$$ 

We call $\mu_a$ a fuzzy equivalence class, or just an equivalence class, of $\mu$ determined by the element $a$.

The set $A/\mu = \{\mu_a \mid a \in A\}$ is called the factor set of $A$ w.r.t. $\mu$. Its cardinality $|A/\mu|$ is called the index of $\mu$, in notation $\text{ind}(\mu)$.

A fuzzy subset $f \in \mathcal{F}(A)$ is said to be extensional w.r.t. $\mu$ if

$$f(x) \otimes \mu(x, y) \leq f(y),$$

for all $x, y \in A$. 

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A fuzzy automaton over \( \mathcal{L} \), or simply a fuzzy automaton, is a triple \( \mathcal{A} = (A, X, \delta) \), where

- \( A \) and \( X \) are sets, called respectively a set of states and an input alphabet,
- \( \delta : A \times X \times A \rightarrow L \) is a fuzzy subset of \( A \times X \times A \), called a fuzzy transition function.

We will always assume that the input alphabet \( X \) is finite, but from methodological reasons we will allow the set of states \( A \) to be infinite.

A fuzzy automaton whose set of states is finite is called a finite fuzzy automaton.
Let $X^*$ denote the free monoid over the alphabet $X$.

The mapping $\delta$ can be extended up to a mapping $\delta^* : A \times X^* \times A \to L$ as follows: If $a, b \in A$ and $e \in X^*$ is the empty word, then

$$\delta^*(a, e, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases},$$

and if $a, b \in A$, $u \in X^*$ and $x \in X$, then

$$\delta^*(a, ux, b) = \bigvee_{c \in A} \delta^*(a, u, c) \otimes \delta(c, x, b).$$

We have that for all $a, b \in A$ and $u, v \in X^*$,

$$\delta^*(a, uv, b) = \bigvee_{c \in A} \delta^*(a, u, c) \otimes \delta^*(c, v, b).$$
If $\delta$ is a crisp subset of $A \times X \times A$, i.e., $\delta : A \times X \times A \rightarrow \{0, 1\}$, then $\mathcal{A}$ is an ordinary crisp **nondeterministic automaton**.

Moreover, if $\delta$ is a mapping of $A \times X$ into $A$, then $\mathcal{A}$ is an ordinary **deterministic automaton**.

Evidently, in these two cases we have that $\delta^*$ is also a crisp subset of $A \times X^* \times A$, and a mapping of $A \times X^*$ into $A$, respectively.

Let $\mathcal{A} = (A, X, \delta)$ be a fuzzy automaton.

Then $\hat{\delta}$, the crisp part of $\delta$, is a crisp subset of $A \times X \times A$, and $\hat{\mathcal{A}} = (A, X, \hat{\delta})$ is a nondeterministic automaton, called the **crisp part** of the fuzzy automaton $\mathcal{A}$. 
A fuzzy language is any fuzzy subset of a free monoid.

A fuzzy automaton \( \mathcal{A} = (A, X, \delta) \) is said to recognize a fuzzy language \( f \in \mathcal{F}(X^*) \), by a fuzzy set \( \sigma \) of initial states and a fuzzy set \( \tau \) of final states, if for any \( u \in X^* \),

\[
f(u) = \bigvee_{a,b \in A} \sigma(a) \otimes \delta^*(a, u, b) \otimes \tau(b).
\]

Here we consider only those fuzzy automata having a single crisp initial state \( \{a_0\} \).

In this case, \( \mathcal{A} \) is said to recognize a fuzzy language \( f \in \mathcal{F}(X^*) \) by a crisp initial state \( a_0 \) and a fuzzy set \( \tau \) of final states, if for any \( u \in X^* \),

\[
f(u) = \bigvee_{b \in A} \delta^*(a_0, u, b) \otimes \tau(b).
\]
In particular, if $\mathcal{A}$ is a deterministic automaton, i.e., $\delta : A \times X \rightarrow A$, then it recognizes a fuzzy language $f \in \mathcal{F}(X^*)$ by a crisp initial state $a_0$ and a fuzzy set $\tau$ of final states, if for any $u \in X^*$,

$$f(u) = \tau(\delta^*(a_0, u)).$$
A fuzzy equivalence relation $\mu$ on a semigroup $S$ is

- a fuzzy left congruence, if $\mu(a, b) \leq \mu(xa, xb)$, for all $a, b, x \in S$,
- a fuzzy right congruence, if $\mu(a, b) \leq \mu(ax, bx)$, for all $a, b, x \in S$,
- a fuzzy congruence, if it is both fuzzy left and right congruence.

For a fuzzy equivalence relation $\mu$ on a semigroup $S$ we define fuzzy relations $\mu^0_l$, $\mu^0_r$ and $\mu^0$ on $S$ by

$$
\mu^0_l(a, b) = \bigwedge_{x \in S^1} \mu(xa, xb), \quad \mu^0_r(a, b) = \bigwedge_{x \in S^1} \mu(ax, bx),
$$

$$
\mu^0(a, b) = \bigwedge_{x, y \in S^1} \mu(xay, xby),
$$

for all $a, b \in S$. 

We have that

1. $\mu_0^l$ is the largest fuzzy left congruence on $S$ contained in $\mu$;
2. $\mu_0^r$ is the largest fuzzy right congruence on $S$ contained in $\mu$;
3. $\mu_0^l$ is the largest fuzzy congruence on $S$ contained in $\mu$;
4. if $\mu$ is a fuzzy right (left) congruence, then $\mu^0 = \mu_0^l$ ($\mu^0 = \mu_0^r$).

Therefore, the mappings $\mu \mapsto \mu_0^l$, $\mu \mapsto \mu_0^r$ and $\mu \mapsto \mu^0$ are opening operators on the lattice of fuzzy equivalence relations on $S$, so

1. $\mu_0^l$ is called the fuzzy left congruence opening of $\mu$,
2. $\mu_0^r$ is the fuzzy right congruence opening of $\mu$,
3. $\mu^0$ is the fuzzy congruence opening of $\mu$.  

Now, we will consider fuzzy right congruences on free monoids. Let $\mu$ be a fuzzy right congruence on a free monoid $X^*$ and let $A_\mu = X^*/\mu$. We define a mapping $\delta_\mu : A_\mu \times X \times A_\mu \to L$ by

$$\delta_\mu(\mu_u, x, \mu_v) = \mu_{ux}(v),$$

for all $u, v \in X^*$ and $x \in X$.

The mapping $\delta_\mu$ is well-defined, and $A_\mu = (A_\mu, X, \delta_\mu)$ is a fuzzy automaton, called a fuzzy right congruence automaton associated with $\mu$.

The transition function can be extended to a function $\delta^*_\mu : A_\mu \times X^* \times A_\mu \to L$ by:

$$\delta^*_\mu(\mu_u, p, \mu_v) = \mu_{u_p}(v) = \mu(u_p, v),$$

for all $u, v \in X^*$ and $p \in X^+$. 

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Note that $\delta^*_\mu$ can be also characterized as follows:

$$\delta^*_\mu(\mu_u, p, \mu_v) = \bigwedge_{w \in X^*} \mu_{up}(w) \leftrightarrow \mu_v(w) = \bigvee_{w \in X^*} \mu_{up}(w) \otimes \mu_v(w),$$

for all $u, v, p \in X^*$.

These equalities can be interpreted as

"$\delta^*_\mu(\mu_u, p, \mu_v)$ is the degree of equality of the classes $\mu_{up}$ and $\mu_v$", or

"$\delta^*_\mu(\mu_u, p, \mu_v)$ is the degree of intersection of the classes $\mu_{up}$ and $\mu_v$"

A fuzzy right congruence automaton $A_\mu$ is usually considered as a fuzzy automaton with a crisp initial state $\mu_e$, and then we write

$$A_\mu = (A_\mu, X, \mu_e, \delta_\mu).$$
When we recognize fuzzy languages by $A_\mu$ we always assume that $A_\mu$ starts from the crisp initial state $\mu_e$.

We say that the automaton $A_\mu$ recognizes a fuzzy language $f \in \mathcal{F}(X^*)$ by a fuzzy set of final states $\tau \in \mathcal{F}(A_\mu)$ if

$$f(u) = \bigvee_{\xi \in A_\mu} \delta^*_\mu(\mu_e, u, \xi) \otimes \tau(\xi) = \bigvee_{w \in X^*} \delta^*_\mu(\mu_e, u, \mu_w) \otimes \tau(\mu_w),$$

for each $u \in X^*$.

Our main result is

**Theorem 2.** Let $\mu$ be a fuzzy right congruence on a free monoid $X^*$. A fuzzy language $f \in \mathcal{F}(X^*)$ is recognized by $A_\mu$ if and only if $f$ is extensional with respect to $\mu$. 
As known, to any crisp right congruence $\pi$ on a free monoid $X^*$ we can associate a crisp deterministic automaton $A_\pi = (A_\pi, X, \lambda_\pi)$, where $A_\pi = X^*/\pi$ and a mapping $\lambda_\pi : A_\pi \times X \rightarrow A_\pi$ is defined by

$$\lambda_\pi(\pi_u, x) = \pi_{ux},$$

for all $u \in X^*$ and $x \in X$.

Also, $\lambda_\pi$ can be extended up to $\lambda_\pi^* : A_\pi \times X^* \rightarrow A_\pi$ so that

$$\lambda_\pi^*(\pi_u, v) = \pi_{uv},$$

for all $u, v \in X^*$. 
We prove the following:

**Theorem 3.** Let $\mu$ be a fuzzy right congruence on $X^*$ and let $\hat{\mu}$ be its crisp part. Then

(a) $A_{\hat{\mu}}$ is the crisp part of $A_\mu$;
(b) any $f \in \mathcal{F}(X^*)$ recognized by $A_\mu$ is also recognized by $A_{\hat{\mu}}$.

**Theorem 4.** For any fuzzy language $f \in \mathcal{F}(X^*)$ the following is true:

(a) A fuzzy relation $\rho_f$ on $X^*$ defined by

$$\rho_f(u, v) = \bigwedge_{w \in X^*} f(uw) \leftrightarrow f(vw), \text{ for any } u, v \in X^*,$$

is the greatest fuzzy right congruence on $X^*$ such that $f$ is extensional w.r.t. to it;
(b) $A_{\hat{\rho}_f}$ is a minimal deterministic automaton recognizing $f$. 
For a fuzzy language $f \in \mathcal{F}(X^*)$ and $u \in X^*$, a fuzzy language $f_u \in \mathcal{F}(X^*)$ defined by

$$f_u(v) = f(uv), \quad \text{for each } v \in X^*,$$

is called a derivative or a (right quotient) of $f$ with respect to $u$.

Let $A_f$ be the set of all derivatives of $f$, i.e., $A_f = \{f_u \mid u \in X^*\}$, and define a mapping $\delta_f : A_f \times X \times A_f \to L$ by

$$(5) \quad \delta_f(f_u, x, f_v) = \bigwedge_{w \in X^*} f_{ux}(w) \leftrightarrow f_v(w),$$

for all $u, v \in X^*$ and $x \in X$. 

We prove:

**Theorem 5.** For any $f \in \mathcal{F}(X^*)$, the mapping $\delta_f$ is well-defined and $A_f = (A_f, X, \delta_f)$ is a fuzzy automaton isomorphic to $A_{\rho_f}$.

For a fuzzy language $f \in \mathcal{F}(X^*)$, we also define a mapping $\lambda_f : A_f \times X \to A_f$ by

\[(6) \quad \lambda_f(fu, x) = fu_x,\]

for any $u \in X^*$ and $x \in X$.

Evidently, $\lambda_f$ can be extended up to $\lambda_f^* : A_f \times X^* \to A_f$ so that

\[(7) \quad \lambda_f^*(fu, v) = fu_v,\]

for all $u, v \in X^*$. 

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We also prove:

**Theorem 6.** For any fuzzy language $f \in \mathcal{F}(X^*)$, the mapping $\lambda_f$ is well-defined and $B = (A_f, X, \lambda_f)$ is a deterministic automaton isomorphic to $\hat{A}_f$.

Moreover, $B$ is the crisp part of $A_f$, that is $B = \hat{A}_f$.

**Theorem 7.** For any fuzzy language $f \in \mathcal{F}(X^*)$, both $A_f$ and $\hat{A}_f$ recognize $f$ with the crisp initial state $f$ and the fuzzy set of final states $\tau \in \mathcal{F}(A_f)$ defined by

$$\tau(g) = g(e),$$

for any derivative $g \in A_f$. 
Given a fuzzy automaton $\mathcal{A} = (A, X, \delta)$ and a state $a \in A$.

A fuzzy relation $\varrho_a$ on the free monoid $X^*$ defined by

$$
\varrho_a(u, v) = \bigwedge_{b \in A} \delta^*(a, u, b) \leftrightarrow \delta^*(a, v, b),
$$

for $u, v \in X^*$, is called **Nerode’s fuzzy relation** determined by $a$.

If $\mathcal{A}$ is an initial fuzzy automaton with a crisp initial state $a_0$, then
the fuzzy relation $\varrho_{a_0}$ is denoted by $\varrho_\mathcal{A}$ and called a **Nerode’s fuzzy relation** of the fuzzy automaton $\mathcal{A}$.

We prove the following:

**Theorem 8.** For any state $a$ of a fuzzy automaton $\mathcal{A} = (A, X, \delta)$, the
Nerode’s fuzzy relation $\varrho_a$ is a fuzzy right congruence on $X^*$. 

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Theorem 9. Any fuzzy language $f \in F(X^*)$ recognized by a fuzzy automaton $\mathcal{A}$ is also recognized by the fuzzy automaton $\mathcal{A}_{\vartheta}\mathcal{A}$.

To a fuzzy automaton $\mathcal{A} = (A, X, \delta)$, we also assign a fuzzy relation $\vartheta\mathcal{A}$ on the free monoid $X^*$ defined by

$$\vartheta\mathcal{A}(u, v) = \bigwedge_{a \in A} \varrho_a(u, v) = \bigwedge_{a, b \in A} \delta^*(a, u, b) \leftrightarrow \delta^*(a, v, b),$$

for $u, v \in X^*$, which is called Myhill's fuzzy relation of the fuzzy automaton $\mathcal{A}$.

Theorem 10. For any fuzzy automaton $\mathcal{A} = (A, X, \delta)$, the Myhill’s fuzzy relation $\vartheta\mathcal{A}$ is a fuzzy congruence on $X^*$. 
Theorem 11. Let $\mu$ be a fuzzy right congruence on $X^*$. Then

(a) Nerode’s fuzzy right congruence of $A_\mu$ coincide with $\mu$;

(b) Myhill’s fuzzy congruence of $A_\mu$ is the fuzzy congruence opening of $\mu$. 

Let $\mathcal{A} = (A, X, a_0, \delta)$ be a fuzzy automaton with a crisp initial state $a_0$.

We denote by $(L_\mathcal{A}, \lor, \otimes)$ the subalgebra of the reduct $(L, \lor, \otimes)$ of $L$ generated by the set $\{\delta(a, x, b) \mid a, b \in A, x \in X\}$.

For any $u \in X^*$ let a mapping $\Delta_u : A \to L_\mathcal{A}$ be defined by

$$\Delta_u(a) = \delta^*(a_0, u, a),$$

for each $a \in A$, let $A_\Delta = \{\Delta_u \mid u \in X^*\}$ and let $\lambda_\Delta : A_\Delta \times X \to A_\Delta$ be defined by

$$\lambda_\Delta(\Delta_u, x) = \Delta_{ux},$$

for all $u \in X^*$ and $x \in X$. 
We have the following

**Theorem 12.** Let $\mathcal{A} = (A, X, a_0, \delta)$ be a fuzzy automaton with a crisp initial state $a_0$. Then

(a) the mapping $\lambda_\Delta$ is well-defined and $\mathcal{A}_\Delta = (A_\Delta, X, \lambda_\Delta)$ is an automaton isomorphic to $\mathcal{A}_{\hat{\rho}_\mathcal{A}}$;

(b) $\text{ind}(\mathcal{Q}_\mathcal{A}) = \text{ind}(\hat{\mathcal{Q}}_\mathcal{A}) \leq |L_\mathcal{A}|$. 
By this we deduce the following:

**Theorem 13.** The following conditions are equivalent:

(i) The reduct $(L, \lor, \otimes)$ of $\mathcal{L}$ is a locally finite algebra;

(ii) Nerode’s fuzzy right congruence of any finite fuzzy automaton over $\mathcal{L}$ has a finite index;

(iii) Myhill’s fuzzy congruence of any finite fuzzy automaton $\mathcal{L}$ has a finite index.

As a consequence, a result of Li and Pedrycz (Fuzzy Sets and Systems 156 (2005), 68–92) one obtains, which says that (i) is equivalent to

(iv) Any fuzzy language recognizable by a finite fuzzy automaton, is also recognizable by a finite deterministic automaton (over $\mathcal{L}$).
Finally, the second main result is:

**Theorem 14.** For a fuzzy language $f \in \mathcal{F}(X^*)$, the following five conditions are equivalent if and only if the algebra $(L, \vee, \otimes)$ is locally finite:

(i) $f$ is a recognizable fuzzy language;

(ii) $f$ is extensional with respect to a fuzzy right congruence of finite index;

(iii) $f$ is extensional with respect to a fuzzy congruence of finite index;

(iv) the syntactic fuzzy right congruence $\rho_f$ has a finite index;

(v) the syntactic fuzzy congruence $\vartheta_f$ has a finite index.
Concluding Remarks

(1) Syntactic right congruences, syntactic congruences and derivatives of fuzzy languages have been considered in

- Shen (Information Sciences 88 (1996), 149-168)

Here, fuzzy languages were studied in terms of fuzzy right congruences and fuzzy congruences for the first time.

Nerode’s fuzzy right congruence and Myhill’s fuzzy congruence of a fuzzy automaton are also new concepts.

(2) The concept of extensionality, which play an outstanding role in our research, has important applications in fuzzy control, fuzzy clustering, and other fields.