Vagueness measure

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Abstract

The basic idea of fuzzy set was the introduction of membership function. This function approximates the classical characteristic function. It is interesting question how close are we from the characteristic function, when we use a certain membership function. The measure called fuzziness measure.

In our article we give an operator dependent fuzziness measure called vagueness measure.

1 Introduction

Our opinion is one of the most important terms, because on this basis we can prove "convergence theorems" in the sence "if less fuzziness the input variables then less fuzziness in the result".

In the fuzzy literature we do not find such theorems, because membership function, operators and fuzziness measures not interrelated and so it is hopeless to prove such convergence theorem.

In the pliant concept we have distending function instead of membership function based on the operator , and now we will establish vagueness measure also on the generator function of operators. The basis on this consistent concept we can prove such convergence theorem.

First we make a closer look to the fuzziness measure: Let $\mu(x)$ the membership function and $d(\mu)$ the fuzziness measure.

Fuzziness measure

We give a list of desirable properties for a measure of fuzziness. Various combinations of these can be found in the literature. It is remarkable that different authors demand different properties for measure d.

(P1) *Sharpness*: $d(\mu) = 0$ iff $\mu(x)\epsilon\{0,1\}$ for all $x\epsilon X$ (i.e. μ is "sharp").

(P2) *Maximality*: $d(\mu)$ is maximum iff $\mu(X) = \{\frac{1}{2}\}$.

(P3) *Resolution*: $d(\mu^*) \leq d(\mu)$ if μ^* is any sharpened version of μ, that is: $\mu^*(x) \leq \mu(x)$ if $\mu(x) \leq \frac{1}{2}$ and $\mu^*(x) \geq \mu(x)$ if $\mu(x) \geq \frac{1}{2}$.

(P4) *Symmetry* (about $\frac{1}{2}$): $d(\mu) = d(\eta(\mu))$.

(P5) *Valuation*: the function d is valuation on $[0,1]^X$, i.e.:

$$d(\mu_1 \cup \mu_2) + d(\mu_1 \cap \mu_2) = d(\mu_1) + d(\mu_2).$$

(P6) *Generalized additivity*: there exist mappings $s, t : [0,1] \to [0, \infty)$ such that:

$$d(\mu_1 \mu_2) = d(\mu_1) t(P(\mu_2)) + d(\mu_2) s(P(\mu_1)).$$

where

$$P(\mu) = \sum_{i=1}^{n} \mu(x_i) \qquad \text{or} \qquad P(\mu) = \int \mu(x) dx$$

for all $\mu_1 \epsilon [0,1]^X$ and $\mu_2 \epsilon [0,1]^Y$, X and Y any finite sets.

The properties (P1)-(P4) are natural requirements for a measure of fuzziness. All measures introduced so far satisfy these properties. But the meanings of (P5) and (P6) are not completely clear and therefore their place in such a list is debatable.

The survey the existing fuzziness measures

We give in this section a brief summary about the various fuzziness measures investigated so far and about their properties.

1. *DeLuca* and *Termini* (1972) [2]
   Measure:

$$d(\mu) = -K \sum_{i=1}^{N} \{\mu(x_i) log(\mu(x_i)) + (1 - \mu(x_i) log(1 - \mu(x_i)))\}$$

2. *Kaufmann* (1975)

   (a) Measure (using the generalized relative Hamming distance):

$$d(\mu) = \frac{2}{N} \sum_{i=1}^{N} |\mu(x_i) - \mu_{\frac{1}{2}}(x_i)|$$

   the $\frac{1}{2}$-cut of μ is defined by:

$$\mu_{\frac{1}{2}}(x) = 0, \qquad \text{if} \qquad \mu(x) < \frac{1}{2},$$

$$\mu_{\frac{1}{2}}(x) = 1, \qquad \text{if} \qquad \mu(x) \geq \frac{1}{2}$$

(b) Measure (using the generalized relative Euclidean distance):

$$d(\mu) = \frac{2}{N^{\frac{1}{2}}} \left\{ \sum_{i=1}^{N} (\mu(x_i) - \mu_{\frac{1}{2}}(x_i))^2 \right\}^{\frac{1}{2}}$$

3. *Loo* has proposed a general mathematical form for d, thus a large class of measures.

   Measure:

$$d(\mu) = F \left\{ \sum_{i=1}^{N} c_i F_i(\mu(x_i)) \right\}$$

   where $c_i \epsilon R^+$; for all i F_i is real-valued function such that $F_i(0) = F_i(1) = 0$,
   $F_i(u) = F_i(1-u)$ for all $u \epsilon [0,1]$, F_i is strictly increasing on $[0, \frac{1}{2}]$; and F is a positive, increasing function.

   Properties: (P1)-(P4) and if F is linear, then (P5) also holds.

   The fuzziness measure introduced by DeLuca and Termini is obtained as special case if $F(x) = x$ and for all $i:\ c_i = K$ and

$$F_i(x) = -(xlog(x) + (1-x)log(1-x))$$

4. *Trillas* and *Riera* (1978) have proposed another large class of fuzziness measures.

   Measure:

$$d(\mu) = \sum_{i=1}^{N} w(x_i) F(\mu(x_i))$$

   where μ is a real-valued function on $[0,1]$, $F(x) = 0$ iff $x \epsilon \{0,1\}$, $F(x)$ is nondecreasing on $[0, \frac{1}{2}]$, $F(x)$ is nonincreasing on $[\frac{1}{2}]$, and w is a positive weight function from X to R.

5. *Emptoz* (1981) [4]
   Measure:

$$d(\mu) = \frac{1}{N} \sum_{i=1}^{N} F(\mu(x_i))$$

   where F is a real-valued function on $[0,1]$ such that $F(0) = 0$,
   $F(x) = F(1-x)$ and F is increasing on $[0, \frac{1}{2}]$.

6. *Ebanks* (1983) [3]
   Measure:

$$d(\mu) = \sum_{i=1}^{N} \mu(x_i)(1 - \mu(x_i))$$

where $\mu \epsilon [0,1]^X$.

In the light of fuzziness measure we will introduced the vagueness measure.

2 Vagueness measure induced by operators

In the pliant system the logical values are basically coming from inequalities. If we are at the border of the inequality i.e. just the equality are fullfiled we are not sure to be inside or outside a region. If we move away from the border we are more sure to be inside or outside the region. Why are we vague on the border? Because small changes radically change the logical value. If we want stabile statements we have to avoid to be just on the border. It is important to measure the vagueness and also important to know how depends the vagueness on the input values.

As we mentioned the idea and construction of vagueness measure are coming from fuzziness measure. In 1972 DeLuca and Termini [1] introduced fuzziness measure which is used for membership function. In pliant concept we will use the vagueness measure to distending function on inflated region.

On the property of $d(\mu)$ we define $\mathcal{V}(x)$ the following way.

Definition 1. *Vagueness measure in pliant system is:*

$$\mathcal{V}^*(\delta(\underline{x})) = \frac{1}{n}\sum_{i=1}^{n} \bar{c}(\delta(x_i), \eta(\delta(x_i))) \tag{1}$$

and

$$\mathcal{V}(x) = \bar{c}(x, \eta(x)) \tag{2}$$

$\mathcal{V}(x)$ called the vagueness function.

The normalized vagueness function is:

$$\mathcal{V}^N(x) = \frac{1}{\bar{c}(\nu_0, \eta(\nu_0))} = \bar{c}(x, \eta(x)) \tag{3}$$

Because

$$\mathcal{V}(x) = \bar{c}(x, \eta(x)) = f^{-1}\left(\frac{1}{2}\left(f(x) + f(\nu_0)\frac{f(\nu)}{f(x)}\right)\right) \tag{4}$$

is the representation of vagueness function and we want that the maximum value should be at ν_0, so we have to find the minimum of:

$$Y = X + \frac{A}{X} \tag{5}$$

which is at $X = \sqrt{A}$ i.e. $f(x) = \sqrt{f(\nu_0)f(\nu)}$, and the maximum is at ν_0 so $f(\nu_0) = f(\nu)$, and we get

$$\mathcal{V}_{\nu_0}(x) = f^{-1}\left(\frac{1}{2}\left(f(x) + \frac{f^2(\nu_0)}{f(x)}\right)\right) \tag{6}$$

if $f(\nu_0) = 1$

$$\mathcal{V}(x) = f^{-1}\left(\frac{1}{2}\left(f(x) + \frac{1}{f(x)}\right)\right) \tag{7}$$

Let $\delta(x)$ distending function given in discrete values

$$\underline{x} = \{x_1, x_2, \dots x_n\}.$$

We give a list of properties for the measure of vagueness:

1. Sharpness (No vagueness)

   $\mathcal{V}(x) = 0 \qquad iff \qquad x \epsilon \{0, 1\}$

2. Maximality (maximal vagueness)

   $\frac{1}{\nu_0}\mathcal{V}(x) = 1 \qquad iff \qquad x = \nu_0$

3. Symmetry

   $\mathcal{V}(x) = \mathcal{V}(\eta(x))$

4. Monotonicity

   $\mathcal{V}(x_1) < \mathcal{V}(x_2)$ if

   $x_1 < x_2$ and $x_1 \le \nu_0$

   or

   $x_1 > x_2$ and $x_1 \ge \nu_0$

Vagueness measure: Dombi operator case

We have defined the vagueness function is the following way:

$$\mathcal{V}(x) = f^{-1}\left(\frac{1}{2}\left(f(x) + \frac{1}{f(x)}\right)\right)$$

Let $\quad f(x) = \left(\frac{1-x}{x}\right)^\alpha \quad$ then

$$F_\alpha(x) = \frac{1}{1 + \left(\frac{1}{2}\left(\frac{1-x}{x}\right)^\alpha + \frac{1}{2}\left(\frac{x}{1-x}\right)^\alpha\right)^{\frac{1}{\alpha}}} \tag{8}$$

If $\alpha = 1$ then

$$\mathcal{V}(x) = 2x(1-x) \tag{9}$$

Vagueness measure of the sigmoid function if $\alpha = 1$

The sigmoid function is:

$$\delta^{(\lambda)}(x) = \frac{1}{1+e^{-\lambda x}}$$

$$\mathcal{V}(\delta^{(\lambda)}(x)) = 2\delta^{(\lambda)}(x)(1-\delta^{(\lambda)}(x)) \tag{10}$$

$$\mathcal{V}^*(\delta^{(\lambda)}(x)) = 2\int\limits_{-\infty}^{\infty} \frac{(\delta^{(\lambda)}(x))'}{\lambda} dx = \frac{2}{\lambda}\left[\delta^{(\lambda)}(x)\right]_{-\infty}^{\infty} = \frac{2}{\lambda} \tag{11}$$

Vagueness measure and aggregation:

Theorem 2.

$$a(w_1, \mathcal{V}(x_1); w_2, \mathcal{V}(x_2) \dots w_n, \mathcal{V}(x_n)) \quad \leq \quad \mathcal{V}(a(w_1, x_1; w_2, x_2 \dots w_n, x_n))$$

where $\sum w_i = 1 \qquad w_i \geq 0$

Proof.

$$f^{-1}\left(\prod_{i=1}^{n}\left(\frac{1}{2}\left(f(x_i)+\frac{1}{f(x_i)}\right)\right)^{w_i}\right) \quad \leq \quad f^{-1}\left(\frac{1}{2}\left(\prod_{i=1}^{n} f^{w_i}(x_i) + \frac{1}{\prod\limits_{i=1}^{n} f^{w_i}(x_i)}\right)\right)$$

It is enough to show

$$\left(a+\frac{1}{a}\right)^{w} \quad \leq \quad a^{w} + \frac{1}{a^{w}} \qquad\qquad w \epsilon [0,1] \qquad a > 0$$

Let: $w = \frac{1}{n}$

$$\left(a+\frac{1}{a}\right)^{\frac{1}{n}} \quad \leq \quad a^{\frac{1}{n}} + \frac{1}{a^{\frac{1}{n}}}$$

and this is true.

$w = \frac{m}{n}$ where $n > m$

$$\left(a+\frac{1}{a}\right)^{m} \quad \leq \quad \left(a^{\frac{m}{n}} + \frac{1}{a^{\frac{m}{n}}}\right)^{n}$$

and it is true.

If w is real number we can approximate by rationals.

$\blacksquare$

3 Vagueness measure and entropy

In the case of multiplicative pliant we define the vagueness function using the conjunctive operator and negation

$$\mathcal{V}(x) = \bar{c}(x, \eta(x)) \tag{12}$$

The normalized $\mathcal{V}(x)$ is

$$\mathcal{V}^N(x) = \frac{1}{\bar{c}(\nu_0, \eta(\nu_0))} = \bar{c}(x, \eta(x)) \tag{13}$$

Additive pliant case

$$c[x, \eta(x)] = 0$$

and (13) has no meaning.

We will use the following notation:

$$[x] = \begin{cases} 1 \text{ if } x > 1 \\ x \text{ if } 0 \leq x \leq 1 \\ 0 \text{ if } x < 0 \end{cases} \tag{14}$$

First we try to find a proper vagueness measure for the Lukasiewicz operator case.

$$c[x, y] = [x + y - 1]$$

The generator function of $c[x, y]$ is $f(x) = 1 - x$.

Let define $c_\alpha[x, y]$ using the $(1 - x)^\alpha$ generator function, where $\alpha > 0$.

$$c_\alpha[x, y] = 1 - [((1 - x)^\alpha + (1 - y)^\alpha)^{\frac{1}{\alpha}}]$$

So

$$c_1[x, y] = [x + y - 1]$$

Let use $\eta(x) = 1 - x$. Then

$$c_\alpha[x, 1 - x] = 1 - ((1 - x)^\alpha + x^\alpha)^{\frac{1}{\alpha}} \neq 0$$

Let define the Vagueness measure for additive pliant case based on (13) in the following way:

$$\mathcal{V}[x] = \lim_{\alpha \to 1} \mathcal{V}_\alpha(x) = \lim_{\alpha \to 1} \frac{1 - ((1 - x)^\alpha + x^\alpha)^{\frac{1}{\alpha}}}{1 - \left(\frac{1}{2}\right)^{\frac{1}{\alpha} - 1}} \tag{15}$$

Theorem 3.

$$\mathcal{V}[x] = \frac{1}{ln(2)}(xlnx + (1-x)ln(1-x)) \tag{16}$$

i.e. we get the ***Shannon entropy*** *and on the theorem it's belong to the Lukosiewicz logic.*

Proof. To calculate the limes of (15) we will use the L' Hospital rule, and we make the derivation on α.

First we transform the nominator into the following form:

$$\frac{1 - e^{\frac{1}{\alpha}\ln((1-x)^{\alpha}+x^{\alpha})}}{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}-1}}$$

The derivative of the denominator is at $\alpha = 1$.

$$-\left(-\frac{1}{\alpha^2}\ln\left((1-x)^{\alpha}+x^{\alpha}\right) + \frac{1}{\alpha}\frac{1}{(1-x)^{\alpha}+x^{\alpha}}\left[x^{\alpha}\ln x + (1-x)^{\alpha}\ln(1-x)\right]\right)\Big|_{\alpha=1} =$$

$$-\left(x\ln x + (1-x)\ln(1-x)\right)$$

where we are using

$$\frac{d(x^{\alpha})}{d\alpha}\Big|_{\alpha=1} = x^{\alpha}\ln x\Big|_{\alpha=1} = x\ln x \tag{17}$$

The derivative of the denominator is

$$-2\left(\frac{1}{2}\right)^{\alpha}\ln\left(\frac{1}{2}\right)\Big|_{\alpha=1} = \ln 2$$

■

The Theorem 3 can be generalized substituting x by $f(x)$, then we get:

$$\mathcal{V}[x] = -\frac{1}{\ln(2)}(f(x)\ln(f(x)) + (1-f(x))\ln(1-f(x))) \tag{18}$$

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