The pliant concept and the Generalized Dombi operator

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Abstract **– *This paper studies a special operator system called pliant operators, for which $f_c(x)f_d(x) = 1$, where $f_c(x)$ and $f_d(x)$ are the generator functions of the conjunction and the disjunction. In the second part we give a generalization of the Dombi operator which involves most well-known operators. We can get the Dombi, product, Einstein, Hamacher, min-max and drastic operators as special cases. The conjunctive and disjunctive operators differ only in the sign of a parameter: i.e. if it is positive we get the conjunctive operator, if it is negative it is the disjunctive operator. The DeMorgan identity is also examined. We show also that the operator is isomorph with the multiplicative utility function.***

Keywords **– *Fuzzy operators, t-norm, t-conorm, negation, multiplicative utility, Einstein operator, Hamacher operator***

I. INTRODUCTION

In fuzzy theory there has been quite many operators and membership functions introduced. There is also a wide choice of unary operators, e.g. negation. In this heterogen environment there is a need to examine operator structures that fulfill consistency properties, such as the DeMorgan identity. The min-max class introduced by Zadeh fulfills the DeMorgan identity. Consistent operators can also be constructed within the nilpotent operator class and the strict operator class. Pliant systems consist of an operator family from each of the latter two classes, defined as follows. Let $f_c(x)$ be a generator function of conjunction and let $f_d(x)$ be a generator function of disjunction. The *additive pliant system* is defined by the $f_c(x) + f_d(x) = 1$ equation. The Łukasiewicz operator family is an additive pliant system. The *multiplicative pliant system* is defined by the $f_c(x) \cdot f_d(x) = 1$ equation. The Dombi operator family [1] is a multiplicative pliant system. As we shall see in the following sections there is a strong correspondence between the additive and the multiplicative pliant systems.

II. DEMORGAN IDENTITY AND NEGATION

Definition 1: $n(x)$ is a negation iff $n : [0, 1] \to [0, 1]$ satisfies the following conditions:

C1:	$n(0) = 1, n(1) = 0$	(Boundary conditions)
C2:	$n(x) < n(y)$ for $x > y$	(Monotonicity)
C3:	$n(x)$ is continuous	(Continuity)
C4:	$n(n(x)) = x$	(Involutivness)

A. *Examination of the Generalized DeMorgan Identity*

Consistent many valued (fuzzy) operators require the validity of certain Boole identities. The most important one is DeMorgan law. Esteva [2] and Dombi [3] were the first two researchers, who carefully studied the DeMorgan identity. It corresponds to the conjunction, disjunction and negation. It is well-known, that conjunctive and disjunctive operators, which are strict monotonously increasing, associative, Archimedian, and fulfilling the boundary conditions have the following form:

$$c(x, y) = f_c^{-1}(f_c(x) + f_c(y)) \tag{1}$$

$$d(x, y) = f_d^{-1}(f_d(x) + f_d(y)) \tag{2}$$

where $f_c(x)$ and $f_d(x)$ are the generator functions of the conjunctive and disjunctive operators. The shape of the function can be found in Dombi's paper, and the result is based on Aczél's theorem.

In the following, we examine the relation of $f_c(x)$, $f_d(x)$ and $n(x)$, if $c(x, y)$, $d(x, y)$ and $n(x)$ fulfills the DeMorgan law. First let's generalize the conjunction and disjunction operators.

Let

$$c(w_1, x_1; w_2, x_2; \ldots; w_n, x_n) = f_c^{-1}\left(\sum_{i=1}^{n} w_i f_c(x_i)\right) \tag{3}$$

$$d(w_1, x_1; w_2, x_2; \ldots w_n, x_n) = f_d^{-1}\left(\sum_{i=1}^{n} w_i f_d(x_i)\right) \tag{4}$$

The DeMorgan law is:

$$c(w_1, n(x_1); \ldots; w_n, n(x_n)) = n(d(w_1, x_1; \ldots; w_n, x_n)) \tag{5}$$

Theorem 2 (DeMorgan Law) Let c and d have the former form. The generalized DeMorgan law is valid iff

$$f_c^{-1}(x) = n(f_d^{-1}(ax)) \tag{6}$$

where $a \neq 0$.

Remark 3: On the basis of Theorem 2 by the given $f_c(x)$ and $n(x)$, $f_d(x)$ can be determined, so that c, d and n is a DeMorgan triple. Similar to the above mentioned, by a given $f_d(x)$ and $n(x)$, $f_c(x)$ can be determined.

B. *Negations from the DeMorgan Law*

Naturally arises the question, if $f_c(x)$ and $f_d(x)$ are given, then what kind of condition ensure that $n(x)$ is a negation (i.e. fullfils C1-C4). From Theorem 2 we know that the necessary and sufficient condition of the DeMorgan Law is (6). Using the $x = f_c^{-1}(x)$ substitution, we get

$$x = n(f_d^{-1}(af_c(x))), \qquad a \neq 0 \tag{7}$$

From this, we get

$$n(x) = f_c^{-1}\left(\frac{1}{a}f_c(x)\right), \qquad a \neq 0 \tag{8}$$

This negation fulfills (C1-C3). The most important question is C4, the involutivness: $n(x) = n^{-1}(x)$.

Theorem 4 (Involutive negation) $n(x)$ given by 8 is involutive iff

$$f_c(x) = \frac{1}{a}k(f_d(x)), \qquad a \neq 0 \tag{9}$$

where $k : (0, \infty) \to (\infty, 0)$ is a strictly decreasing function with the property

$$k^{-1}(x) = k(x). \tag{10}$$

We can get a new representation theorem for the negation.

Theorem 5 (General form of the negation)
c, d and n form a DeMorgan triple iff

$$f_c(x) = \frac{1}{a}k(f_d(x)) \tag{11}$$

and

$$n(x) = f^{-1}(k(f(x))) \tag{12}$$

where $f(x) = f_c(x)$ or $f(x) = f_d(x)$ and $k(x)$ is a strictly decreasing function with the property

$$k(x) = k^{-1}(x) \tag{13}$$

From (12) it is easy to get

$$k(x) = f(n(f^{-1}(x))) \tag{14}$$

i.e. if $f(x)$ and $n(x)$ is given, then $k(x)$ is determined by (14).

Another interesting question whether (12) is a general representation form of the negation? The following theorem ensures, that all negation has the form (12).

While Trillas's theorem represents negations (from our point of view) for the nilpotent class of t-norms and t-conorms, our next result gives a representation theorem of the strict t-norms and t-conorms.

Theorem 6 (Representation theorem of negation) For all given $n(x)$ and $k(x)$, there exist an $f(x)$ such that

$$n(x) = f^{-1}(k(f(x))) \tag{15}$$

where $k(x)$ is a strictly decreasing function with the property $k(x) = k^{-1}(x)$ and $f(x)$ is the generator function of conjunctive, or disjunctive operator.

Remark 7: The most frequently used negation has an important role. If

$$k(x) = f\left(1 - f^{-1}(x)\right) \tag{16}$$

then

$$n(x) = 1 - x \tag{17}$$

Lemma 8: A DeMorgan triple can be built by using only one operator's generator function and choosing a $k(x)$, i.e. it is valid that

$$n(x) = f_c^{-1}\left(k\left(f_c(x)\right)\right) \tag{18}$$

$$c(x,y) = f_c^{-1}\left(f_c(x) + f_c(y)\right) \tag{19}$$

$$d(x,y) = f_c^{-1}\left(k\left(k\left(f_c(x)\right) + k\left(f_c(y)\right)\right)\right) \tag{20}$$

form a DeMorgan triple.

C. *Examples for DeMorgan Systems*

Using the above results, we can get the classical and also new operator systems.

- If $f_c(x) = -\ln(x)$ and $n(x) = 1 - x$ then

$$c(x,y) = xy$$
$$d(x,y) = x + y - xy$$
$$k(x) = -\ln(1 - e^{-x})$$

- If $f_c(x) = -\ln(x)$ and $k(x) = \frac{1}{x}$ then

$$c(x,y) = xy$$
$$d(x,y) = e^{1/\ln\left(e^{1/(\ln x) + 1/(\ln y)}\right)}$$
$$n(x) = e^{1/\ln x}$$

D. *Parametrial Form of the Negation*

From C1-C4 follows, that there exists a ν_* fix point of negation where

$$n(\nu_*) = \nu_* \tag{21}$$

It is another possible characterization of the negation, if we give a ν decision value for a given ν_0 (usually $\nu_0 = 1/2$). If x is less than the decision value, the negated value is larger than the threshold and vice versa:

$$x < \nu \quad \text{then} \quad n(x) > \nu_0$$
$$x > \nu \quad \text{then} \quad n(x) < \nu_0$$

If $x = \nu$, then

$$n(\nu) = \nu_0 \tag{22}$$

If $n(x)$ has ν_* fix point, we use the notation $n_{\nu_*}(x)$ and if the decision value is ν, then $n_\nu(x)$. Let's characterize the negation with the ν_*, ν_0 and ν parameters.

Lemma 9: The parametrial form of the negation is

$$n(x) = f^{-1}\left(f(\nu_*)\frac{k\left(f(x)\right)}{k\left(f(\nu_*)\right)}\right) \tag{23}$$

$$n(x) = f^{-1}\left(f(\nu_0)\frac{k\left(f(x)\right)}{k\left(f(\nu)\right)}\right). \tag{24}$$

III. PLIANT DEMORGAN SYSTEMS

From Dombi's result [1] we know, that if $f(x)$ is a generator function, then $f^\alpha(x)$ is a generator function, too. As we've seen $k(x)$ plays an important role in DeMorgan systems. Let us define the multiplicative pliant system with one of the simplest $k(x)$.

Definition 10: If $f_c(x) \cdot f_d(x) = 1$ that is $k(x) = 1/x$ and $f_\alpha(x) = f^\alpha(x)$ then we call the generated connectives multiplicative pliant system.

Remark 11: The first pliant system was introduced by Roychowdhury [4], who defines it only with $k(x) = 1/x$.

Theorem 12: The general form of the multiplicative pliant system is

$$o_\alpha(x,y) = f^{-1}\left(f^\alpha(x) + f^\alpha(y)\right)^{1/\alpha} \tag{25}$$

$$n_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu_0)\frac{f(\nu)}{f(x)}\right) \quad \text{or} \tag{26}$$

$$n_{\nu_*}(x) = f^{-1}\left(\frac{f^2(\nu_*)}{f(x)}\right) \tag{27}$$

where $f(x)$ is the generator function of either the conjunctive or the disjunctive operator. If $f(x) = f_c(x)$, then depending on the value of α the operator is

$$\begin{array}{ll} \alpha \geq 0 & o_\alpha(x,y) = c(x,y) \\ \alpha \leq 0 & o_\alpha(x,y) = d(x,y) \\ \alpha = \infty & o_{+\infty}(x,y) = \min(x,y) \\ \alpha = -\infty & o_{-\infty}(x,y) = \max(x,y) \end{array}$$

Remark 13: It is important, that in the multiplicative pliant system the negation is independent of the value and the sign of α. (In other words it is independent of whether the generator function is conjunctive or disjunctive.)

Theorem 14: If $g(x) = f^\alpha(x)$ is the generator function then the negation does not change.

A. Dombi's Operator Class and Pliant Notation

We can get the generalized Dombi operator class [3] by using

$$f(x) = \frac{1-x}{x} \tag{28}$$

then

$$o_\alpha(x_1, x_2, \ldots, x_n) = \frac{1}{1 + \left(\sum_{i=1}^{n}\left(\frac{1-x_i}{x_i}\right)^\alpha\right)^{1/\alpha}}$$

Its pliant notation:

$$\{x \prec \nu\}_{\nu_0} = \frac{1}{1 + \frac{1-\nu}{\nu}\frac{1-\nu_0}{\nu_0}\frac{x}{1-x}}$$

if $\nu_0 = 1/2$ then

$$\{x \prec \nu_*\} = \frac{1}{1 + \left(\frac{1-\nu_*}{\nu_*}\right)^2\frac{x}{1-x}}$$

If $\alpha \geq 0$, then $o_\alpha(x,y) = c_\alpha(x,y)$, if $\alpha \leq 0$, then $o_\alpha(x,y) = d_\alpha(x,y)$, if $\nu_0 = 1/2$

$$n_\nu(x) = \frac{1}{1 + \frac{1-\nu}{\nu}\frac{x}{1-x}}$$

if $\nu_* = 1/2$ then

$$n(x) = 1 - x$$

IV. THE GENERALIZED DOMBI OPERATOR

From the work of Dombi [3] we know that the generator function of the strict monotone conjunctive (disjunctive) operator class have the following properties:

$$f_c(1) = 0 \qquad f_d(0) = 1 \tag{29}$$

$$\lim_{x\to 0} f_c(x) = -\infty \qquad \lim_{x\to 1} f_d(x) = +\infty \tag{30}$$

where $f_c(x)$ ($f_d(x)$) is strictly monotonously decreasing (increasing).

Definition 15: The generator functions of the Generalized Dombi operator are

$$f_c(x) = \ln\left(1 + \frac{\gamma_c}{1-\gamma_c}\left(\frac{1-x}{x}\right)^\alpha\right) \qquad \alpha > 0 \tag{31}$$

$$f_d(x) = \ln\left(1 + \frac{\gamma_d}{1-\gamma_d}\left(\frac{1-x}{x}\right)^\alpha\right) \qquad \alpha < 0 \tag{32}$$

where $\gamma_c, \gamma_d \in [0,1]$. From

$$c(\mathbf{x}) = f_c^{-1}\left(\sum_{i=1}^{n} f_c(x_i)\right) \tag{33}$$

$$d(\mathbf{x}) = f_d^{-1}\left(\sum_{i=1}^{n} f_d(x_i)\right) \tag{34}$$

and

$$f_c^{-1}(x) = \frac{1}{1 + \left(\frac{1-\gamma_c}{\gamma_c}\left(e^x - 1\right)\right)^{1/\alpha}} \qquad \alpha > 0 \tag{35}$$

$$f_d^{-1}(x) = \frac{1}{1 + \left(\frac{1-\gamma_d}{\gamma_d}\left(e^x - 1\right)\right)^{1/\alpha}} \qquad \alpha < 0 \tag{36}$$

the operators are

$$c^{(\alpha)}_{GD,\gamma_c}(\mathbf{x}) = \frac{1}{1 + D_{\gamma_c}(\mathbf{x})} \qquad \alpha > 0 \tag{37}$$

$$d^{(\alpha)}_{GD,\gamma_d}(\mathbf{x}) = \frac{1}{1 + D_{\gamma_d}(\mathbf{x})} \qquad \alpha < 0 \tag{38}$$

where

$$D_\gamma(\mathbf{x}) = \left(\frac{1-\gamma}{\gamma}\left(\prod_{i=1}^{n}\left(1 + \frac{\gamma}{1-\gamma}\left(\frac{1-x_i}{x_i}\right)^\alpha\right)\right) - 1\right)^{1/\alpha} \tag{39}$$

and $\gamma_c, \gamma_d \in [0,1]$.

Definition 16: The function $c(x,y)$ is a strict conjunctive operator iff

1. $c:[0,1]\times[0,1]\rightarrow[0,1]$ is continuous
2. associative
3. strictly monotonously increasing in both x and y
4. compatible with the boolean ”and” i.e.

- $c(0,0)=c(1,0)=c(0,1)=0$
- $c(1,1)=1$.

The function $d(x,y)$ is a strict disjunctive operator iff it fulfills (1)-(3) and

4'. compatible with the boolean ”or” i.e.

- $d(0,0)=0$
- $d(1,0)=d(0,1)=d(1,1)=1$.

It is easy to check that $c^{(\alpha)}_{GD,\gamma_c}(\mathbf{x})$ and $d^{(\alpha)}_{GD,\gamma_d}(\mathbf{x})$ are conjunctive and disjunctive operators i.e. fulfill (1)-(4) and (1)-(3),(4') respectively. Equations (37) and (38) are the same and so the Generalized Dombi operator class is:

$$o^{(\alpha)}_{GD,\gamma}(\mathbf{x})=\frac{1}{1+\left(\frac{1-\gamma}{\gamma}\left(\prod_{i=1}^{n}\left(1+\frac{\gamma}{1-\gamma}\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)\right)-1\right)^{1/\alpha}} \tag{40}$$

Theorem 17:

$$o^{(\alpha)}_{GD,\gamma}(\mathbf{x})=c^{(\alpha)}_{GD,\gamma_c}(\mathbf{x}) \text{ iff } \alpha>0 \tag{41}$$

$$o^{(\alpha)}_{GD,\gamma}(\mathbf{x})=d^{(\alpha)}_{GD,\gamma_d}(\mathbf{x}) \text{ iff } \alpha<0 \tag{42}$$

Theorem 18: The following equation is valid for the generator functions f_c and f_d:

$$f_c(x)=f_d(1-x) \tag{43}$$

A. The Dombi operator case

The Dombi operator has the form

$$o^{(\alpha)}_{D}(\mathbf{x})=\frac{1}{1+\left(\sum_{i=1}^{n}\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)^{1/\alpha}} \tag{44}$$

and if $\alpha>0$ then the operator is conjunctive and if $\alpha<0$ then the operator is disjunctive.

Theorem 19: The Dombi operator is a special case of the Generalized Dombi operator if $\gamma=0$.

B. The product operator case

The product operator is the most widely used among the applications of fuzzy sets. Zadeh in his first paper also suggested to use it. It is also called probabilistic operator because the probabilities of independent events is the product of the event probabilities. It has the following form:

$$c_P(\mathbf{x})=\prod_{i=1}^{n}x_i \tag{45}$$

$$d_P(\mathbf{x})=1-\prod_{i=1}^{n}(1-x_i)\,. \tag{46}$$

Theorem 20: The product operator is a special case of the Generalized Dombi operator if $\gamma=1/2$ and $\alpha=\pm1$.

C. The Einstein operator case

Einstein in his famous work on special relativity theory examined how two velocities have to be added. His result was

$$v=\frac{v_1+v_2}{1+\frac{v_1\cdot v_2}{c^2}}. \tag{47}$$

Let us introduce the relative velcities to c as $x=v_1/c$, $y=v_2/c$ and $z=v/c$, then

$$d_E(x,y)=z=\frac{x+y}{1+xy}. \tag{48}$$

It is easy to check that z is a disjunctive operator. Because (48) can be derived from (47) it is called Einstein operator. The corresponding conjunctive operator can be built by using the DeMorgan identity with the negation $n(x)=1-x$:

$$\begin{aligned}c_E(x,y)&=1-d_E(1-x,1-y)=\\&=1-\frac{(1-x)(1-y)}{1+(1-x)(1-y)}=\\&=\frac{xy}{1+(1-x)(1-y)}.\end{aligned} \tag{49}$$

Theorem 21: The Einstein operator is a special case of the Generalized Dombi operator if $\gamma=2/3$ and $\alpha=\pm1$.

Using this result, the n-ary Einstein operators are

$$c^{(1)}_{GD,2/3}(\mathbf{x})=\frac{1}{1+\frac{1}{2}\left(\prod_{i=1}^{n}\left(1+2\frac{1-x_i}{x_i}\right)-1\right)} \tag{50}$$

$$d^{(-1)}_{GD,2/3}(\mathbf{x})=\frac{1}{1+2\left(\prod_{i=1}^{n}\left(1+2\frac{x_i}{1-x_i}\right)-1\right)^{-1}} \tag{51}$$

and we can give the general law of additivity of velocities in the framework of special relativity theory.

Theorem 22: The general law of Einstein additivity of velocities is

$$v=\frac{c}{1+2\left(\prod_{i=1}^{n}\left(1+2\frac{v_i}{c-v_i}\right)-1\right)^{-1}}. \tag{52}$$

D. The Hamacher operator case

Hamacher [5] was one of the first who discussed that finding new logical operators can be done using the solutions of associative functional equations. With the help of the generator function of the operator, infinitely many operators can be constructed. To restrict the solution space Hamacher added a new requirement. Namely he looked only for operators which can be written in rational form (a proportion of two polinoms). Kuwagaki [6] showed that in this case the generator function can only have the following forms:

$$f^{-1}(x)=\frac{ax+b}{cx+d} \text{ or } f^{-1}(x)=\frac{ae^x+b}{ce^x+d} \tag{53}$$

Hamacher showed that to have conjunctive or disjunctive operators equations (53) have the following solutions

$$f_c^{-1}(x)=\frac{e^x}{1+(1-\gamma)e^x} \qquad 0<\gamma \tag{54}$$

$$f_d^{-1}(x)=\frac{e^x-1}{\gamma'+e^x} \qquad -1<\gamma' \tag{55}$$

The Hamacher operator is

$$c_{H,\gamma}(x,y) = f_c^{-1}(f_c(x)+f_c(y)) = \\ = \frac{xy}{\gamma+(1-\gamma)(x+y-xy)} \quad (56)$$

$$d_{H,\gamma'}(x,y) = f_d^{-1}(f_d(x)+f_d(y)) = \\ = \frac{x+y-(1-\gamma')xy}{1+\gamma' xy} \quad (57)$$

where $0 < \gamma$ and $-1 < \gamma'$.

Theorem 23: The Hamacher operator class is a special case of the Generalized Dombi operator if $\alpha = \pm 1$ and $\gamma \in (0,1)$.

Using the new type of generator functions we can write the Hamacher operators in a new form

$$c_H(\mathbf{x}) = \frac{1}{1+\frac{1-\gamma_c}{\gamma_c}\left(\prod_{i=1}^n\left(1+\frac{\gamma_c}{1-\gamma_c}\frac{1-x_i}{x_i}\right)-1\right)} \quad (58)$$

and

$$d_H(\mathbf{x}) = \frac{1}{1+\left(\frac{1-\gamma_d}{\gamma_d}\left(\prod_{i=1}^n\left(1+\frac{\gamma_d}{1-\gamma_d}\frac{x_i}{1-x_i}\right)-1\right)\right)^{-1}} \quad (59)$$

Using the Generalized Dombi operator then

$$o_\gamma^{(\alpha)}(\mathbf{x}) = \frac{1}{1+\left(\frac{1-\gamma}{\gamma}\left(\prod_{i=1}^n\left(1+\frac{\gamma}{1-\gamma}\left(\frac{1-x_i}{x_i}\right)^\alpha\right)\right)-1\right)^{1/\alpha}} \quad (60)$$

where $\alpha \in \{0,1\}$ is a common form for the Hamacher operators. So

$$o_{GD,\gamma}^{(1)}(\mathbf{x}) = c_H(\mathbf{x}) \quad (61)$$

and

$$o_{GD,\gamma}^{(-1)}(\mathbf{x}) = d_H(\mathbf{x}) \quad (62)$$

where $\gamma \in (0,1)$.

E. The drastic operator case

The drastic operators are used if we want to go as close as possible to the two valued logic. Because from the solution of the associative equation it is known that

$$c(x,1) = c(1,x) = x \quad (63)$$

$$c(x,0) = c(0,x) = 0 \quad (64)$$

$$d(x,1) = d(1,x) = 1 \quad (65)$$

$$d(x,0) = d(0,x) = x \quad (66)$$

so the drastic operator in the conjunctive case takes the value 0 if $x,y \in [0,1)$ and in the disjunctive case takes the value 1 if $x,y \in (0,1]$. So the drastic operators are

$$c_{Dr}(x,y) = \begin{cases} x \text{ if } y=1 \\ y \text{ if } x=1 \\ 0 \text{ otherwise} \end{cases} \quad (67)$$

$$d_{Dr}(x,y) = \begin{cases} x \text{ if } y=0 \\ y \text{ if } x=0 \\ 1 \text{ otherwise} \end{cases} \quad (68)$$

Theorem 24: The drastic operator class is a special case of the Generalized Dombi operator if $\gamma = 1$.

F. The min and max operator case

The most widely used operators in fuzzy theory are the min and the max operator. They have many advantages: they are easy to calculate, and they can be extended into a lattice structure. In practice the strict operators are more intensively used. The reason is that the result is determined only by one variable and the other have no influence, opposite to the strict monotonously increasing operators like the Generalized Dombi operator class. We show here that as a limit we can get the min and max operators.

Theorem 25: The min and max operators are the limits of the Generalized Dombi operator if $\gamma = 0$ and $\alpha \to \infty$ or $\alpha \to -\infty$.

G. The Aczél-Alsina operator case

The Aczél-Alsina operator is (see Klement, Mesiar and Pap [7])

$$c(x,y) = \exp\left[-\left((-\log x)^\lambda + (-\log y)^\lambda\right)^{1/\lambda}\right] \quad (69)$$

The Aczél-Alsina operator can not be derived from the the Generalized Dombi operator. We have to modify it by introducing a third parameter λ'. It is known from the properties of the generator functions that the power of the generator function is also a generator function of an operator. Let us take the generator function of the Generalized Dombi operator to the λ'th power:

$$f(x) = \left(-\ln\left(1+\frac{\gamma}{1-\gamma}\left(\frac{1-x}{x}\right)^\alpha\right)\right)^{\lambda'} \quad (70)$$

Theorem 26: Equation (70) generates the Aczél-Alsina operator if and only if $\alpha = 1$, $\gamma = 1/2$ in the conjunctive case, and $\alpha = -1$, $\gamma = 1/2$ in the disjunctive case.

H. Summary of special cases

Table H summarizes the special cases of the Generalized Dombi operator.

Type of operator	Value of γ	Value of α	
		conj.	disj.
Dombi	0	$0<\alpha$	$\alpha<0$
Product	$1/2$	1	-1
Einstein	$2/3$	1	-1
Hamacher	$\gamma\in(0,1)$	1	-1
Drastic	1	$0<\alpha$	$\alpha<0$
Min-max	0	∞	$-\infty$

Table I. Summary of special cases

V. THE NEGATION

The negation is a unary operator. Its most frequently used form is

$$n(x)=1-x. \tag{71}$$

Sugeno also introduced a negation:

$$n(x)=\frac{1-x}{1+\lambda x} \qquad x>-1. \tag{72}$$

The usual requirements for a negation are:

1. $n:[0,1]\to[0,1]$ is continuous
2. strictly decreasing
3. $n(0)=1$ and $n(1)=0$
4. $n(x)$ is involutive i.e. $n(n(x))=x$

Trillas gave the general representation theorem of the negation:

$$n(x)=f^{-1}\left(1-f(x)\right) \tag{73}$$

where $f(x)$ is a continuously increasing strictly monotone function, $f(0)=0$ and $f(1)=1$. According to this infinitely many negations exist. Hamacher in his work showed that the rational form of involutive negations is also (72) (it is also called Hamacher negation). In this paper we modify the form of (72) and we give the semantic meaning of the technical parameter. Two types of characterizations will be given.

If the negation fulfills axioms (1)-(4) then there exists a fix point ν'_* such that

$$n(\nu_*)=\nu_*. \tag{74}$$

If we fix a neutral value (usually it is $\nu_0=1/2$) then there exists a ν value such that

$$n(\nu)=\nu_0 \tag{75}$$

and λ can be expressed by ν_* and ν, ν_0.

Theorem 27: The Dombi form of the negation is

$$n_{\nu_*}(x)=\frac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\right)^2\left(\frac{1-x}{x}\right)^{-1}} \tag{76}$$

$$n_{\nu,\nu_0}(x)=\frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\left(\frac{1-x}{x}\right)^{-1}} \tag{77}$$

VI. THE DEMORGAN LAW

It is natural to demand the validity of the DeMorgan law in a consistent logical system. In this section we examine the necessary and sufficient conditions of it. We suppose that the conjunctive and the disjunctive operators have the same α. The three operators are:

$$c^{(\alpha)}_{GD,\gamma_c}(\mathbf{x})=\frac{1}{1+D_{\gamma_c}(\mathbf{x})} \qquad \alpha>0 \tag{78}$$

$$d^{(\alpha)}_{GD,\gamma_d}(\mathbf{x})=\frac{1}{1+D_{\gamma_d}(\mathbf{x})} \qquad \alpha<0 \tag{79}$$

$$n_{\nu,\nu_0}(x)=\frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\left(\frac{1-x}{x}\right)^{-1}} \tag{80}$$

or

$$n_{\nu_*}(x)=\frac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\right)^2\left(\frac{1-x}{x}\right)^{-1}} \tag{81}$$

where

$$D_\gamma(\mathbf{x})=\left(\frac{1-\gamma}{\gamma}\left(\prod_{i=1}^{n}\left(1+\frac{\gamma}{1-\gamma}\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)\right)-1\right)^{1/\alpha} \tag{82}$$

and $\gamma_c,\gamma_d,\nu,\nu_0,\nu_*\in(0,1)$.

Theorem 28: The Generalized Dombi operator class (i.e. equations (78), (79) and (80),(81)) is a DeMorgan triple if and only if

$$\frac{1-\gamma_c}{\gamma_c}\cdot\frac{\gamma_d}{1-\gamma_d}=\left(\frac{1-\nu_0}{\nu_0}\cdot\frac{1-\nu}{\nu}\right)^{\alpha} \tag{83}$$

or

$$\frac{1-\gamma_c}{\gamma_c}\cdot\frac{\gamma_d}{1-\gamma_d}=\left(\frac{1-\nu_*}{\nu_*}\right)^{2\alpha} \tag{84}$$

Corollary 29: If the negation is

$$n(x)=1-x \tag{85}$$

i.e. $\nu=\nu_0=\nu_*=1/2$ then (78) and (79) form DeMorgan triples with $n(x)$ if and only if

$$\gamma=\gamma_c=\gamma_d. \tag{86}$$

A. The DeMorgan law of special cases

In Theorem 28 we did not examine the case of $\gamma=0$ (Dombi operator), $\gamma=1$ (drastic operator) and $\alpha=\pm\infty$ (min-max operator).

Theorem 30: The Dombi operators form a DeMorgan triple with the negations (80) and (81) with the same α for all $\nu_*,\nu,\nu_0\in(0,1)$.

Theorem 31: The min-max operators and the drastic operator are DeMorgan triples with any involutive negation.

Corollary 32: The min-max operators form DeMorgan triples with the negations (80) and (81) for all $\nu_*,\nu,\nu_0\in(0,1)$.

VII. THE WEIGHTED GENERALIZED DOMBI OPERATOR

In the area of multicriteria decision making not only the values of the criteria determine the decision. The general form of the weighting is

$$o(x_1,w_1;x_2,w_2;\dots;x_n,w_n)=f^{-1}\left(\sum_{i=1}^{n}w_i f(x_i)\right). \quad (87)$$

Using (87) and the generator function of the operator we get

$$o_{GD}(\mathbf{x},\mathbf{w})=\frac{1}{1+\frac{1-\gamma}{\gamma}\left(D(\mathbf{x},\mathbf{w})\right)^{1/\alpha}}, \quad (88)$$

where

$$D_\gamma(\mathbf{x},\mathbf{w})=\prod_{i=1}^{n}\left(1+\frac{\gamma}{1-\gamma}\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)^{w_i}-1 \quad (89)$$

Theorem 33: In the Dombi operator case ($\gamma=0$) the weighted operator is

$$o_D(\mathbf{x},\mathbf{w})=\frac{1}{1+\left(\sum_{i=1}^{n}w_i\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)^{1/\alpha}}. \quad (90)$$

A. The weighted Min-max operators

The min-max operators are the limit of the strict monotone operators. The min or max value does not change by the weights. Let $\min(x_1,\dots,x_n)=x_j$, then

$$\lim_{\alpha\to\infty}o^{(\alpha)}(\mathbf{x})=\lim_{\alpha\to\infty}f^{-1}\left(\left(\sum_{i=1}^{n}w_i f^{\alpha}(x_i)\right)^{1/\alpha}\right)=$$
$$=f^{-1}\left(w_j^{1/\alpha}f(x_j)\sum_{i=1}^{n}\left(1+\frac{w_i}{w_j}\left(\frac{f(x_i)}{f(x_j)}\right)^{\alpha}\right)^{1/\alpha}\right)=$$
$$=x_j \quad (91)$$

Because $w^{1/\alpha}\to 1$ and if $A<1$ and $K>0$ then

$$\lim_{\alpha\to\infty}\left(1+KA^{\alpha}\right)^{1/\alpha}=1. \quad (92)$$

B. The weighting of the Drastic operators

The weighted Drastic operators are not logical operators. We will show that the result is

$$a_D(\mathbf{x})=\frac{1}{1+\prod_{i=1}^{n}\left(\frac{1-x_i}{x_i}\right)^{w_i}} \quad (93)$$

which is the weighted aggregation operator introduced by Dombi [1].

Theorem 34: The weighting of the Drastic operator getting as a limit of the Generalized Dombi operator is the weighted aggregation operator (93), and it is the same in the conjunctive and disjunctive cases.

VIII. AGGREGATION

The aggregation operator can be built in the following manner:

$$a(\mathbf{x})=f^{-1}\left(\prod_{i=1}^{n}f(x_i)\right). \quad (94)$$

The generator function of the Generalized Dombi operator is

$$f_c(x)=\ln\left(1+\frac{\gamma}{1-\gamma}\left(\frac{1-x}{x}\right)^{\alpha}\right) \quad (95)$$

so

$$a_{GD}(\mathbf{x})=\frac{1}{1+\left(\frac{1-\gamma}{\gamma}\exp\left[D(\mathbf{x},\mathbf{w})\right]\right)^{1/\alpha}} \quad (96)$$

where

$$D(\mathbf{x},\mathbf{w})=\prod_{i=1}^{n}\ln\left(1+\frac{\gamma}{1-\gamma}\left(\frac{1-x}{x}\right)^{\alpha}\right)-1 \quad (97)$$

The weighted form is

$$a_{GD}(\mathbf{x},\mathbf{w})=\frac{1}{1+\left(\frac{1-\gamma}{\gamma}\exp\left[D(\mathbf{x},\mathbf{w})\right]\right)^{1/\alpha}} \quad (98)$$

where

$$D(\mathbf{x},\mathbf{w})=\prod_{i=1}^{n}\ln^{w_i}\left(1+\frac{\gamma}{1-\gamma}\left(\frac{1-x}{x}\right)^{\alpha}\right)-1 \quad (99)$$

IX. UNARY OPERATORS OF THE GENERALIZED DOMBI OPERATOR

$$\kappa^{(\beta)}(x)=f^{-1}\left(\beta f(x)\right)=$$
$$=\frac{1}{1+\left(\frac{1-\gamma}{\gamma}\left(\left(1+\frac{\gamma}{1-\gamma}\left(\frac{1-x}{x}\right)^{\alpha}\right)^{\beta}-1\right)\right)^{1/\alpha}} \quad (100)$$

$$c\left(\kappa^{(\beta_1)}(x_1),\dots,\kappa^{(\beta_n)}(x_n)\right)=$$
$$=\frac{1}{1+\left(\frac{1-\gamma_c}{\gamma_c}\left(\prod_{i=1}^{n}\left(1+\frac{\gamma_c}{1-\gamma_c}\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)^{\beta_i}-1\right)\right)^{1/\alpha}} \quad (101)$$

$$c\left(x_1,w_1;\dots;x_n,w_n\right)=$$
$$=\frac{1}{1+\left(\frac{1-\gamma_c}{\gamma_c}\left(\prod_{i=1}^{n}\left(1+\frac{\gamma_c}{1-\gamma_c}\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)^{w_i}-1\right)\right)^{1/\alpha}} \quad (102)$$

X. THE MULTIPLICATIVE REPRESENTATION OF THE GENERALIZED DOMBI OPERATOR

It is known that the solution of the associative equation is

$$o(x,y) = f^{-1}\left(f(x) + f(y)\right) \tag{103}$$

Let

$$f(x) = \ln(g(x)) \tag{104}$$

then

$$f^{-1}(x) = g^{-1}\left(e^{x}\right). \tag{105}$$

For the operator is valid

$$\begin{aligned} o(x,y) &= f^{-1}\left(f(x) + f(y)\right) = \\ &= g^{-1}\left(e^{\ln(g(x))+\ln(g(y))}\right) = \\ &= g^{-1}\left(e^{\ln(g(x)g(y))}\right) = g^{-1}\left(g(x)g(y)\right). \end{aligned} \tag{106}$$

This is the multiplicative form of the solution of the associative equation. Using the fact that

$$f_{GD}(x) = \ln\left(1 + \frac{\gamma}{1-\gamma}\left(\frac{1-x}{x}\right)^{\alpha}\right) \tag{107}$$

so

$$g_{GD}(x) = 1 + \frac{\gamma}{1-\gamma}\left(\frac{1-x}{x}\right)^{\alpha} \tag{108}$$

and

$$g_{GD}^{-1}(x) = \frac{1}{1 + \left(\frac{1-\gamma}{\gamma}(x-1)\right)^{-1/\alpha}} \tag{109}$$

From this

$$\begin{aligned} o_{GD}(\mathbf{x}) &= g_{GD}^{-1}\left(\prod_{i=1}^{n} g_{GD}(x_i)\right) = \\ &= \frac{1}{1 + \left(\frac{1-\gamma}{\gamma}\left(1 + \frac{\gamma}{1-\gamma}\left(\frac{1-x_i}{x_i}\right)^{\alpha} - 1\right)\right)} \end{aligned} \tag{110}$$

XI. THE GENERALIZED DOMBI OPERATOR AND THE MULTIPLICATIVE DECISION FUNCTION

Let $A_1, A_2, \ldots, A_n$ be mutually exclusive events so

$$p(A_i \cap A_j) = 0 \quad \text{if } i \neq j \tag{111}$$

and

$$\sum_{i=1}^{n} p(A_i) = 1 \tag{112}$$

where $p(A_i)$ is the probability that event A_i will occur. Associated with each event is a reward or consequence c_i (not necessarily money) to which one associates some value or utility $u(c_i)$. The expected utility is

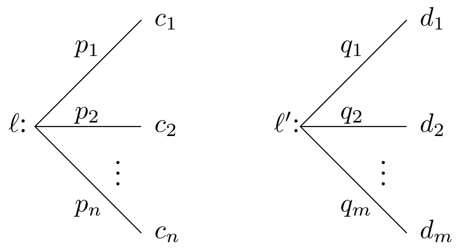
$$E = \sum_{i=1}^{n} p(A_i)u(c_i) \tag{113}$$

The Expected Utility Rule is the following. If x and y are lotteries, then x is preferred to y if and only if

$$\sum_{i=1}^{n} p(A_i)u(c_i) > \sum_{j=1}^{m} p(A'_j)u(c_j) \tag{114}$$

where the first sum is over the events and consequences of lottery x and the second sum is over the events and consequences of lottery y.

Suppose K is a set of consequences and L is a collection of lotteries with consequences in K. Let P be a binary strict preference relation on L. Any function $u : K \to \mathbb{R}$ will be called a value function on K. We say that the triple (K, L, P) satisfies the Expected Value Hypothesis if there is a value function u on K that whenever ℓ and ℓ' are in L and



are lotteries then

$$\ell P \ell' \Leftrightarrow E(\ell) = \sum_{i=1}^{n} p_i u(c_i) > \sum_{j=1}^{m} q_j u(d_j) = E(\ell') \tag{115}$$

In this case u is a value function satisfying the Expected Value (EV) Hypothesis. $E(\ell)$ is called the expected value of lottery ℓ.

Suppose $\succ$ is a strict preference relation on K. We would like to find an order preserving utility function on $(K, \succ)$ that is $u : K \to \mathbb{R}$ satisfying

$$c \succ d \Leftrightarrow u(c) > u(d) \tag{116}$$

In a sense each element in the set K is a lottery, for we can identify a consequence c in K with the lottery that gives c with probability 1. This lottery will be denoted by $\ell(c)$. We shall assume that each such lottery is in L. If P is a strict preference relation on L, we also assume that for all $c, d \in K$

$$c \succ d \Leftrightarrow \ell(c) P \ell(d). \tag{117}$$

If these two assumptions hold, we say (L, P) extends $(K, \succ)$ and then any value function u on K satisfying the EV Rule is an order preserving utility function for $(K, \succ)$. For

$$c \succ d \Leftrightarrow \ell(c) P \ell(d) \Leftrightarrow E[\ell(c)] > E[\ell(d)] \Leftrightarrow u(c) > u(d). \tag{118}$$

We shall assume the EV Hypothesis, that is, that there is a value function u on K satisfying the EV Rule. We shall not assume that u is known, but only that it exists.

Assume that there are two elements of K, c_*, and c^*, so that

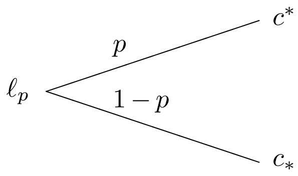
$$c^* \succ c_* \tag{119}$$

and so that for all c in K,

$$\sim (c \succ c^*) \quad \text{and} \quad \sim (c_* \succ c). \tag{120}$$

That is, c^* is strictly preferred to c_*, nothing is strictly preferred to c^*, and c_* is not strictly preferred to anything. The consequences c^* and c_* can be thought of as the “best” and “worst” consequences in K. Since $c^* \succ c_*$ and u is an order preserving utility function, we have $u(c^*) > u(c_*)$, and hence $u(c^*) - u(c_*) > 0$.

Given c in K, let $\ell(c)$ be the lottery with only one consequence, c. We are assuming that $\ell(c)$ always belongs to L. Consider the lottery



Since (L, P) extends $(K, \succ)$, if p is 1, you either prefer ℓ_p to $\ell(c)$ or are indifferent between these lotteries. If p is 0, you either prefer $\ell(c)$ to ℓ_p or are indifferent between these lotteries. As we let p gradually increase from 0 to 1, it is reasonable to assume that we can find some number $p = p(c)$ so that you are indifferent between $\ell_{p(c)}$ and $\ell(c)$. Then $p(c)$ defines an order-preserving utility function over K. For

$$\begin{aligned} E(\ell_{p(c)}) &= p(c)u(c^*) + [1 - p(c)]u(c_*) = \\ &= p(c)[u(c^*) - u(c_*)] + u(c_*). \end{aligned} \tag{121}$$

Since you are indifferent between $\ell_{p(c)}$ and $\ell(c)$, the EV Hypothesis implies that

$$E[\ell(c)] = E\left[\ell_{p(c)}\right] \tag{122}$$

or

$$u(c) = p(c)[u(c^*) - u(c_*)] + u(c_*). \tag{123}$$

Since $u(c^*) - u(c_*) \neq 0$, we may divide by $u(c^*) - u(c_*)$, and we obtain

$$p(c) = \frac{u(c) - u(c_*)}{u(c^*) - u(c_*)} = \alpha u(c) + \beta, \tag{124}$$

where

$$\alpha = \frac{1}{u(c^*) - u(c_*)} \tag{125}$$

and

$$\beta = \frac{-u(c_*)}{u(c^*) - u(c_*)}. \tag{126}$$

Since $\alpha > 0$, it follows that for all c and d in A,

$$u(c) > u(d) \Leftrightarrow p(c) > p(d). \tag{127}$$

Since u is an order-preserving utility function over K, we have

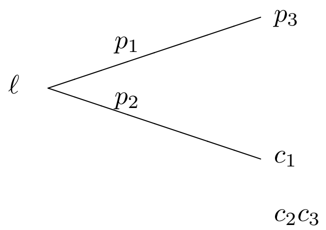
$$c \succ d \Leftrightarrow p(c) > p(d). \tag{128}$$

Thus p is an order-preserving utility function over K. Notice that computation of the utility function p only assumes that u exists, and does not require knowledge of u. Also, there is no need to assume that the individual whose utility function is being calculated actually computes expected values or utilities to make decisions, but only to assume that he acts *as if* he makes decisions on the basis of expected utilities. Finally, calculation of the number $p(c)$ is required only n times if there are n consequences being compared.

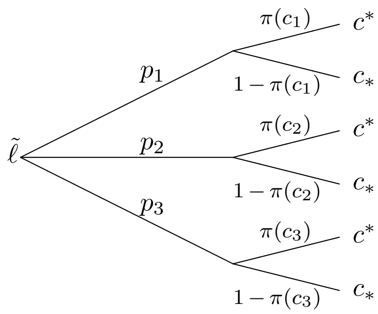
Suppose an individual is considering how much money to invest on a risky venture if his payoff is \$1000 if things go well and \$0 otherwise. He can calculate his utility of, say, \$400 by finding that value of $p = p(\$400)$ for which he is indifferent between having \$400 for certain and having a p-probability of obtaining \$1000, a $(1-p)$-probability of \$0. Note that $p(\$n)$ is almost certainly not $np(\$1)$. For example, $p(\$500)$ is almost certainly greater than $1/2$, so $p(\$1000) < 2p(\$500)$.

A. *The Basic Reference Lottery Ticket*

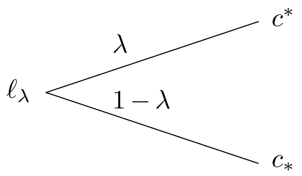
Let us fix the same two consequences c_* and c^* described before. A ticket to enter the lottery will be called π-*basic reference lottery ticket* or π-*brlt* for short. To make a choice between two complicated lotteries ℓ and ℓ', Raiffa suggests that we reduce them to π-*brlt*’s. Namely, if c is a consequence and $\ell(c)E\ell_{\pi(c)}$, then replace c is lotteries ℓ and ℓ' by the lottery $\ell_{\pi(c)}$. For example, the lottery



becomes the lottery



Using standard properties of tree diagrams, we can replace $\tilde{\ell}$ by the simple λ-*brlt*



where $\lambda = p_1\pi(c_1) + p_2\pi(c_2) + p_3\pi(c_3)$. We have $\ell E \ell_\lambda$. Similarly, we reduce an alternative lottery ℓ' to a δ-*brlt* ℓ_δ, with $\ell' E \ell_\delta$. Then, if preference P is strict weak, we should have

$$\ell P \ell' \Leftrightarrow \ell_\lambda P \ell_\delta \tag{129}$$

If the EV Hypothesis holds, we should have

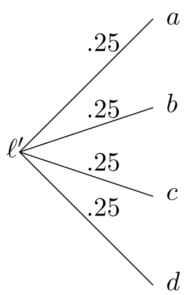
$$\ell_\lambda P \ell_\delta \Leftrightarrow \lambda > \delta \tag{130}$$

since $c^* \succ c_*$. The point of this procedure is that you can make choices between lotteries or acts or choices without knowing a utility function over consequences.

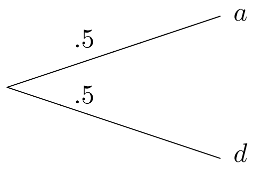
To illustrate how to make decisions using the idea of π-*brlt* let us give some examples. Suppose a small businessman faces decreasing sales in his present location and considers moving to a new location. Suppose for want of further information, he thinks it is equally likely that one of the following will happen if he moves:

$a =$ he will increase his sales,
$b =$ his sales will stay at their present level,
$c =$ his sales will continue to decrease,
$d =$ he will face bankruptcy.

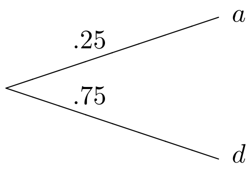
Thus, the businessman faces a choice between the lotteries $\ell(c)$ and



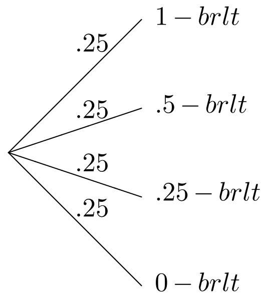
Suppose he chooses $c^* = a$ and $c_* = d$, and, on reflection, he is indifferent between $\ell(b)$ and



a .5-brlt; and he is indifferent between $\ell(c)$ and



a .25-brlt. Then he is indifferent between ℓ' and



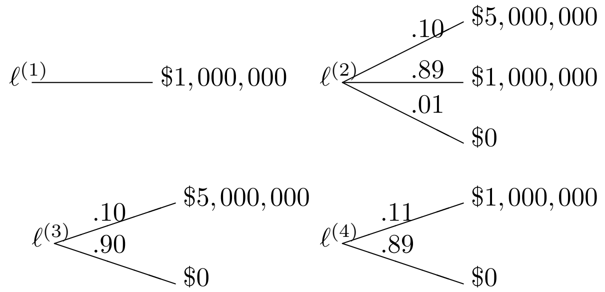
which is the same as a .4375-brlt, since

$$.25(1) + .25(.5) + .25(.25) + .25(0) = .4375. \tag{131}$$

Since .25 is smaller than $.4375$, the small businessman should choose ℓ' over $\ell(c)$ – that is, he should choose to move.

B. The Allais Paradox

Consider the following four lotteries:



We shall pose two problems: Problem 1 is to choose between $\ell^{(1)}$ and $\ell^{(2)}$, and Problem 2 is to choose between $\ell^{(3)}$ and $\ell^{(4)}$. The French economist Allais, and others, have reported that most subjects prefer $\ell^{(1)}$ to $\ell^{(2)}$ and $\ell^{(3)}$ to $\ell^{(4)}$. To quote Raiffa, most subjects reason as follows: "In Problem 1, I have a choice between \$1,000,000 for certain and a gamble where I might end up with \$0. Why gamble? In Problem 2, there is a good chance that I will end up with \$0 no matter what I do. The chances of getting \$5,000,000 are almost as good as getting \$1,000,000, so I imght as well go for the \$5,000,000 and choose" $\ell^{(3)}$ over $\ell^{(4)}$.

XII. QUASI-ADDITIVE UTILITY FUNCTIONS

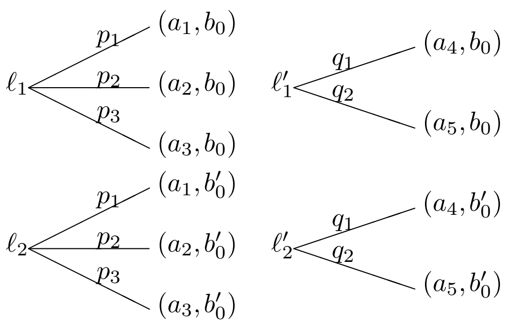
One of the necessary conditions on $(A_1 \times A_2, \succ)$ for the existence of an additive order-preserving utility function was independence: for all $a, a' \in A_1$ and $b_0, b_0' \in A_2$,

$$(a, b_0) \succ (a', b_0) \Leftrightarrow (a, b_0') \succ (a', b_0'), \tag{132}$$

and for all $a_0, a_0' \in A_1$ and $b, b' \in A_2$

$$(a_0, b) \succ (a_0, b') \Leftrightarrow (a_0', b) \succ (a_0', b'). \tag{133}$$

Suppose again that there is an *additive* value function u which satisfies the EV Hypothesis for $(A_1 \times A_2, L, R)$. Such a function u is an additive order-preserving utility function for $(A_1 \times A_2, \succ)$ if (L, R) extends $(K, \succ)$. If u exists, a stronger condition to be called strong independence (sometimes called utility independence) holds. We say that $(A_1 \times A_2, \succ, L, R)$ satisfies strong independence (on the first component) if, for all $b_0 \in A_2$, whenever ℓ and ℓ' are lotteries all of whose consequences have the form (a, b_0), then preferences for ℓ versus ℓ' do not change if the common value b_0 is changed in *every consequence* to the same b_0' in A_2, and the probabilities $p(a, b_0)$ and $p(a, b_0')$ are the same. A similar definition applies to the second component, and we say that *strong independence* holds if strong independence holds ob both components. To give an example, we observe that strong independence says that one prefers lottery ℓ_1 to lottery ℓ_1' if and only if one prefers lottery ℓ_2 to lottery ℓ_2', where the lotteries ℓ_1, ℓ_1', ℓ_2 and ℓ_2' are given by



Trivially, strong independence implies independence, if we assume that (L, R) extends $(K, \succ)$.

Under the EV Hypothesis, the condition of strong independence does not imply that there is an additive value function u on $A_1 \times A_2$ satisfying the EV Hypothesis or that there is an additive order-preserving utility function u on $(A_1 \times A_2, \succ)$. However, under certain simple assumptions, we shall conclude the existence of a u that is almost additive. Let us say a real-valued function u on $A_1 \times A_2$ is *quasi-additive* if there are real valued functions u_i on $A_i (1 = 1, 2)$ and a real number λ so that for all $(a, b) \in A_1 \times A_2$,

$$u(a, b) = u_1(a) + u_2(b) + \lambda u_1(a) u_2(b). \tag{134}$$

The third term on the right hand side represents an interaction effect. We shall see that under certain additional assumptions, strong independence implies the existence of a quasi-additive value or utility function.

Nonadditive representations for utility, in particular the quasi-additive representation, have had a wide variety of applications, for example, to transportation, medical decision-making, management, solid waste disposal, air pollution and urban services.

We say that the component A_1 is bounded if there are $a_*, a^* \in A_1$, such that for all $a \in A_1$,

$$a^* \succsim_1 a \succsim_1 a_*, \tag{135}$$

where

$$a \succsim_1 a' \Leftrightarrow (\exists b \in A_2) \left[(a, b) \succsim (a', b) \right]. \tag{136}$$

A similar definition applies on the second component.

Theorem 35: Suppose $(A_1 \times A_2, \succ, L, R)$ satisfies the following conditions:

1. The EV Hypothesis
2. String independence
3. Each component A_i is bounded
4. Continuity

Then there is a quasi-additive value function u on $A_1 \times A_2$ which satisfies the EV Hypothesis. If in addition (L, R) extends $(K, \succ)$, then u is a quasi-additive order-preserving utility function for $(A_1 \times A_2, \succ)$.

Remark 36: Under the EV Hypothesis, strong independence is a necessary condition for quasi-additivity.

Corollary 37: Under the hypothesis of Theorem 35, either there is an additive value function u on $A_1 \times A_2$ which satisfies the EV Hypothesis, or there is a multiplicative value function u on $A_1 \times A_2$ which satisfies the EV Hypothesis.

XIII. ISOMORPHISM WITH THE MULTIPLICATIVE MULTIATTRIBUTE UTILITY FUNCTION

In their seminal treatment of multiattribute utility (MAU) theory, Keeney and Raiffa (1976) show how certain conditions of independence among attributes yield the so called multiplicative multiattribute utility form

$$K u(\mathbf{x}) + 1 = \prod_{i=1}^{n} (k k_i u_i(x_i) + 1) \tag{137}$$

which can also be expanded as

$$\begin{aligned} u(\mathbf{x}) = & \sum_{i=1}^{n} k_i u_i(x_i) + k \sum_{i<j} k_i k_j u_i(x_i) u_j(x_j) + \\ & + k^2 \sum k_i k_j k_l u_i(x_i) u_j(x_j) u_l(x_l) + \ldots \\ & + k^{n-1} k_1 k_2 \ldots k_n u_1(x_1) \ldots u_n(x_n) \end{aligned} \tag{138}$$

The so called constant k is a function of all k_is, that aside from single-attribute utilities u_i just u free constant have to be estimated. k satisfies

$$1 + k = \prod_{i=1}^{n} (1 + k k_i). \tag{139}$$

In the particular case, if $k = 0$ from (138) and (139) we get

$$u(\mathbf{x}) = \sum_{i=1}^{n} k_i u_i(x_i) \tag{140}$$

and

$$\sum_{i=1}^{n} k_i = 1. \tag{141}$$

So the multiplicative model reduces to the additive form.

Use of the multiplicative model requires that the condition of mutual utility independence is satisfied. A subset of criteria to be independent of its complement. The criteria are said to be mutually independent if every subset of the criteria is utility independent of its complement. The multiplicative MAU model is able to represent fairly rich preference structures including nonlinearities in the attributes and interactions between attributes without restricting to unrealistic behavioral assumption. Furthermore it is quite tractable in practice. The assessment required for its calibration are neither prohibitive in number nor unduly difficult for the decision maker. In short, the multiplicative MAU model constitutes a good compromise of flexibility and practicality for application to real problems. It has been used in numerous applications.

In the following we will show that the Generalized Dombi operator is isomorph with the multiplicative multiattribute utility function which can be written in the following form:

$$u_M(\mathbf{x}) = \frac{1}{k}\left(\prod_{i=1}^{n}(1+kk_iu_i(x_i)) - 1\right) \tag{142}$$

The Generalized Dombi operator has the form

$$o_{GD,\gamma}^{(\alpha)}(\mathbf{x}) = \frac{1}{1+D_\gamma(\mathbf{x})} \tag{143}$$

where

$$D_\gamma(\mathbf{x}) = \left(\frac{1-\gamma}{\gamma}\left(\prod_{i=1}^{n}\left(1+\frac{\gamma}{1-\gamma}\left(\frac{1-x_i}{x_i}\right)^\alpha\right)\right)-1\right)^{1/\alpha} \tag{144}$$

Let us denote

$$k = \left(\frac{1-\gamma}{\gamma}\right)^{-1} \qquad z_i = k_iu_i(x_i) \tag{145}$$

and we will use the generator function of the Dombi operator i.e.

$$h(x) = \left(\frac{1-x}{x}\right)^\alpha \quad \text{and} \quad h^{-1}(x) = \frac{1}{1+x^{1/\alpha}} \tag{146}$$

Theorem 38: h(x) is isomorph transformation between the Generalized Dombi operator and the multiplicative multiattribute utility function, i.e. it is valid

$$u_M(\mathbf{x}) = h\left(o_{GD}\left(h^{-1}(z_i)\right)\right) \tag{147}$$

$$o_{GD}(\mathbf{z}) = h^{-1}\left(u_M\left(\mathbf{h}(\mathbf{x})\right)\right) \tag{148}$$

A. Isomorphism in the case of $k = 0$

Theorem 39: If $k = 0$ i.e. in the case of additive utility, (140) is isomorph with the Dombi operator.

REFERENCES

[1] J. Dombi, "Basic concepts for a theory of evaluation: The aggregation operator," *European Journal of Operation Research*, vol. 10, pp. 282–293, 1982.

[2] F. Esteva, "On some isomorphisms of demorgan algebras of fuzzy sets," *BUSEFAL*, vol. 8, pp. 49–60, 1981.

[3] J. Dombi, "General class of fuzzy operators, the demorgan class of fuzzy operators and fuzziness included by fuzzy operators," *Fuzzy Sets and Systems*, vol. 8, pp. 149–168, 1982.

[4] S. Roychowdhury, "New triangular operator generators for fuzzy systems," *IEEE Trans. of Fuzzy Systems*, vol. 5, pp. 189–198, 1997.

[5] H. Hamacher, *Über logische Aggregationen nicht-binär explizierter Entscheidungskriterien*, Rita G. Fischer Verlag, Frankfurt, 1978.

[6] Akira Kuwagaki, "Sur l'équation fonctionnelle rationnelle de la fonction inconnue de deux variables," *Memoirs of the College of Science, Vol. XXVIII, Mathematics No. 2*, 1952.

[7] E.P. Klement, R. Mesiar and E. Pap, *Triangular Norms*, Kluwer, 2000.