New control algorithm and defuzzification

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***Abstract*— Fuzzy defuzzification is one of the most important part of the fuzzy control. Several approaches exist. Mamdani uses the α-cuts and builds the union of the membership function. The resulted function is the starting point of the defuzzification process. In this article we define more natural way to get the membership function by using fuzzy arithmetics. The defuzzification is the optimum value of the resulted membership function. The main idea is that the membership function a soft inequality which we call distending function and we handled the left and right hand side of the inequalities..**

I. Introduction

In real world applications we often need to deal with imprecise quantities. They can be results of measurements or vague statements, e.g. I have about 40 dollars in my pocket, she is approximately 170cm tall. In arithmetics we can use $a < x$ and $x < b$ inequalites to characterize such quantities, e.g. if I have about 40 dollars then my money is probably more than 35 dollars and less than 45 dollars.

Fuzzy numbers can also used to represent imprecise quantities. Fuzzy numbers are created by *softening* the $a < x$ and $x < b$ inequalities, i.e. replacing the crisp characteristic function with two functions and by applying a fuzzy conjunction operator we get a soft interval. We refer to the softened inequalities as distending function.

We call the membership funtion corresponding to the $x < a$ interval the left side of the fuzzy number and denote it as μ_l. Similarly we refer to the membership function corresponding to the $x < b$ interval as the right side of the fuzzy number and denote it as μ_r.

Naturally one would like to execute arithmetic operations over fuzzy numbers. Fuzzy arithmetic operations are generally carried out using the α-cut method. In Section II we propose a new and efficient method for arithmetic calculations. The next two sections discuss the arithmetic operations and their properties for two classes of fuzzy membership funtions. Section III investigates additive pliant functions, i.e. membership functions represented as lines. Section IV presents multiplicative pliant functions, i.e. membership functions based on pliant inequalities. Finally, in section V we present the control algorithm and defuzzification.

II. Fuzzy arithmetics

Fuzzy arithmetic operations are based on the extension principle of arithmetics. In arithmetics we can find the result of an arithmetic operation by measuring the distance of the operands from the zero point than applying the operation on these distances. Fig. 1 presents this idea in case of addition.

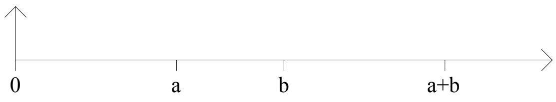

Fig. 1. Arithmetic addition

In fuzzy arithmetics we deal with fuzzy numbers. Fuzzy numbers are mappings from real numbers to the $[0, 1]$ real interval. Operations are executed by creating an α-cut for all $\alpha \in [0, 1]$ and using the arithmetic principle to get the resulting value for each α value. Fig. 2 demonstrates fuzzy addition with fuzzy numbers represented as lines. The dotted triangle number is the sum of the two other triangle numbers.

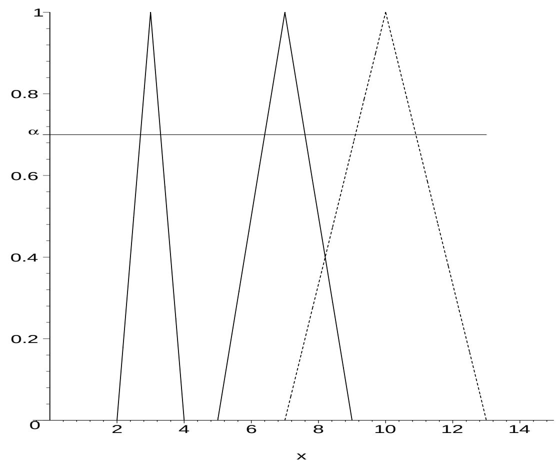

Fig. 2. Fuzzy addition with α-cut.

This way we can have all the well-known unary ($-x$, x^2) and binary operations ($x + y$, xy, $x \, mod \, y$) available as fuzzy operations. However the calculation of fuzzy operations with α-cut is tedious and often impractical. In this paper a new efficient method is proposed which is equivalent with the α-cut.

Fuzzy numbers are often composed of two strictly monotone functions, i.e. the left side denoted as μ_l, and the right side denoted as μ_r of the fuzzy number. Fuzzy operations can be carried out by first applying them to the left sides than to the right sides of the operands.

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This separation allows us to treat fuzzy numbers as strictly monotone functions when dealing with fuzzy arithmetic operations. In the following we omit the subscript from μ_l and μ_r and simply write μ with the inherent assumption that we shall only do arithmetic operations with functions representing the same side of fuzzy numbers.

In the next we introduce a new terms called distending function.

Definition 2.1: Distending function is a measure expressing the truth of an inequality, i.e.

$$\mu_a(x) = truth(a < x)$$

In the next we will distinguish two type of distending function, linear and sigmoid function. If we are using the first one we call it an additive pliant system. If we deal with sigmoid function we call it multiplicative pliant system. If we want to express a fuzzy member we can do it by using logical connectives.

$$truth(a < x < b) = truth(a < x) \wedge truth(x < b) \qquad (1)$$

where $\wedge$ stands for continuous valued conjunction operator. The expression (1) is membership function in a classical sense because the height usual is not 1.

Lemma 2.2: Let $\mu_1, \mu_2, \ldots, \mu_n$ $(n \geq 1)$ be strictly monotone functions representing fuzzy inequalities and let F be an n-ary fuzzy operation over them. If

$$\mu = F(\mu_1, \mu_2, \ldots, \mu_n),$$

then

$$\begin{aligned} \mu(z) &= \left(F\left({\mu_1}^{-1}, {\mu_2}^{-1}, \ldots, {\mu_n}^{-1}\right)\right)^{-1}(z) \iff \qquad (2) \\ \mu(z) &= \sup_{F(x_1,x_2,\ldots,x_n)=z} \min\left\{\mu_1(x_1), \mu_2(x_2), \ldots, \mu(x_n)\right\}. \end{aligned}$$

Proof: It can be easily verified that the method is equivalent with the α-cut. ■

Fig. 3 and 4 visualize the equivalence for the addition of lines. The left side shows the result of lines added together using α-cut with the result presented as the dotted line. On the right side we have simply added together the inverse functions of the two operands. The result is also presented as a dotted line. It can be seen from the figures that the result of the α-cut is indeed the inverse of the result in the right hand side figure.

We can state a theorem regarding the properties of fuzzy operations.

Theorem 2.3: Let $\mu_1, \mu_2, \ldots, \mu_n$ $(n \geq 1)$ be strictly monotone functions representing fuzzy inequalities and let F be an n-ary fuzzy operation over them. If

$$F\left({\mu_1}^{-1}, {\mu_2}^{-1}, \ldots, {\mu_n}^{-1}\right)$$

is strictly monotone then F has all the properties as its non-fuzzy interpretation.

Proof: It can be derived from Eq. 2 in Lemma 2.2. ■

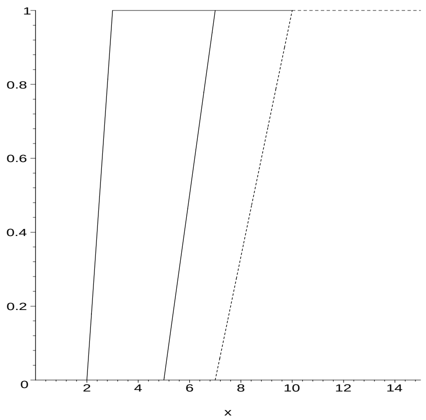

Fig. 3. α-cut addition

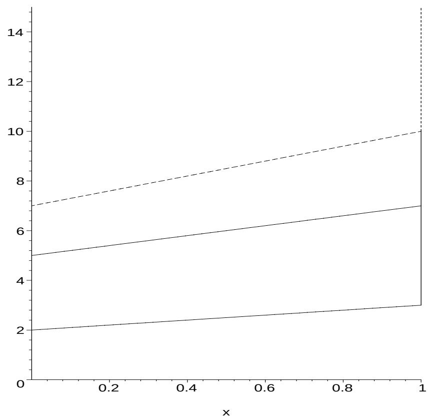

Fig. 4. Inverse of addition.

III. Additive Pliant

Triangle fuzzy numbers are commonly used to represent approximate values. A triangle fuzzy number has one line on each side. We can add triangle fuzzy numbers by first adding their left lines and than adding their right lines together.

Lemma 2.2 let us derive a general formula for adding lines.

Definition 3.1: We say that a line $l_a^{(m)}(x)$ is given by its mean value if

$$l_a^{(m)}(x) = \max\left(\min\left(m(x-a) + \frac{1}{2}, 0\right), 1\right) = [a <_m x]$$

as shown in Fig. 5.

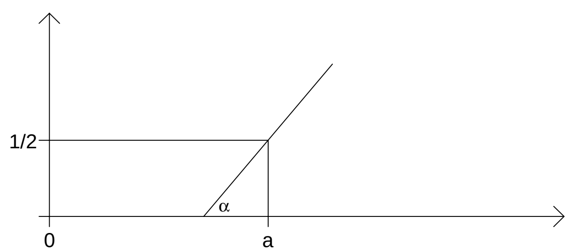

Fig. 5. Line given by its mean value a and tangent $m = \tan\alpha$.

The following properties can be seen from the Figure 5

$$\text{if} \quad a < x \quad \text{then} \quad [a <_m x] > \frac{1}{2},$$
$$\text{if} \quad a = x \quad \text{then} \quad [a <_m x] = \frac{1}{2},$$
$$\text{if} \quad a > x \quad \text{then} \quad [a <_m x] < \frac{1}{2}.$$

Interpretation of $l_a^{(m)}(x)$

$$l_a^{(m)}(x) = truth(a <_m x) = [a <_m x]$$

where m is the sharpness of the inequality.
The inverse of $l_a^{(m)}(x)$ denoted as $l^{-1}(y)$ can be calculated easily

$$l^{-1}(y) = \frac{y - \frac{1}{2}}{m} + a.$$

When we apply an arithmetic operation to pliant inequalities we need to make sure that the operation is meaningful, i.e. the pliant inequalities represent the same sides of the fuzzy numbers. The following criteria formulates this requirement.

Criteria 3.2:

$$\text{sgn}(m_1) = \text{sgn}(m_2) = \ldots = \text{sgn}(m_n)$$

must always hold.

A. Addition rule

Theorem 3.3: Let $l_i(x) = m_i(x - a_i) + \frac{1}{2}$ ($i \in \{1, \ldots, n\}$) lines given by their mean values. The fuzzy sum of l_i lines denoted as l is also a line and can be given as

$$l_a^{(m)}(x) = l_1(x) \oplus \ldots \oplus l_n(x) = m(x - a) + \frac{1}{2}$$

where

$$\frac{1}{m} = \sum_{i=1}^{n} \frac{1}{m_i} \quad \text{and} \quad a = \sum_{i=1}^{n} a_i.$$

Proof: Using Lemma 2.2 gives us

$$\begin{aligned} l^{-1}(y) &= \left(l_1{}^{-1}(y) + \ldots + l_n{}^{-1}(y)\right) \\ &= \sum_{i=1}^{n} \left(\frac{y - \frac{1}{2}}{m_i} + a_i\right) = \sum_{i=1}^{n} \left(\frac{y - \frac{1}{2}}{m_i}\right) + \sum_{i=1}^{n} a_i = \\ &= \left(y - \frac{1}{2}\right) \sum_{i=1}^{n} \frac{1}{m_i} + \sum_{i=1}^{n} a_i \end{aligned}$$

From here we have

$$l_a^{(m)}(x) = \frac{1}{\sum_{i=1}^{n} \frac{1}{m_i}} \left(x - \sum_{i=1}^{n} a_i\right) + \frac{1}{2}.$$

Substituting $\frac{1}{m}$ and a into the equation we get the desired result

$$l_a^{(m)}(x) = m\,(x - a) + \frac{1}{2}.$$

■

B. Subtraction

Calculations for subtraction yields

$$l_a^{(m)}(x) = l_1 \ominus l_2 = \frac{1}{\frac{1}{m_1} - \frac{1}{m_2}} \left(x - (a_1 - a_2)\right) + \frac{1}{2}.$$

Note: l does not exist when $m_1 = m_2$.
It is an important property that the result of the operation is also a line, i.e. the operation is closed for lines.

C. Multiplication by constant

Theorem 3.4: Let $l_a^{(m)}(x) = m(x - a) + \frac{1}{2}$ lines. The multiplication of l lines by w is also a line

$$l^*(x) = m^*(x - a^*) + \frac{1}{2}$$

where $mj = \frac{m}{w}$ and $a^* = wa$.

Proof: Using the inverse function and multiplying

$$wb^{-1}(x) = w\left(\frac{y - \frac{1}{2}}{m} + a\right)$$
$$y = \frac{m}{w}(x - wa) + \frac{1}{2}$$

■

D. Properties of Operations

Theorem 3.5: Fuzzy addition is commutative and associative over lines.

Proof: The properties can be easily seen from the construction of $\frac{1}{m}$ and a in Theorem 3.3. ■

IV. Multiplicative Pliant

Let us start by introducing a special fuzzy inequality, the *pliant inequality* and examine its most important properties.

Pliant Inequality Model

Definition 4.1: A pliant inequality is given as a sigmoid function of

$$truth(a < x) = \sigma_a^{(\lambda)}(x) = \frac{1}{1 + e^{-\lambda(x-a)}} = \{a <_\lambda x\}$$

where a is the mean value, i.e. $\sigma_a^{(\lambda)}(a) = \frac{1}{2}$.

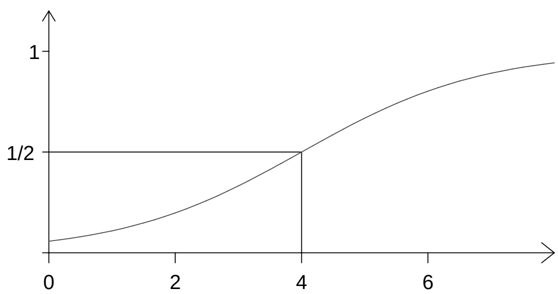

Fig. 6. Pliant inequality with $\lambda = 0.7$ and $a = 4$ parameters.

The following properties can be seen from the Figure 6

$$\text{if} \quad a < x \quad \text{then} \quad \{a <_\lambda x\} > \frac{1}{2},$$
$$\text{if} \quad a = x \quad \text{then} \quad \{a <_\lambda a\} = \frac{1}{2},$$
$$\text{if} \quad a > x \quad \text{then} \quad \{a <_\lambda x\} < \frac{1}{2}.$$

Definition 4.2: The inverse function of $\sigma_a^{(\lambda)}(x)$ is denoted as $\left(\sigma_a^{(\lambda)}\right)^{-1}(x)$ and can be calculated easily. Let

$$\sigma_a^{(\lambda)}(x) = \frac{1}{1 + e^{-\lambda(x-a)}} = \omega,$$

then

$$\begin{aligned} 1 &= \omega(1 + e^{-\lambda(x-a)}) = \omega + \omega e^{-\lambda(x-a)} \\ \frac{1-\omega}{\omega} &= e^{-\lambda(x-a)} \\ \ln\left(\frac{1-\omega}{\omega}\right) &= -\lambda(x-a) \\ x = \left(\sigma_a^{(\lambda)}\right)^{-1}(\omega) &= \frac{1}{-\lambda}\ln\left(\frac{1-\omega}{\omega}\right) + a. \end{aligned}$$

A. Addition

Theorem 4.3: Fuzzy addition is closed over pliant inequalities and the addition function can be given as

$$\sigma_{a_1}^{(\lambda_1)} \oplus \ldots \oplus \sigma_{a_n}^{(\lambda_n)} = \sigma_a^{(\lambda)} \qquad n \geq 1$$

where

$$\frac{1}{\lambda} = \sum_{i=1}^{n} \frac{1}{\lambda_i} \quad \text{and} \quad a = \sum_{i=1}^{n} a_i.$$

Proof: We prove by induction, if $i = 1$ then the statement is trivially true. Now let us assume that it holds for $i = n - 1$ and prove it for $i = n$,

$$\mu = \underbrace{\sigma_{a_1}^{(\lambda_1)} \oplus \ldots \oplus \sigma_{a_{n-1}}^{(\lambda_{n-1})}}_{\sigma_{a'}^{(\lambda')}} \oplus \sigma_{a_n}^{(\lambda_n)} = \sigma_{a'}^{(\lambda')} \oplus \sigma_{a_n}^{(\lambda_n)}$$

where

$$\frac{1}{\lambda'} = \sum_{i=1}^{n-1} \frac{1}{\lambda_i} \quad \text{and} \quad a' = \sum_{i=1}^{n-1} a_i.$$

Now by using Lemma 2.2 we have

$$\begin{aligned} &\mu^{-1}(z) = \\ &= \left(\sigma_{a'}^{(\lambda')}\right)^{-1}(z) + \left(\sigma_{a_n}^{(\lambda_n)}\right)^{-1}(z) = \\ &= \frac{1}{-\lambda'}\ln\left(\frac{1-z}{z}\right) + a' + \frac{1}{-\lambda_n}\ln\left(\frac{1-z}{z}\right) + a_n = \\ &= \left(\sum_{i=1}^{n-1}\frac{1}{-\lambda_i} + \frac{1}{-\lambda_n}\right)\ln\left(\frac{1-z}{z}\right) + \left(\sum_{i=1}^{n-1} a_i + a_n\right) = \\ &= \frac{1}{-\lambda}\ln\left(\frac{1-z}{z}\right) + a. \end{aligned} \tag{3}$$

If $\sum_{i=1}^{n} \frac{1}{\lambda_i} \neq 0$ then $\mu^{-1}(z)$ is a strictly monotone function and inverse of a pliant inequality. Therefore $\mu(x)$ is a pliant inequality with λ and a parameters:

$$\begin{aligned} &\mu(x) = \\ &= \left(\sigma_{a_1}^{(\lambda_1)} \oplus \ldots \oplus \sigma_{a_n}^{(\lambda_n)}\right)(x) = \\ &= \frac{1}{1 + e^{-\lambda}(x - (a))} = \sigma_a^{(\lambda)}(x). \end{aligned} \tag{4}$$

If $\sum_{i=1}^{n} \frac{1}{\lambda_i} = 0$ then the addition function does not exist since $\mu^{-1}(z) = a$ is a constant thus has no inverse. ■

B. Subtraction

We can derive subtraction from addition and negation.

Lemma 4.4: Fuzzy negation is closed over pliant inequalities and the negation function can be given as

$$\ominus\sigma_a^{(\lambda)} = \sigma_{-a}^{(-\lambda)}$$

Proof: Let

$$\mu = \ominus\sigma_a^{(\lambda)}$$

by using Lemma 2.2 we have

$$\begin{aligned} \mu^{-1}(z) &= -\left(\left(\sigma_a^{(\lambda)}\right)^{-1}(z)\right) = \\ &= \frac{1}{\lambda}\ln\left(\frac{1-z}{z}\right) - a = -\frac{1}{-\lambda}\ln\left(\frac{1-z}{z}\right) + (-a). \end{aligned}$$

Therefore

$$\mu(x) = \ominus\sigma_a^{(\lambda)}(x) = \frac{1}{1 + e^{-(-\lambda)(x-(-a))}} = \sigma_{-a}^{(-\lambda)}(x). \tag{5}$$

■

Theorem 4.5: Fuzzy subtraction is closed over pliant inequalities and the subtraction function can be given as

$$\sigma_{a_1}^{(\lambda_1)} \ominus \sigma_{a_2}^{(\lambda_2)} = \sigma_{a_1 - a_2}^{\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)}.$$

Proof:
Let

$$\mu = \sigma_{a_1}^{(\lambda_1)} \ominus \sigma_{a_2}^{(\lambda_2)}$$

by using Lemma 2.2 and Lemma 4.4 we have

$$\begin{aligned} \mu^{-1} &= \left(\sigma_{a_1}^{(\lambda_1)}\right)^{-1} - \left(\sigma_{a_2}^{(\lambda_2)}\right)^{-1} = \\ &= \left(\sigma_{a_1}^{(\lambda_1)}\right)^{-1} + \left(-\left(\sigma_{a_2}^{(\lambda_2)}\right)^{-1}\right) = \\ &= \left(\sigma_{a_1}^{(\lambda_1)}\right)^{-1} + \left(\ominus\sigma_{a_2}^{(\lambda_2)}\right)^{-1} = \\ &= \left(\sigma_{a_1}^{(\lambda_1)} \oplus \left(\ominus\sigma_{a_2}^{(\lambda_2)}\right)\right)^{-1}, \end{aligned}$$

therefore

$$\left(\sigma_{a_1}^{(\lambda_1)} \ominus \sigma_{a_2}^{(\lambda_2)}\right) = \sigma_{a_1}^{(\lambda_1)} \oplus \left(\ominus\sigma_{a_2}^{(\lambda_2)}\right). \tag{6}$$

Substituting Eq. 5 and Eq. 4 into Eq. 6 we get the desired result

$$\left(\sigma_{a_1}^{(\lambda_1)} \ominus \sigma_{a_2}^{(\lambda_2)}\right) = \sigma_{a_1}^{(\lambda_1)} \oplus \sigma_{-a_2}^{(-\lambda_2)} = \sigma_{a_1-a_2}^{\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)}. \quad (7)$$

■

Note: The function does not exist in case of $\lambda_1 - \lambda_2 = 0$.

C. Multiplication by constant

Theorem 4.6: Fuzzy multiplication is closed over multiplication of a constant.

$$w \circ \sigma_a^\lambda = \sigma_{a^*}^{\lambda^*}$$

where

$$a^* = wa^* \quad \text{and} \quad \lambda^* = \frac{\lambda}{w}.$$

Proof: As we have seen:

$$(\sigma_a^\lambda)^{-1}(z) = -\frac{1}{\lambda}\ln\left(\frac{1-z}{z}\right) + a.$$

Multiplication by w

$$w \circ \sigma_a^\lambda = -\frac{w}{\lambda}\left(\ln\left(\frac{1-z}{z}\right) + a\right)$$

and taking the inverse we get the result. ■

D. Properties of Operations

Theorem 4.7: Fuzzy addition is commutative and associative over pliant inequalities.

Proof: These properties can be easily seen from the construction of $\frac{1}{m}$ and a in Theorem 4.3. ■

Distending interval can be decomposed to left hand side and right hand side inequalities. In the previous sections we have used this decomposition. After then we carry out the arithmetic operation we need to combine the left hand side and the right hand side results using a fuzzy conjunction operator.

V. Fuzzy control and defuzzification

Let us suppose that we have the following rule base system:

$$\text{if} \quad \mathcal{L}_j(x) \quad \text{then} \quad \{a_j <_{\lambda_l^j} y <_{\lambda_r^j} b_j\} \quad j = 1\ldots m.$$

For the defuzzification we devided every rule into two:

$$\begin{array}{lll} \text{if} & \mathcal{L}_j(\mathbf{x}) & \text{then} \quad \{y < a_j\} \\ \text{if} & \mathcal{L}_j(\mathbf{x}) & \text{then} \quad \{y > b_j\} \end{array}$$

i.e. we use the left hand side and right hand side of the inequality separately.

If we want to get the fuzzy interval then we use a fuzzy conjunctive operator, i.e.

$$[a_j <_{m_l^j} y <_{m_r^j} b_j] \equiv [a_j <_{m_l^j} y] \wedge [y <_{m_r^j} b_j],$$
$$\{a_j <_{\lambda_l^j} y <_{\lambda_r^j} b_j\} \equiv \{a_j <_{\lambda_l^j} y\} \wedge \{y <_{\lambda_r^j} b_j\}.$$

In fuzzy control we have to evaluate the rules on certain $\mathbf{x}^*$ value. Let denote the validity of j-th rule by w_j:

$$w_j = \frac{\mathcal{L}_j(\mathbf{x}^*)}{\sum_{i=1}^{n} \mathcal{L}_i(\mathbf{x}^*)}.$$

Now we can use the fuzzy result of the arithmetic chapter. We can determine the weighted left (and right) hand sided distending function in the additive and multiplicative cases.

On the basis of the previous result it is easy to show that the convex combination of the parameters of distending functions at additive and multiplicative case are:

$$a = \sum_{i=1}^{n} w_i a_i \qquad b = \sum_{i=1}^{n} w_i b_i$$
$$\frac{1}{\lambda^l} = \sum_{i=1}^{n} \frac{w_i}{\lambda_i^l} \qquad \frac{1}{\lambda^r} = \sum_{i=1}^{n} \frac{w_i}{\lambda_i^r}.$$

Similar way we can get the result for linear function. We get the same result.

$$a = \sum_{i=1}^{n} w_i a_i \qquad b = \sum_{i=1}^{n} w_i b_i$$
$$\frac{1}{m^l} = \sum_{i=1}^{n} \frac{w_i}{m_i^l} \qquad \frac{1}{m^r} = \sum_{i=1}^{n} \frac{w_i}{m_i^r}$$

We show the calculation multiplicative pliant case.

Let us use Dombi operator to construct the distending interval.

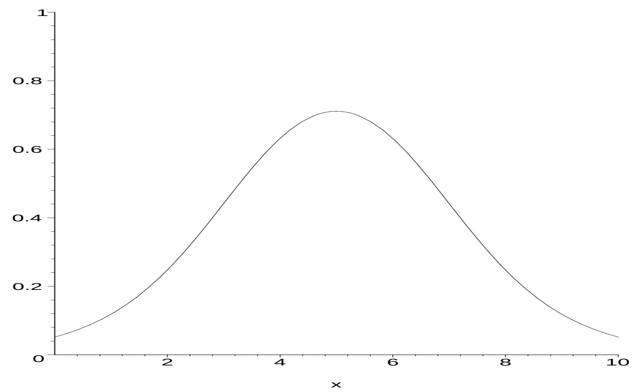

Fig. 7. Conjunction with the Dombi operator.

This case the result is a smooth curve that reaches its maximum value. The Dombi operator also retains the parameter values of the two functions as the following calculations show if $\alpha = 1$.

$$c\left(\sigma_a^{(\lambda_l)}, \sigma_b^{(\lambda_r)}\right) = \frac{1}{1 + \frac{1-\sigma_a^{(\lambda_l)}}{\sigma_a^{(\lambda_l)}} + \frac{1-\sigma_b^{(\lambda_r)}}{\sigma_b^{(\lambda_r)}}} = \frac{1}{1 + e^{-\lambda_l(y-a)} + e^{-\lambda_r(y-b)}}.$$

This enables us to easily decompose the result for further arithmetic operations. These properties makes the Dombi operator a good choice for constructing pliant numbers based on pliant inequalities.

Defuzzification of the resulted membership function can be done by finding the maximum values.

$$y^* = arg\max \frac{1}{1 + e^{-\lambda_l(x-a)} + e^{-\lambda_r(y-b)}}$$

Using the derivate of the function we get:

$$y^* = \frac{a+b}{\lambda_l + \lambda_r} + \frac{1}{\lambda_l + \lambda_r}(\ln(\lambda_l) - l(\lambda_r))$$

If $\lambda_l = \lambda_r$, then

$$y^* = \frac{a+b}{2}.$$

REFERENCES

[1] H. T. Nguyen, M. Sugeno, *Fuzzy Systems Modeling and Control,* Kluwer Academic Publishers, 1998.