

# Modalities in pliant system

József Dombi  
 Department of Informatics  
 University of Szeged  
 H-6720 Szeged, Árpád tér 2., Hungary  
 dombi@inf.u-szeged.hu

**Abstract**—There are several attempt to achieve unary operators is continuous valued logic and in fuzzy theory. They play an important rule modeling natural language expression, inference algorithm, fuzzy control etc. While other areas as conjunction, disjunction, aggregation, etc are theoretically deeply studied, the modalities (in fuzzy theory) hedges preserved its heuristic nature. In our paper we give a general form of the modalities by extension of the definition of the logical modalities and by using different negations. To carry out this we need such strict monotone operator system, where infinite number of negation are consistent with the logical connectives, which is the pliant system.

## I. INTRODUCTION

The pliant system were introduced in [1], [2]. The main issue of these articles that the operators having the form if

$$c(x_1, x_2, \dots, x_n) = f^{-1} \left( \sum_{i=1}^n f(x_i) \right) \quad (1)$$

then

$$d(x_1, x_2, \dots, x_n) = f^{-1} \left( \frac{1}{\sum_{i=1}^n f(x_i)} \right) \quad (2)$$

$$a(x_1, x_2, \dots, x_n) = f^{-1} \left( \prod_{i=1}^n f(x_i) \right) \quad (3)$$

$$\eta_{\nu_0}(x) = f^{-1} \left( f(\nu_0) \frac{f(x)}{f(\nu)} \right) \quad (4)$$

$$\eta_{\nu_*}(x) = f^{-1} \left( \frac{f^2(\nu_*)}{f(x)} \right) \quad (5)$$

where  $c(x_1, x_2, \dots, x_n)$  is a strict monotone t-norm, introduced by a generator function  $f(x)$ ,  $d(x_1, \dots, x_n)$  a strict monotone t-conorm. (In this article we call it conjunctive and disjunctive operator.)  $a(x_1, \dots, x_n)$  is a representable uninorm (in this article we call it aggregation).  $\eta_{\nu_0}$  and  $\eta_{\nu_*}$  are involutive negations. (In this article we call them negations.)

The pliant system have several fine properties as independent form the neutral value of the negation ( $\nu$ ) the system builds De Morgan class, and the pan operator of the conjunctive and disjunctive operator are identity and gives the aggregation operator. The aggregation is self De Morgan using (4) or (5) negation. The theorems are strict, that necessity and sufficient conditions ensures the form of such system.

From our point of view the negations are important.

## II. NEGATION AND ITS PROPERTIES

The negation operator in the pliant system has the form:

$$\eta_{\nu, \nu_0}(x) = f^{-1} \left( f(\nu_0) \frac{f(x)}{f(\nu)} \right) \quad (6)$$

$$\eta_{\nu_*}(x) = f^{-1} \left( \frac{f^2(\nu_*)}{f(x)} \right) \quad (7)$$

Because the generator function is determined up to the multiplicative constant, we can choose that

$$f(\nu_0) = 1$$

and so

$$\eta_{\nu}(x) = f^{-1} \left( \frac{f(x)}{f(\nu)} \right) \quad (8)$$

If  $f(\nu_0) = 1$  and if  $\nu_0 = \nu$ , we get:

$$\eta(x) = f^{-1} \left( \frac{1}{f(x)} \right) \quad (9)$$

*Definition 2.1:*

$$\eta_1(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases} \quad \eta_0(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases}$$

$\eta_0(x)$  is the most strict negation, while  $\eta_1(x)$  is the most non-strict negation (see figure).

$\nu$  is the neutral value of the negation and can be understand as the strictness of the negation i.e. if  $\nu_1 < \nu_2$  then  $\eta_{\nu_1}(x)$  is called more strict negation than  $\eta_{\nu_2}(x)$

*Theorem 2.2:* The negation  $\eta(x) = f^{-1} \left( f(\nu_0) \frac{f(x)}{f(\nu)} \right)$  has the following properties:

- continuous
- strictly monotonous decreasing
- Correspondence principle is valid

$$\eta_{\nu, \nu_0}(0) = 1 \quad \eta_{\nu, \nu_0}(1) = 0$$

- Involutivness is valid

$$\begin{aligned} \eta_{\nu, \nu_0}(\eta_{\nu, \nu_0}(x)) &= x \\ \eta_{\nu}(\eta_{\nu}(x)) &= x \\ \eta_{\nu_*}(\eta_{\nu_*}(x)) &= x \end{aligned}$$

- Neutral value property

### III. MODALITIES INDUCED BY NEGATIONS

The linguistic hedges “very” or “very very” express the modal hedges necessity, and similarly the hedge “more or less” express the possibility hedge.

From this starting point, the hedges used in fuzzy logic based on the extension of modal logic into continuous case. We start with the negation and we use two types of it, one is strict the other is less strict. We show that this two negations can express the modal hedges.

The modal logic which is a field of mathematical logic [?] can be considered from the viewpoint of syntax as a logical system obtained by adding logical symbols and inference rules. From the viewpoint of semantics the modal logic can also be considered as a part logical system.

Zadeh introduced the modifier functions of fuzzy sets called linguistic hedges. A number of studies [] have been made which discuss the fuzzy logic and fuzzy reasoning with linguistic truth values. However, there has not been given a systematic view in the construction of linguistic hedges, which have mutually reverse effects, such as in the case of “very” and “more or less”.

This issue relates the linguistic hedges with mutually reverse effects to the modal operators with mutually reverse modal concepts and constructs the linguistic modal hedges called necessity and possibility hedges. The construction is based on the fact that modal operator can be realized by combining two kinds of negations i.e. negation in the reverse of classical logic and negation in the reverse of intuitionistic logic, which is a strict negation.

One can consider the necessity hedge  $\Box x$  and the possibility hedge  $\Diamond x$ , which have mutually reverse effects.

We will show that both operators are unary functions, and because we show that both has a common form in the pliant system we will denoted by  $\tau_\nu(x)$ . Depends on  $\nu$  value we get the necessity or the possibility hedges.

As we mentioned above in the modal logic we have two more operations compare to the classical logic: i.e. necessity and possibility, and in the modal logic there are two basic identities:

$$impossible(x) = necessity(not(x)) \quad (10)$$

$$possible(x) = not(impossible(x)) \quad (11)$$

In our context we modelize  $impossible(x)$  with a more strict negation than  $not(x)$ .

In the intuitionistic logic is another negation also take into consideration, because the law of Excluded Middle does not hold.  $\sim_1 x$  does not imply  $x$  is not, although “there exists a

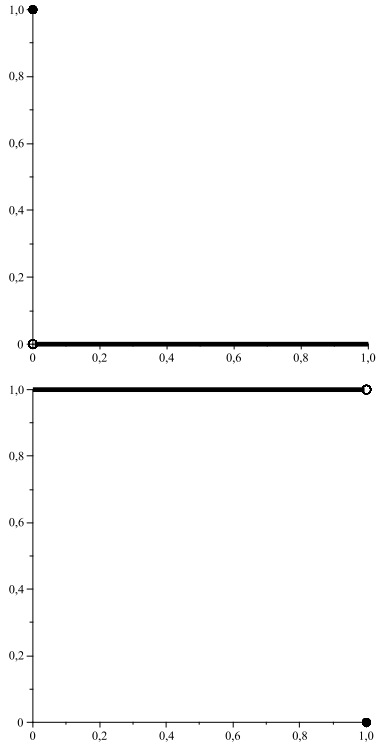


Fig. 1.  $\eta_0$  strict and  $\eta_1$  non-strict negation

$$\eta_{\nu, \nu_0}(\nu) = \nu_0$$

$$\eta_\nu(\nu) = \nu_0$$

$$\eta_{\nu_*}(\nu_*) = \nu_*$$

#### f) Interchangability

$$\eta_{\nu, \nu_0}(x) = \eta_{\nu_0, \nu}(x)$$

**Proof.** Trivial using the representation of negation.

In Figure 2 we can see the negation function

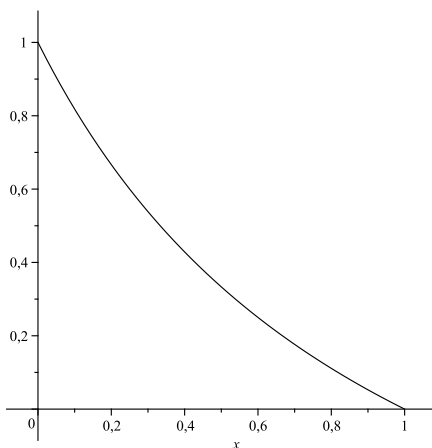


Fig. 2.  $\eta(x)$

way that  $x$  is contradiction.” In other words  $\sim_1 x$  is stronger negation than  $\sim_2 x$ .  $\sim_1 x$  in modal logic is ” $x$  is impossible”.

$$\sim_1 x = \Box \sim_2 x \quad (12)$$

In the modal logic  $\Box x$  means that  $x$  is necessary valid. If we negated  $x$ , then ”necessary not<sub>1</sub>  $x$ ”,  $\Box \sim_1$  has the meaning ”impossible” and we suppose ”not<sub>2</sub> impossible” is ”possible”, so

$$\Diamond x = \sim_2 \Box \sim_1 x \quad (13)$$

the possibility operator is defined by (13).

Here we used  $\sim$  for negation.

If in (10) is replaced by  $\sim_2 x$  and using the fact that  $\sim_2 x$  is involutive, then we get:

$$\Box x = \sim_1 (\sim_2 x), \quad (14)$$

and using (13) we get:

$$\Diamond x = \sim_2 (\sim_1 x). \quad (15)$$

It is also obvious in modal logic that:

$$\Box x \leq x \leq \Diamond x \quad (16)$$

Only in pliant logic are several negations  $\Box$  and we can differentiate between them by the neutral value. If  $\nu$  is small we can say that negation is strict, otherwise is not strict. Using this we can translate (14) and (15).

On the above consideration we can introduce the necessity and possibility modifiers.  
On the basis of (10) we get:

$$\eta_{\nu_1}(x) = \tau_N(\eta_{\nu_0}(x)) \quad (17)$$

where  $\nu_1 < \nu_0$ , and  $\tau_N(x)$  the necessity operator.

Because  $\eta_{\nu_0}(x)$  is involutive

$$\tau_N(x) = \eta_{\nu_1}(\eta_{\nu_0}^{-1}(x)) = \eta_{\nu_1}(\eta_{\nu_0}(x)). \quad (18)$$

Using (6)

$$\tau_N(x) = f^{-1}(f(\nu_0) \frac{f(\nu_1)}{f(\nu_0)f(\nu_0)} f(x))$$

and so we can write in the following form:

$$\tau_N(x) = f^{-1} \left( f(\nu_0) \frac{f(x)}{f(\nu_N)} \right) \quad (19)$$

where

$$\nu_N = f^{-1} \left( \frac{f(\nu_0)}{f(\nu_N)} \right)$$

We can summarize our consideration in the following theorem:

*Theorem 3.1:* Let  $\tau_N(x)$  and  $\tau_P(x)$  the necessity and possibility modifier then on the basis of (14, 15)

$$\tau_I(x) = f^{-1} \left( f(\nu_0) \frac{f(x)}{f(\nu_I)} \right)$$

(where  $I = \{N, P\}$ )

has a common form where  $\nu_P < \nu_0 < \nu_N$

A more general concept can be established:

*Definition 3.2:* Modal hedge is

$$\tau_{\nu_1, \nu_2}(x) = \eta_{\nu_1}(\eta_{\nu_2}(x)) \quad (20)$$

where  $\nu_1$  and  $\nu_2$  are neutral values if  $\nu_1 < \nu_2$ , then  $\tau_{\nu_1, \nu_2}(x)$  is necessity operator and if  $\nu_2 < \nu_1$ , then  $\tau_{\nu_1, \nu_2}(x)$  is possibility operator.

In the pliant concept:

$$\eta_{\nu}(x) = f^{-1} \left( f(\nu_0) \frac{f(\nu)}{f(x)} \right) \quad (21)$$

By the definition of (20) we get

$$\tau_{\nu_1, \nu_2}(x) = f^{-1} \left( f(\nu_1) \frac{f(x)}{f(\nu_2)} \right), \quad (22)$$

(22) can be written on the following form

$$\tau_{\nu, \nu_0}(x) = f^{-1} \left( f(\nu_0) \frac{f(x)}{f(\nu)} \right) \quad (23)$$

We call (23) general form of the hedges and in this case it is easy to show that if  $\nu_0 < \nu$ , the  $\tau_{\nu}(x)$  is the necessity and  $\nu < \nu_0$  is the possibility hedge. If  $f(\nu_0) = 1$

$$\tau_{\nu}(x) = f^{-1} \left( \frac{f(x)}{f(\nu)} \right) \quad (24)$$

*Definition 3.3:* We call necessity and possibility hedges are duals if  $\nu_1 = \eta(\nu_2)$ , i.e.

$$\nu_1 = f^{-1} \left( \frac{1}{f(\nu_2)} \right)$$

From this the necessity operator is

$$\tau_N(x) = f^{-1} \left( \frac{f(x)}{f(\nu)} \right),$$

and the possibility operator is

$$\tau_p = f^{-1}(f(\nu)f(x)).$$

We will use the following notation:

- If  $\nu > \nu_0$  and  $f(\nu_0) = 1$

$$\square(x) = f^{-1}\left(\frac{f(x)}{f(\nu)}\right) = \tau_\nu(x) \quad \text{necessity} \quad (25)$$

$$\diamond(x) = f^{-1}(f(\nu)f(x)) = \tau_{\eta(\nu)}(x) \quad \text{possibility} \quad (26)$$

- If  $\nu < \nu_0$

$$\diamond(x) = f^{-1}\left(\frac{f(x)}{f(\nu)}\right) = \tau_\nu(x) \quad \text{necessity}$$

$$\square(x) = f^{-1}(f(\nu)f(x)) = \tau_{\eta(\nu)}(x) \quad \text{possibility}$$

**Definition 3.4:**

$$\square(\square(\dots \square(x))\dots) = \square^k(x) \quad (27)$$

$$\diamond(\diamond(\dots \diamond(x))\dots) = \diamond^k(x) \quad (28)$$

**Definition 3.5:**

$$\text{drastic necessity} \quad \square_1(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases} \quad (29)$$

$$\text{drastic possibility} \quad \diamond_0(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases} \quad (30)$$

see Figure 3

On the basis of (25) and (26) it is easy to verify the following properties.

**Properties of modalities:**

- 1)  $\square(\diamond(x)) = x$
- 2)  $\diamond(\square(x)) = x$
- 3)  $\square_1(\diamond_0(x)) = \diamond_1(x)$
- 4)  $\diamond_0(\square_1(x)) = \square_0(x)$
- 5)  $\square^n(\square^m(x)) = \square^{n+m}(x)$
- 6)  $\diamond^n(\diamond^m(x)) = \diamond^{n+m}(x)$
- 7)  $\diamond^n(\square^m(x)) = \begin{cases} \diamond^{n-m}(x) & \text{if } n - m > 0 \\ x & \text{if } n = m = 0 \\ \square^{m-n}(x) & \text{if } n - m < 0 \end{cases}$
- 8)  $\square^n(\diamond^m(x)) = \begin{cases} \square^{n-m}(x) & \text{if } n - m > 0 \\ x & \text{if } n = m = 0 \\ \diamond^{m-n}(x) & \text{if } n - m < 0 \end{cases}$

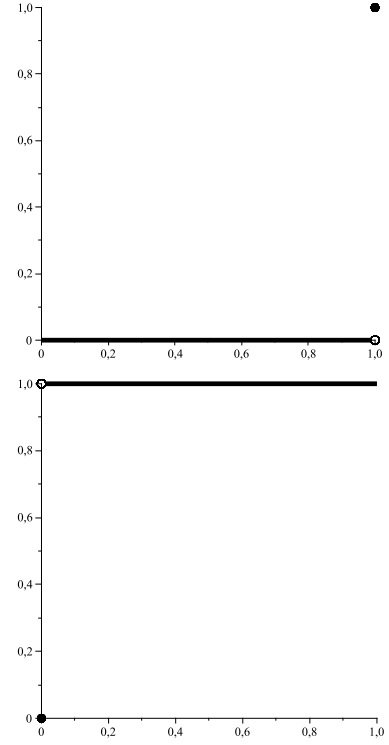


Fig. 3. The drastic necessity:  $\square_1(x)$  and the drastic possibility:  $\diamond_0(x)$

#### A. Composition of modalities and negations

The following properties hold for the composition of two modifiers:

1.  $\eta_{\nu_1}(\eta_{\nu_2}(x)) = \tau_\nu(x)$  where  $\nu = \tau_{\nu_1}(\nu_2)$
2.  $\tau_{\nu_1}(\tau_{\nu_2}(x)) = \tau_\nu(x)$  where  $\nu = \tau_{\nu_2}(\nu_1)$
3.  $\tau_{\nu_1}(\eta_{\nu_2}(x)) = \eta_\nu(x)$  where  $\nu = \tau_{\nu_1}(\nu_2)$
4.  $\eta_{\nu_1}(\tau_{\nu_2}(x)) = \eta_\nu(x)$  where  $\nu = a(\nu_1, \nu_2)$

**Proof.** Using the interpretation of the operators above, we get the results by calculation.

#### IV. COMPOSITION OF MODALITIES

We can make use a composition of several modalities.

**Theorem 4.1:** Let  $\tau_{\nu_1}(x) \dots \tau_{\nu_n}(x)$  modalities and  $f(\nu_*) = 1$ , then:

$$\tau_{\nu_1}(\tau_{\nu_2}(\tau_{\nu_3}(\dots \tau_{\nu_n}(x)))) = \tau_\nu(x)$$

where

$$\nu = a(\nu_1 \dots \nu_n)$$

**Proof.** Using (24)

$$\tau_{\nu_1}(\tau_{\nu_2}(\tau_{\nu_3}(\dots \tau_{\nu_n}(x)))) = f^{-1}\left(\frac{f(x)}{\prod_{i=1}^n f(\nu_i)}\right) = f^{-1}\left(\frac{f(x)}{f(\nu)}\right)$$

From this

$$\nu = f^{-1} \left( \prod_{i=1}^n f(\nu_i) \right) = a(\nu_1 \dots \nu_n)$$

*Corollary 1:* If the necessity operator is dual to the possibility operator, then

$$\tau_{\nu_1}(\tau_{\nu_2}(x)) = x$$

i.e. we get the identical operator, because  $\nu_2 = \eta(\nu_1)$ .

**Proof.** If the necessity operator is dual to the possibility operator, then  $\nu_1 = \eta(\nu_2)$  and in this case

$$a(\nu_1, \nu_2) = \nu_0$$

and  $f(\nu_0) = 1$

$$\tau(x) = f^{-1} \left( \frac{f(x)}{f(\nu_0)} \right)$$

## V. LIMES OF MODALITIES

We introduce the drastic modifiers:

$$\tau_1(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x < 1 \end{cases}$$

and

$$\tau_0(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

*Theorem 5.1:* Let

$$\tau_{\nu, \nu_0}(\tau_{\nu, \nu_0}(\tau_{\nu, \nu_0} \dots \tau_{\nu, \nu_0}(x))) = \underbrace{\tau \circ \tau \circ \dots \circ \tau}_M(x) = \tau^M(x)$$

If  $\nu_0 < \nu$  then:

$$\lim_{M \rightarrow \infty} \tau^M(x) = \tau_1(x)$$

If  $\nu_0 > \nu$  then:

$$\lim_{M \rightarrow \infty} \tau^M(x) = \tau_0(x)$$

**Proof.** Because

$$\tau^M(x) = f^{-1} \left( \frac{f(\nu_0) f(x)}{f(\nu)^M} \right) \quad (31)$$

where

$$\nu_M = f^{-1} \left( f(\nu_0) \left( \frac{f(\nu)}{f(\nu_0)} \right)^M \right) \quad (32)$$

If  $\nu < \nu_0$  then  $f(\nu_0) > f(\nu)$

$$\lim_{M \rightarrow \infty} \nu_M = 0 \quad (33)$$

Similar way we get the case if  $\nu_0 < \nu$ .

If  $\nu = \nu_0$  the  $\nu_M = \frac{1}{2}$

If we are using the interpretation of  $\tau(x) = \Box x$  or  $\Diamond x$  (i.e.

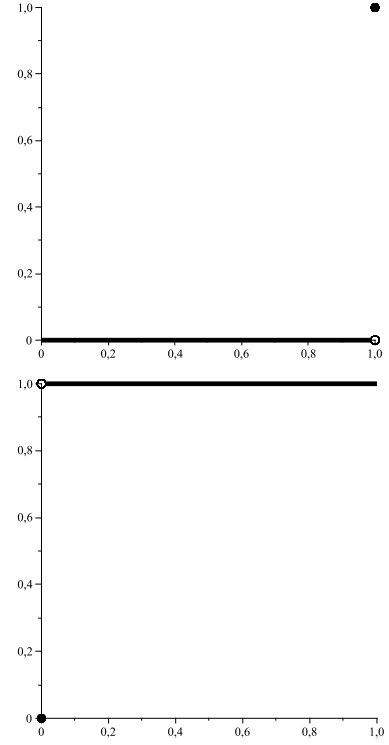


Fig. 4. The  $\tau_1$  and  $\tau_0$  drastic modifiers

necessity, or possibility  $x$ ), then we can introduce different necessity and possibility operators.

$$\begin{aligned} \Box^2 x &= \Box(\Box(x)) \\ \Diamond^2 x &= \Diamond(\Diamond(x)) \end{aligned} \quad (34)$$

More general:

$$\begin{aligned} \Box^M(x) &= \underbrace{(\Box \dots \Box)}_M(x) \quad \text{and} \quad \lim_{M \rightarrow \infty} \nu_M = 1 \\ \Diamond^M(x) &= \underbrace{(\Diamond \dots \Diamond)}_M(x) \quad \text{and} \quad \lim_{M \rightarrow \infty} \nu_M = 0 \end{aligned} \quad (35)$$

### A. Logic over modalities

We can calculate a logical expression over the modalities.

*Theorem 5.2:* Let  $\tau_{\nu_i}(x)$  modalities and then

a)

$$c(\tau_{\nu_1}(x), \tau_{\nu_2}(x), \dots, \tau_{\nu_n}(x)) = \tau_\nu(x) \quad (36)$$

$$\bar{c}(\tau_{\nu_1}(x), \tau_{\nu_2}(x), \dots, \tau_{\nu_n}(x)) = \tau_{\bar{\nu}}(x) \quad (37)$$

where

$$\nu = d(\nu_1, \nu_2, \dots, \nu_n) \quad (38)$$

$$\bar{\nu} = \bar{d}(\nu_1, \nu_2, \dots, \nu_n) \quad (39)$$

b)

$$d(\tau_{\nu_1}(x), \tau_{\nu_2}(x), \dots, \tau_{\nu_n}(x)) = \tau_\nu(x) \quad (40)$$

$$\bar{d}(\tau_{\nu_1}(x), \tau_{\nu_2}(x), \dots, \tau_{\nu_n}(x)) = \tau_{\bar{\nu}}(x) \quad (41)$$

where

$$\nu = c(\nu_1, \nu_2, \dots, \nu_n) \quad (42)$$

$$\bar{\nu} = \bar{c}(\nu_1, \nu_2, \dots, \nu_n) \quad (43)$$

c) 
$$\bar{a}(\tau_{\nu_1}(x), \dots, \tau_{\nu_n}(x)) = \tau_{\bar{\nu}}(x) \quad (44)$$

where

$$\bar{\nu} = a(\nu_1, \nu_2, \dots, \nu_n) \quad (45)$$

**Proof.** We show only the mean operator cases

a)

$$\bar{c}(\tau_{\nu_1}(x), \dots, \tau_{\nu_n}(x)) = f^{-1}\left(\frac{1}{n} \sum_{i=1}^n \frac{f(x)}{f(\nu_i)}\right) = f^{-1}\left(\frac{f(x)}{f(\bar{\nu})}\right) = \tau_{\bar{\nu}}(x) \quad \bar{c}(\mathbf{x}) = f^{-1}\left(\frac{1}{n} \sum_{i=1}^n f(x_i)\right) \quad (46)$$

where

$$\frac{1}{f^\lambda(\nu)} = \frac{1}{n} \sum_{i=1}^n \frac{1}{f^\lambda(\nu_i)} \quad (47)$$

From this we get (39).

b)

$$\bar{d}(\tau_{\nu_1}(x) \dots \tau_{\nu_n}(x)) = f^{-1}\left(\frac{n}{\sum_{i=1}^n \frac{f(x)}{f(\nu_i)}}\right) = f^{-1}\left(\frac{f(x)}{f(\nu)}\right) \quad (48)$$

where

$$\frac{1}{f(\bar{\nu})} = \frac{n}{\sum_{i=1}^n f(\nu_i)} \quad (49)$$

From this we get (41)

c)

$$\bar{a}(\tau_{\nu_1}(x) \dots \tau_{\nu_n}(x)) = f^{-1}\left(\left(\prod_{i=1}^n \frac{f(x)}{f(\nu_i)}\right)^{\frac{1}{n}}\right) \quad (50)$$

where

$$\frac{1}{f(\nu)} = \frac{1}{\left(\prod_{i=1}^n f(\nu_i)\right)^{\frac{1}{n}}} \quad (51)$$

From this we get (45)

## B. Modalities and negation

*Theorem 5.3:*

$$\tau_{\nu, \nu_0}(x) = \tau_{\eta(\nu_0), \eta(\nu)}(x) \quad (52)$$

**Proof.** Using the expression

$$\begin{aligned} \tau_{\nu, \nu_0}(x) &= f^{-1}\left(f(\nu_0) \frac{f(x)}{f(\nu)}\right) \\ \eta(x) &= f^{-1}\left(\frac{1}{f(x)}\right) \end{aligned} \quad (53)$$

we get (52).

## C. Modalities and Mean operator

Let us suppose  $x_i > \nu_0$ . We want to find a transformation

$$\tau_\nu(c(x_1, x_2, \dots, x_n)) > \nu_0$$

and similar way if  $x_i < \nu_0$

$$\tau_\nu(c(x_1, x_2, \dots, x_n)) < \nu_0$$

$\tau_\nu$  called normalization

*Theorem 5.4:* The normalized operator is

$$\tau_\nu(c(x_1, x_2, \dots, x_n)) = \bar{c}(x_1, x_2, \dots, x_n)$$

where

$$\bar{c}(\mathbf{x}) = f^{-1}\left(\frac{1}{n} \sum_{i=1}^n f(x_i)\right)$$

called mean operator.

**Proof.** From the normalization requirement

$$\tau_\nu(c(\nu_0, \nu_0, \dots, \nu_0)) = \tau_\nu(f^{-1}(nf(\nu_0))) = \nu_0$$

so:

$$f^{-1}\left(f(\nu_0) \frac{nf(\nu_i)}{f(\nu)}\right) = \nu_0$$

and

$$f(\nu) = nf(\nu_0)$$

The normalization transformation

$$\tau_\nu(x) = f^{-1}\left(f(\nu_0) \frac{f(x)}{f(\nu)}\right) = f^{-1}\left(f(\nu_0) \frac{f(x)}{nf(\nu_0)}\right) = f^{-1}\left(\frac{1}{n} f(x)\right)$$

Similar consideration is valid to the disjunction operator. The  $\tau_\nu(x)$  transformation depends on the number of the arguments of the operator.

*Theorem 5.5:* There exist  $\nu_1$  and  $\nu_2$

$$\begin{aligned} \tau_{\nu_1}(c(\mathbf{x})) &= \bar{c}(\mathbf{x}) \quad \text{and} \\ \tau_{\nu_2}(\bar{c}(\mathbf{x})) &= c(\mathbf{x}) \end{aligned} \quad (54)$$

**Proof.** Let

$$\nu_1 = f^{-1}(n) \quad \nu_2 = f^{-1}\left(\frac{1}{n}\right)$$

*Remark 1:*  $\nu$  depends on the arguments of  $c$  and  $\bar{c}$ .

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- [2] J. Dombi. DeMorgan systems with infinite number of negation. *Information Science*, under referee process