On Products and Sums:

The Feferman-Vaught Approach

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Background:

Algorithmic Model Theory

- Construction and representation of structures with emphasis on algorithmic aspects (in particular: model checking: "$\mathfrak{A} \models \varphi$?")

- Model constructions:
  - Subset construction
  - Unfolding
  - Interpretations
  - Products, sums

Sources

- A. Mostowski (JSL 1952)
- S. Feferman, R. Vaught (Fund. Math. 1959)
- ⤳ Books by Chang, Keisler and Hodges
- S. Shelah (Ann. Math. 1975),
  cf. also W. Th. LNCS 1261

- S. Wöhrle, W. Th. (LICS 2004)
- A. Rabinovich (to appear)
- A. Bes (to appear)

Plan

I Prologue (done)

II Model-theoretic Preliminaries

III FV-Theorems for Products

IV FV-Theorem for Ordered Sums

V Tree Iterations

VI Epilogue

II Model-theoretic Preliminaries

Some Logics

Format of structures: $\mathfrak{A} = (A, R_1^A, \dots, R_m^A)$

Example: Transition graphs $(V, (E_a)_{a \in \Sigma})$

- FO (first-order logic)
- FO(R) over transition graphs
  Include atomic formula $Reach_\Gamma(x,y)$ $(\Gamma \subseteq \Sigma)$
- TC (transitive closure logic)
  $TC_{zz'}[\bigvee_{a \in \Gamma} E_a zz'](x,y)$
- MSO (monadic second-order logic)

Sums

- Given $\mathfrak{A} = (A, R^A, \ldots)$, $\mathfrak{B} = (B, R^B, \ldots)$

  $\mathfrak{A} = (A \dot\cup B, R^{A+B}, \ldots)$

  $$R^{A+B} a_1 \ldots a_n \iff (a_1, \ldots, a_n \in A \wedge R^A a_1 \ldots a_n)$$
  $$\vee (a_1, \ldots, a_n \in B \wedge R^B a_1, \ldots a_n)$$

- $\sum_{i \in I} \mathfrak{A}_i$ similarly

- If the $\mathfrak{A}_i$ are orderings $(A_i, <_i, \ldots)$ and $(I, <)$ is an ordering, obtain ordered sum over $\bigcup_{i \in I} A_i$ with

  $$a < b \iff a, b \in A_i \quad a <_i b$$
  $$\text{or } a \in A_i, b \in A_j \text{ with } i < j$$

Direct Products

- $\mathfrak{A} = (A, R^A, \dots)$ $\qquad \mathfrak{B} = (B, R^B, \dots)$

  $\mathfrak{A} \times \mathfrak{B} = (A \times B, R^{A \times B}, \dots)$

  $R^{A \times B}(a_1, b_1) \dots (a_n, b_n) \iff R^A a_1 \dots a_n, \ R^B b_1 \dots b_n$

- $\prod_{i \in I} \mathfrak{A}_i$ similarly $\qquad [I = \{1, \dots, n\}, \ I = \mathbb{N}]$

- Variant: Reduced product (with filter $\mathcal{F} \subseteq 2^I$)

  $R^{\prod_i \mathfrak{A}/\mathcal{F}} \bar{a}_1 \dots \bar{a}_n \iff \{i \in I \mid R^{A_i} (\bar{a}_1)_i \dots (\bar{a}_n)_i\} \in \mathcal{F}$

Synchronized Products

- $\mathfrak{A}_i = (V_i, (E^i_a)_{a \in \Sigma_i})$ $\quad i = 1, \dots, n$

  $\Sigma_i = \Sigma_i^s \dot{\cup} \Sigma_i^\ell$

  $C \subseteq (\Sigma_1^s \cup \{\varepsilon\}) \times \dots \times (\Sigma_n^s \cup \{\varepsilon\})$ (synchr. constraint)

- $\prod_{i=1}^n{}_C\, \mathfrak{A}_i = (\prod_i V_i, (E_a)_{a \in \cup \Sigma_i})$

  $a \in \bigcup \Sigma_i^\ell$: $E_a((u_1 \dots u_i \dots u_n), (u_1 \dots v_i \dots u_n))$ if $E^i_a(u_i, v_i)$

  $\bar{a} \in C$: $E_{\bar{a}}(\bar{u}, \bar{v})$ if $E^i_{a_i}(u_i, v_i)$ for $i = 1, \dots, n$

- $\prod_{i=1}^n \mathfrak{A}_i$ finitely synchronized : $\bigcup_{\bar{a} \in C} E_{\bar{a}}$ finite

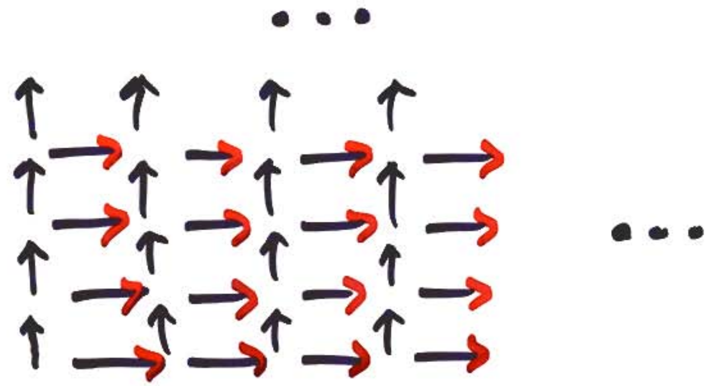
Examples (over factors $(\mathbb{N}, succ)$)

$(V_1, E_1) \cong (V_2, E_2) \cong (\mathbb{N}, succ)$

- $C = \emptyset$ gives asynchronous product

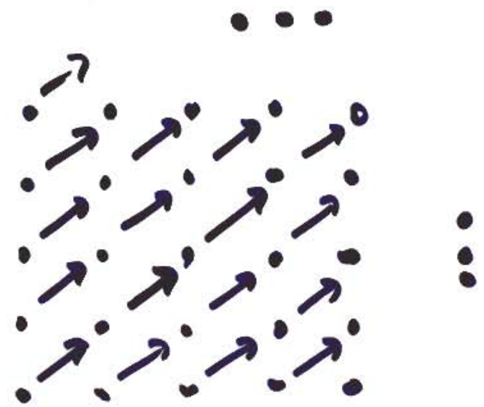
  $\Sigma_1^e = \{1\}$, $\Sigma_2^e = \{2\}$

  Grid

- $C = \{(1,2)\}$

  Bundle

Two Factors or Summands

- $\mathrm{FOTh}(\mathfrak{A} \times \mathfrak{B})$ is determined by $\mathrm{FOTh}(\mathfrak{A})$, $\mathrm{FOTh}(\mathfrak{B})$
- The analogous result holds for ordered sums $\mathfrak{A} = (A, <^A, P_1^A, \dots, P_n^A)$, $\mathfrak{B} = (B, <^B, P_1^B, \dots, P_n^B)$ with unary P_i

Proof • Show $\mathfrak{A}_1 \equiv \mathfrak{B}_1$, $\mathfrak{A}_2 \equiv \mathfrak{B}_2 \Rightarrow \mathfrak{A}_1 \overset{\times}{+} \mathfrak{A}_2 \equiv \mathfrak{B}_1 \overset{\times}{+} \mathfrak{B}_2$

- Use EF-games
- Construct winning strategy over $\mathfrak{A}_1 \overset{\times}{+} \mathfrak{A}_2$, $\mathfrak{B}_1 \overset{\times}{+} \mathfrak{B}_2$ using winning strategies over components in the obvious way

III FV-Theorems for Products

Syntactic Counterpart to Games: Hintikka Formulas

- $(\mathfrak{A}, a_1 \ldots a_m)$ and quantifier rank n determine a formula $\psi^n_{\mathfrak{A}\bar{a}}(x_1, \ldots, x_m)$ such that
$$(\mathfrak{B}, \bar{b}) \models \psi^n_{\mathfrak{A},\bar{a}}(\bar{x}) \iff (\mathfrak{A}, \bar{a}) \equiv_n (\mathfrak{B}, \bar{b})$$

- For given finite signature, m, n there are only finitely many Hintikka formulas $\psi_1, \ldots, \psi_k$

- Each $\varphi(\bar{x})$ of quantifier rank n is equivalent to a disjunction of the corresponding ψ_i (distributive normal form)

Definition of $\psi^n_{\mathfrak{A},\bar{a}}$

- $$\psi^0_{\mathfrak{A},\bar{a}}(\bar{x}) = \bigwedge_{\substack{R \\ \bar{a} \in R^A}} R\bar{x} \;\wedge\; \bigwedge_{\substack{R \\ \bar{a} \notin R^A}} \neg R\bar{x}$$

- $$\psi^{n+1}_{\mathfrak{A},\bar{a}}(\bar{x}) = \bigwedge_{\substack{a \in A \\ \mathfrak{A},\bar{a} \models \psi^n_{\mathfrak{A},\bar{a}a}(\bar{x},x)}} \exists x \; \psi^n_{\mathfrak{A},\bar{a},a}(\bar{x},x)$$

$$\wedge \; \forall x \bigvee_{\substack{a \in A \\ \mathfrak{A},\bar{a},a \models \psi^n_{\mathfrak{A},\bar{a},a}(\bar{x},x)}} \psi^n_{\mathfrak{A},\bar{a},a}(\bar{x},x)$$

Idea of Composition

- Reduce the question "$\prod_{i \in I} \mathfrak{A}_i \models \varphi$?" to questions "$\mathfrak{A}_i \models \psi_j$" and information about the distribution of the $\mathfrak{A}_i$-theories over I
- For second aspect use expansion of the index set : $(I, P_1, \ldots, P_k)$ or boolean models.

(A) Feferman-Vaught-Theorem

Given quantifier rank n, list of all n-types $\tau_1, \dots, \tau_k$

For each φ of quantifier rank n there is a monadic second-order formula $\chi(X_1, \dots, X_k)$ such that for any model $\prod_{i \in I} \mathfrak{A}_i$:

$$\prod_{i \in I} \mathfrak{A}_i \models \varphi \iff (I, P_1, \dots, P_k) \models \chi$$

where $P_j = \{ i \in I \mid \mathfrak{A}_i \models \tau_j \}$

Variant: Weak Powers (Mostowski)

$I = \mathbb{N}$, consider $\prod_{i \in \mathbb{N}} \mathfrak{A}_i$
restricted to finite sequences

Example: $\mathfrak{A}_i = \mathfrak{A} = (\mathbb{N}, +)$

$\mathfrak{A}^* = \prod_{i \in \mathbb{N}} \mathfrak{A}_i$ restricted to sequences ultimately 0

FV Theorem holds with weak monadic logic over $\mathbb{N}$

$$(\mathbb{N} \setminus \{0\}, \cdot) \cong (\mathbb{N}, +)^*$$

$$84 = 2^2 \cdot 3^1 \cdot 5^0 \cdot 7^1 \longmapsto (2,1,0,1)$$

<u>Presburger's Thm</u>: FO Th$(\mathbb{N}, +)$ decidable

FV-Thm $\longrightarrow$ <u>Skolem's Thm</u> FO Th$(\mathbb{N}, \cdot)$ decidable

A FV-Theorem for FO(R)

- Let $\mathfrak{A} = \prod_{i=1}^{n} \mathfrak{A}_i$ be a finitely synchronized product of graphs $\mathfrak{A}_i = (V_i, (E_a^i)_{a \in \Sigma_i})$
- For each FO(R)-formula φ one construct a Boolean formula χ in variables p_j^i $\begin{array}{l} i = 1, \dots, n \\ j = 1, \dots, k_i \end{array}$ and a Boolean model $\mathcal{B}$ such that

$$\mathfrak{A} \models \varphi \iff \mathcal{B} \models \chi$$

where truth values of $p_1^i \cdots p_{k_i}^i$ are computable from $FO(R)Th(\mathfrak{A}_i)$

<u>Consequence</u>: $FO(R)Th(\mathfrak{A}_i)$ decidable for $i = 1, \dots, n$

$\Longrightarrow$ $FO(R)Th(\prod_{i=1}^{n} \mathfrak{A}_i)$ decidable

Idea of Proof

- by induction on φ
- Interesting case: Atomic formula $\underline{Reach_\rho(x,y)}$

1) It suffices to consider paths from x to y which traverse each synchronized transition only once

2) Traverse them in some order (of finitely many possible ones)

3) Reduce to path segments with one synchronizing edge only

4) Evaluation of $Reach_\rho$ over the $\mathfrak{A}_i$ suffices

Limits for Generalization

There are examples of graphs $\mathfrak{A}_1, \mathfrak{A}_2$ with
- FO(R)-theory of $\mathfrak{A}_1, \mathfrak{A}_2$ decidable
- FO(R)-theory of synch. product undecidable

for any of the following situations:

- one factor is allowed to share infinitely many synchronizing edges
- $Reach_\Gamma(x,y)$ is generalized to regular constraints (written $Reach_\varrho(x,y)$ with $\varrho \in RegExpr(\Sigma)$)
- FO(R) is extended to TC-logic or MSO-logic

IV FV-Theorem for Ordered Sums

Shelah's Theorem

For each quantifier <u>alternation</u> depth n, $\bar{k} = (k_1, \ldots, k_n)$, a $\bar{k}$ formula has a (MSO-) quantifier prefix with blocks of length $k_1, \ldots, k_n$

For $\mathfrak{A}_i = (A_i, <_i, Q_1^i, \ldots, Q_m^i)$ $\quad (Q_j^i \subseteq A_i, \; i \in I)$

let $P_j = \{ i \in I \mid (A_i, \leq_i, \bar{Q}^i)$ has the j-th $\bar{k}$ type$\}$

$$\boxed{\begin{array}{l}\text{For } n, m, \bar{k} \text{ there is a tuple } \bar{r} \text{ of length } n \\ \text{such that } \quad MSOTh_{\bar{k}}(\Sigma_{i \in I} \mathfrak{A}_i) \text{ is computable} \\ \text{from } MSOTh_{\bar{r}}(I, <, \bar{P}).\end{array}}$$

V Tree Iterations

Tree Models as Weak Powers

- $\mathfrak{A} = (A, R^A, \dots)$

  Weak tree iteration $\mathfrak{A}^* = (A^*, \underbrace{eql}_{\text{equal length predicate}}, \underbrace{\leq}_{\text{prefix relation}}, R^{A^*}, \dots)$

  with $R^{A^*} w_1 \dots w_n \Leftrightarrow w_i = u_i a_i$ s.t. $R^A a_1, \dots a_n$, $|u_1| = \dots = |u_n|$

- Thm (Bes):

  $FOTh(\mathfrak{A})$ decidable $\Rightarrow$ $FOTh(\mathfrak{A}^*)$ decidable

Definability in $\mathfrak{A}^*$ (Bes)

$R \subseteq (A^*)^n$ is FO-definable in $\mathfrak{A}^*$ $\iff$

R is synchronously recognizable by an $\mathfrak{A}$-automaton

Input

a_1	a_2	a_3	$\cdots$	a_k	$\#$		$\cdots$	$\#$
b_1	b_2	b_3	$\cdots$	b_k	b_{k+1}		$\cdots$	b_ℓ

Run $\quad q_0 \quad q_1 \quad q_2 \quad q_3 \quad \cdots \quad q_k \quad \cdots \quad q_\ell$

Transitions are triples $(p, \varphi(x,y), q)$ with FO-formula φ

Applicable at i-th position if $\mathfrak{A} \models \varphi[a_i, b_i]$

$\overline{\underline{\text{VI}}}$ Epilogue

Summary

- There is a substantial machinery to reduce the model-checking problem for products / sums to the components
- There are severe limitations
- Iteration models arise as products

Perspectives

- Understand more clearly when we can compose a "product theory" from the "factor theories".
- Find stronger means of gluing the theories of the components
- Replace logic by automata