On a consistent fuzzy operator system

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Abstract— **We give a new representation theorem of the negation based on the generator function of the strict operator. We study a certain class of strict monotone operators which build DeMorgan class with infinite negations. We show that the necessary and sufficient condition for this operator class is $f_c(x) f_d(x) = 1$, where $f_c(x)$ and $f_d(x)$ are the generator function of the conjunctive and disjunctive operators.**

In the second part of the article we examine the relationship between Dombi's aggregative operators, uninorms and strict, continuous t-norms and t-conorms. We show that the class of representable uninorms is equivalent to the class of those uninorms which are also aggregative operators. We give new representation theorems for strong negations, and discuss the correspondence between strong negations, aggregative operators and strict, continuous (logical) operators.

We show that in this system the four operators (conjunction, disjunction, aggregation, negation) can be described using only one generator function.

Index Terms— **t-norms, t-conorms, uninorms, conjunctive operators, disjunctive operators, aggregative operators, negations, DeMorgan class, Pan operator, pliant systems**

NORMS

I. INTRODUCTION

Studiing continuous valued logical operators one of the most important question how to choose the element of the logic in a consistent way. One of the most important Boolean identity is the DeMorgan class. Several author studied this identity. The first article appear in 1983 [4]. Esteva in 1984 [9] made a study on some representable DeMorgan algebras. One of the most important research have been done by Garcia and Valvedre [11]. They focus on isomorphism between DeMorgan triplets. In this article the authors are dealing with the main types of fuzzy t-norm and t-conorms, i.e. with the min and max operators, the nilpotent operator and with strict monotonously increasing operators. The main equivalence classis of DeMorgan triplets are fully studied. The next important step have be done by Gehrke C. Walker and E. Walker [12]. They used algebraic approach which is general and very extensive. We have to mentioned also the book of Nguyen and Walker [20]. Here we can find summary of the results on the existence of DeMorgan triplets.

In this part of the we pay attention to the DeMorgan systems which correspond infinite many negations. This type of operators are important because the fix point of the negation (see later equation (6)) can be varied, and this value can be interpreted as decision level, so such logic is very flexible (pliant).

This general characterisation give us the possibility to construct an operator system.

We show in article [7] that building pan operators [18] by using the generator function of conjunctive and disjunctive operators are always the same if the same conditions are valid what we have in this article.

In the introduction part we give an elementary introduction for readers not to be familiar in this topic. In the second part we extended the operators by weights and we describe the relation between conjunctive, disjunctive generator function and negation. This result is reformulation of the known results. In chapter 3 we show that the involutivness of the negation (if given $f_c(x)$ and $f_d(x)$) ensures a $k(x)$ function (see Fig.1). In chapter 4 we give the general form of the negation by using $k(x)$. In chapter 5 we show that all involutive negation can be given in this form. In chapter 6 we give some examples. The main result of the article can be found in chapter 7. We show that DeMorgan triplet is valid with infinite number of negation if and only if $f_c(x) f_d(x) = 1$ (i.e. $k(x) = \frac{1}{x}$) . We call such system as pliant system. In chapter 8 we characterize the pliant operators. In chapter 9 we show that the aggregation (uninorm) is also closely related to the pliant system.

In this sense conjunctive, disjunctive, aggregative operators and the negation are consistent and we get them by using only one generator function.

All the proofs can be found in [7], [8].

II. NEGATION

For the sake of completeness the definition of a negation is the following:

Definition 1: $\eta(x)$ is a negation iff $\eta : [0,1] \to [0,1]$ satisfies the following conditions:

1) $\eta(x)$ is continuous
2) $\eta(0) = 1$, $\eta(1) = 0$ (boundary conditions)
3) $\eta(x) < \eta(y)$ for $x > y$ (monotonicity)
4) $\eta(\eta(x)) = x$ (involutivness)

Most of the article this negation is called strong negation because the involutivness is valid. In this article we left the word ”strong”.

Theorem 1: Let $\eta : [0,1] \to [0,1]$ be a continuous function, then the following are equivalent:

1) η is a negation.

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2) There exists a continuous and strictly monotone function $g : [0,1] \to [-\infty, \infty]$ with $g(\nu_*) = 0$, $\nu_* \in]0,1[$ such that for all $x \in [0,1]$

$$\eta(x) = g^{-1}(-g(x)). \tag{1}$$

If (1) holds, then ν_* is called the neutral value of the negation, i.e. for which $\eta(\nu_*) = \nu_*$.

Proof. The representation theorem of Trillas [24] states that all negations can be written as

$$\eta(x) = \varphi^{-1}\left(1 - \varphi(x)\right), \tag{2}$$

where φ is an automorphism of the unit interval. Let

$$g(x) = \ln\left(\frac{1}{\varphi(x)} - 1\right), \tag{3}$$

where $g(0) = \infty$ if φ is increasing and $g(0) = -\infty$ if φ is decreasing. It is easy to see that $g(x)$ is continuous, strictly monotone and maps $[0,1]$ to $[-\infty, \infty]$. Because

$$g^{-1}(x) = \varphi^{-1}\left(\frac{1}{1+e^x}\right), \tag{4}$$

so

$$\begin{aligned} \eta(x) &= g^{-1}(-g(x)) = \\ &= \varphi^{-1}\left(\frac{1}{1+\exp\left[-\ln\left(1/\varphi(x) - 1\right)\right]}\right) = \\ &= \varphi^{-1}(1-\varphi(x)). \end{aligned} \tag{5}$$

It is easy to see that $\eta(\nu_*) = \nu_*$ is equivalent to $g(\nu_*) = 0$.

$$\eta(\nu_*) = \nu_* \tag{6}$$

Since this value and its negated form are the same, it may be termed a neutral value. Further, since the negation of values smaller than the neutral value gives values larger than the neutral value, and vice versa, the neutral value naturally divides the evaluation interval into two parts. The values larger than ν may be interpreted as the positive or acceptable evaluation range, and those smaller than ν as the negative evaluation range; ν is thus a threshold value, and can be interpreted as an expectation level.

Another possible characterization of the negation, if we give a decision value ν for a given ν_0 (usually $\nu_0 = 1/2$). If x is less than the decision value, the negated value is larger than the threshold and vice versa:

$$\begin{array}{lll} x < \nu & \text{then} & \eta(x) > \nu_0 \\ x > \nu & \text{then} & \eta(x) < \nu_0 \end{array}$$

If $x = \nu$, then

$$\eta(\nu) = \nu_0 \tag{7}$$

If $\eta(x)$ has a fix point ν_*, we use the notation $\eta_{\nu_*}(x)$ and if the decision value is ν, then we use the notation $\eta_{\nu,\nu_0}(x)$. Let's characterize the negation with the ν_*, ν_0 and ν parameters.

III. Operators and DeMorgan law

We generalize the conjunctive and disjunctive operators. Let:

$$\begin{aligned} c(\mathbf{w}, \mathbf{x}) &= c(w_1, x_1; w_2, x_2; \ldots; w_n, x_n) = \\ &= f_c^{-1}\left(\sum_{i=1}^{n} w_i f_c(x_i)\right), \end{aligned} \tag{8}$$

$$\begin{aligned} d(\mathbf{w}, \mathbf{x}) &= d(w_1, x_1; w_2, x_2; \ldots w_n, x_n) = \\ &= f_d^{-1}\left(\sum_{i=1}^{n} w_i f_d(x_i)\right), \end{aligned} \tag{9}$$

where $w_i \geq 0$.

Definition 2: We define the DeMorgan law for general conjunctive and disjunctive operator is:

$$\begin{aligned} c(w_1, \eta(x_1); w_2, \eta(x_2); \ldots; w_n, \eta(x_n)) = \\ = \eta(d(w_1, x_1; w_2, x_2; \ldots; w_n, x_n)), \end{aligned} \tag{10}$$

where $\eta(x)$ is the negation function.

Theorem 2 (DeMorgan Law): The generalized DeMorgan law is valid iff

$$f_c^{-1}(x) = \eta(f_d^{-1}(ax)), \tag{11}$$

where $a \neq 0$.

IV. Negations and the DeMorgan Law

Naturally arises the following question. If f_c and f_d are given, then what kind of condition ensures that η is a negation (i.e. fullfils C1-C4). From Theorem 2 we know that the necessary and sufficient condition of the DeMorgan Law is (11). Substitute the $x := f_d^{-1}(ax)$. Then we have

$$\eta(x) = f_c^{-1}\left(\frac{1}{a} f_d(x)\right), \qquad a \neq 0. \tag{12}$$

Let us give the parametrical form of the negation.

Theorem 3: Parametrical form of the negation is

$$\eta(x) = f_d^{-1}\left(\frac{f_d(\nu_*)}{f_c(\nu_*)} f_c(x)\right), \tag{13}$$

$$\eta(x) = f_c^{-1}\left(\frac{f_c(\nu_*)}{f_d(\nu_*)} f_d(x)\right). \tag{14}$$

Examples for the negations: If

$$\begin{array}{lll} & f_c(x) = \ln(x), & f_d(x) = \ln(1-x), \\ i.e.: & f_c^{-1}(x) = e^x, & f_d^{-1}(x) = 1 - e^x \end{array}$$

then

$$\eta(x) = f_d^{-1}\left(K f_c(x)\right) = 1 - e^{Klnx} = 1 - x^K \tag{15}$$

$$\eta(x) = f_c^{-1}\left(K f_d(x)\right) = e^{Kln(1-x)} = (1-x)^K \tag{16}$$

If $K \neq 1$, then (15) and (16) are not involutive.

This negation fulfills (C1-C3). The most important question is the involutivness C4: $\eta(x) = \eta^{-1}(x)$.

Theorem 4: Let η be given by (12). Then $\eta(x)$ is involutive iff

$$f_c(x) = \frac{1}{a} k(f_d(x)), \qquad a \neq 0, \tag{17}$$

where $k\colon (0,\infty) \to (\infty, 0)$ is a strictly decreasing function and

$$k^{-1}(x) = k(x). \tag{18}$$

V. General form of the negation

We can get a new representation theorem for the negation using Theorem (4).

Theorem 5 (General form of the negation): We have that $c(x,y)$, $d(x,y)$ and $\eta(x)$ is a DeMorgan triple if and only if

$$\eta(x) = f^{-1}(k(f(x))), \tag{19}$$

where $f(x) = f_c(x)$ or $f(x) = f_d(x)$ and $k(x)$ is a strictly decreasing function with the property

$$k(x) = k^{-1}(x). \tag{20}$$

Corollary 1: From (19) it is easy to get

$$k(x) = f(\eta(f^{-1}(x))), \tag{21}$$

i.e. if $f(x)$ and $\eta(x)$ is given, then $k(x)$ is determined by (21).

VI. Representation theorem of negation

Another interesting question is whether (19) is a general representation form of the negation? The following theorem ensures that all negations can be written in (19) form.

While Trillas' theorem [24] represents negations (from our point of view) for the nilpotent class of t-norms and t-conorms, our next result gives a representation theorem for the strict t-norms and t-conorms.

Theorem 6 (Representation theorem of negation): For all given $\eta(x)$ there exists an $f(x)$ such that

$$\eta(x) = f^{-1}(k(f(x))), \tag{22}$$

where $k(x)$ is a strictly decreasing function with the property $k(x) = k^{-1}(x)$ and f is the generator function of a conjunctive, or disjunctive operator.

Remark 1: A DeMorgan triple can be built by using only one operator's generator function and choosing a $k(x)$, i.e. it is valid that

$$\eta(x) = f_c^{-1}\left(k\left(f_c(x)\right)\right) \tag{23}$$

$$c(x,y) = f_c^{-1}\left(f_c(x) + f_c(y)\right) \tag{24}$$

$$d(x,y) = f_c^{-1}\left(k\left(k\left(f_c(x)\right) + k\left(f_c(y)\right)\right)\right) \tag{25}$$

form a DeMorgan triple, and

$$f_c(x) = k\left(f_d(x)\right). \tag{26}$$

VII. Examples for DeMorgan Systems

Using the above results, we can get classical and also new operator systems.

- If $f_c(x) = -\ln(x)$ and $\eta(x) = 1 - x$, then

$$k(x) = f(\eta(f^{-1}(x))) = -\ln(1 - e^{-x}) \quad c(x,y) = xy,$$

$$d(x,y) = x + y - xy.$$

- If $f_d(x) = -\ln(1-x)$ and $\eta(x) = 1 - x$, then

$$k(x) = f_d(\eta(f_d^{-1}(x))) = -\ln(1 - e^{-x}) \quad c(x,y) = xy,$$

$$d(x,y) = x + y - xy.$$

- If $f_c(x) = -\ln(x)$ and $k(x) = \frac{1}{x}$, then

$$c(x,y) = xy \quad d(x,y) = e^{\frac{1}{\ln}\left(e^{\frac{1}{\ln x} + \frac{1}{\ln y}}\right)} \quad \eta(x) = e^{1/\ln x}$$

VIII. Parametrial Form of the Negation

Lemma 1: The parametrial form of the negation is

$$\eta(x) = f^{-1}\left(f(\nu_*)\frac{k\left(f(x)\right)}{k\left(f(\nu_*)\right)}\right) \tag{27}$$

$$\eta(x) = f^{-1}\left(f(\nu_0)\frac{k\left(f(x)\right)}{k\left(f(\nu)\right)}\right). \tag{28}$$

Corollary 2: The negation is independent from the type of the operators (i.e. disjunctive or conjunctive). So we can leave the index c and d in (27) and (28).

IX. Operators with infinite number of negations

In this chapter we will characterize the operator class (conjunctive and disjunctive) for which various negations exist and build DeMorgan class. The fixpoint ν_* or the neutral value ν can be considered as decision threshold. Operators with various negations are useful because the threshold can be changed.

It is easy to see that the min and max operator belongs to this class, as the drastic operator too. The next theorem characterize the strict operator systems which have infinite many negations and build DeMorgan system. It is easy to see that $c(x,y) = xy$, $d(x,y) = x+y-xy$ only with $\eta(x) = 1-x$ build DeMorgan system (i.e. there are no other negation to build a DeMorgan system).

Theorem 7: $c(x,y)$ and $d(x,y)$ build a DeMorgan system for $\eta_{\nu_*}(x)$ for all $\nu_* \epsilon (0,1)$ if and only if

$$f_c(x) f_d(x) = 1. \tag{29}$$

X. Multiplicative Pliant systems

From Dombi's result [3] we know, that if $f(x)$ is a generator function, then $f^{\alpha}(x)$ is a generator function, too. As we've seen $k(x)$ plays an important role in DeMorgan systems. Let us define the multiplicative pliant system with one of the simplest $k(x)$.

Definition 3: If $k(x) = 1/x$, i.e

$$f_c(x) f_d(x) = 1, \tag{30}$$

then we call the generated connectives multiplicative pliant system.

We use the following notation in the pliant system.

$$f^{\alpha}(x) = f_c^{\alpha}(x) = f_c(x)$$

Remark 2: A similar operator system was introduced by Roychowdhury [21]. Theorem 7 gives the necessary and sufficient conditions for a such system.

Theorem 8: The general form of the multiplicative pliant system is

$$o_{\alpha}(x, y) = f^{-1}\left((f^{\alpha}(x) + f^{\alpha}(y))^{1/\alpha} \right) \tag{31}$$

$$\eta_{\nu,\nu_0}(x) = f^{-1}\left(f(\nu_0) \frac{f(\nu)}{f(x)} \right) \quad \text{or} \tag{32}$$

$$\eta_{\nu_*}(x) = f^{-1}\left(\frac{f^2(\nu_*)}{f(x)} \right), \tag{33}$$

where $f(x)$ is the generator function of either the conjunctive or the disjunctive operator. If $f = f_c$, then depending on the value of α, the operator is

$$\begin{aligned} \alpha > 0 \qquad & o_{\alpha}(x, y) = c(x, y) \\ \alpha < 0 \qquad & o_{\alpha}(x, y) = d(x, y) \\ \alpha = \infty \qquad & o_{+\infty}(x, y) = \min(x, y) \\ \alpha = -\infty \qquad & o_{-\infty}(x, y) = \max(x, y) \end{aligned} \tag{34}$$

$$\alpha = 0^+ \qquad o_{0^+} = \begin{cases} x \text{ if } y = 1 \\ y \text{ if } x = 1 \\ 0 \text{ otherwise} \end{cases} \tag{35}$$

$$\alpha = 0^- \qquad o_{0^-} = \begin{cases} x \text{ if } y = 0 \\ y \text{ if } x = 0 \\ 1 \text{ otherwise} \end{cases} \tag{36}$$

This operator called drastic operator.

Remark 3: It is important, that in the multiplicative pliant system the negation is independent of the value and the sign of α. (In other words, it is independent of whether the generator function belongs to the conjunctive or disjunctive operator.)

Remark 4: The limes value of the pliant operators (min, max and drastic) also have the property that they build DeMorgan triplet by infinite many negations.

Theorem 9: If $g(x) = f^{\alpha}(x)$ is the generator function, then the negation does not change in the pliant system.

Uninorms

I. Uninorms and aggregation

The term *uninorm* was introduced by Yager and Rybalov [27] in 1996. Uninorms are generalization of t-norms and t-conorms, by relaxing the constraint on the identity element from $\{0, 1\}$ to the unit interval. Since then many articles dealt with uninorms, both from the theoretical [2], [13], [15]–[17], [19] and the practical point of view [26]. The paper of Fodor, Yager and Rybalov [10] is important since it characterized a subclass of uninorms, called representable uninorms. This characterization is similar to the representation theorem of strict t-norms and t-conorms, in the sense that both originate from the solution of the associativity functional equation given by Aczél [1].

In the paper [3] were introduced *aggregative operators* by selecting a set of minimal concepts which must be fulfilled by an evaluation like operator.

As it is stated in [10], there is a close relationship between Dombi's aggregative operators and uninorms..

We differentiate between logical operators (strict, continuous t-norms and t-conorms) and aggregative operators, where the former means strict, continuous operators. We emphasize that aggregative operators are not logical operators, because its values at $\{0, 1\}$ and $\{1, 0\}$ are not defined.

The first aim of this part is to prove the equivalence between two subclasses of uninorms: representable uninorms and such uninorms which are also aggregative operators. The second one is to show the close correspondence between strong negations, aggregative and logical operators. Our third aim is to introduce and characterize multiplicative pliant operator systems.

This part is organized as follows. Section 2 contains basic definitions and the relationship between uninorms and aggregative operators are shown. We make issue on the role of the neutral value in Section 3. Section 4 discusses the correspondence between strong negations, aggregative and logical operators. Finally, we introduce and give the characterization of so called multiplicative pliant systems with examples in Section 5.

II. Aggregative operators and representable uninorms

In 1982 Dombi [3] defined the aggregative operator in the following way:

Definition 4: An aggregative operator is a strictly increasing function $a : [0, 1]^2 \to [0, 1]$ with the properties:

1) Continuous on $[0, 1]^2 \backslash \{(0, 1), (1, 0)\}$
2) $a(0, 0) = 0$ and $a(1, 1) = 1$ (boundary conditions)
3) $a(x, a(y, z)) = a(a(x, y), z)$ (associativity)
4) There exists a strong negation η such that $a(x, y) = \eta(a(\eta(x), \eta(y)))$ (self De Morgan identity)

Definition 5: A uninorm U is a mapping $U : [0, 1]^2 \to [0, 1]$ having the following properties:

- $U(x, y) = U(y, x)$ (commutativity)
- $U(x_1, y_1) \geq U(x_2, y_2)$ if $x_1 \geq x_2$ and $y_1 \geq y_2$ (monotonicity)

- $U(x, U(y, z)) = U(U(x, y), z)$ (associativity)
- $\exists \nu_* \in [0,1]\ \forall x \in [0,1]\ U(x, \nu_*) = x$ (neutral element)

A uninorm is a generalization of t-norms and t-conorms. By adjusting its neutral element, a uninorm is a t-norm if $\nu_* = 1$ and a t-conorm if $\nu_* = 0$. The following representation theorem of strict, continuous uninorms (or *representable uninorms*) was given by Fodor et al. [10] (see also Klement et al. [14]).

Theorem 10: Let $U : [0,1] \to [0,1]$ be a function and $\nu_* \in]0,1[$. The following are equivalent:

1) U is a uninorm with neutral element ν_* which is strictly monotone on $]0,1[^2$ and continuous on $[0,1]^2 \backslash \{(0,1),(1,0)\}$.
2) There exists a strictly increasing bijection $g_u : [0,1] \to [-\infty, \infty]$ with $g_u(\nu_*) = 0$ such that for all $(x,y) \in [0,1]^2$ we have
$$U(x,y) = g_u^{-1}\left(g_u(x) + g_u(y)\right), \tag{37}$$
where, in the case of a conjunctive uninorm U, we use the convention $\infty + (-\infty) = -\infty$, while, in the disjunctive case, we use $\infty + (-\infty) = \infty$.

If (37) holds, then the function g_u is uniquely determined by U up to a positive multiplicative constant, and it is called an additive generator of the uninorm U.

Note that the function g in the Theorem 1 has the same properties as the uninorm generator function g_u in Theorem 10. We show that a uninorm is representable (i.e. strict and continuous) if and only if it is also an aggregative operator.

Theorem 11: Let U be a uninorm. It is representable if and only if it is an aggregative operator.

Theorem 12: Let $U : [0,1] \times [0,1] \to [0,1]$ be a uninorm. It is an aggregative operator if and only if there exists a continuous and strictly monotone function $g : [0,1] \to [-\infty, \infty]$ with $g(\nu) = 0$, $\nu \in]0,1[$ such that for all $x, y \in [0,1]$
$$U(x,y) = g^{-1}(g(x) + g(y)). \tag{38}$$

III. The neutral value

Theorem 13 (Additive form of negations): Let $\eta : [0,1] \to [0,1]$ be a continuous function, then the following are equivalent:

1) η is a negation with neutral value ν_*.
2) There exists a continuous and strictly monotone function $g : [0,1] \to [-\infty, \infty]$ and $\nu_* \in]0,1[$ such that for all $x \in [0,1]$
$$\eta(x) = g^{-1}(2g(\nu_*) - g(x)). \tag{39}$$

Lemma 2 (Dombi [3]): If g is the additive generator function of an aggregative operator a, then the function displaced by $d \in \mathbb{R}$, $g_*(x) = g(x) + d$ is also a generator function of an aggregative operator with neutral value $\nu_* = g^{-1}(-d)$.

Theorem 14 (Dombi [3]): Let $a : [0,1]^n \to [0,1]$ be a function, a is an aggregative operator with additive generator g and neutral value ν_* if and only if for all $\mathbf{x} \in [0,1]^n$
$$a(\mathbf{x}) = g^{-1}\left(\sum_{i=1}^{n} g(x_i) - (n-1)g(\nu_*)\right). \tag{40}$$

Definition 6: Let a_f and a_g be aggregative operators, with the additive generator functions f and g, respectively. The functions a_f and a_g belong to the same family if $f(x) = g(x) + d$, for all $x \in [0,1]$ and a suitable $d \in \mathbb{R}$. Note, that a_f and a_g not necessarily have the same neutral value.

Theorem 15: Let a be an aggregative operator with generator function g and neutral value ν_*. The only negation for which it fulfills the self De Morgan identity is
$$\eta(x) = g^{-1}\left(2g(\nu_*) - g(x)\right). \tag{41}$$

Definition 7: Let a be an aggregative operator with the additive generator function g and neutral value ν_*. Let us call the negation $\eta(x) = g^{-1}\left(2g(\nu_*) - g(x)\right)$ the corresponding negation of the aggregative operator, and vice versa.

By Theorem 15 every aggregative operator has exactly one corresponding negation, which is a negation fulfilling the self De Morgan identity with it. Conversely every negation has exactly one corresponding aggregative operator.

IV. Conjunctive, disjunctive and aggregative operators

Definition 8: We will use the term conjunctive operator for strict, continuous t-norms, and disjunctive operator for strict, continuous t-conorms. The expression logical operators will refer to both of them.

In the following we show that from any logical operator one can get an aggregative operator by changing the addition to multiplication in their additive generator functional forms.

Theorem 16: The following are equivalent:

1) $o(x,y) = f^{-1}\left(f(x) + f(y)\right)$ is a logical operator.
2) $a(x,y) = f^{-1}\left(f(x)f(y)\right)$ is an aggregative operator.

Definition 9: Let f be the additive generator of a logical operator. The aggregative operator $a(x,y) = f^{-1}\left(f(x)f(y)\right)$ is called the corresponding aggregative operator of the conjunctive or disjunctive ope-rator, and vice versa.

We note that pan-operators, introduced by Wang and Klir [25], with a non-idempotent unit element (see [18] and [23]) have similar properties to a corresponding pair of logical and aggregative operators.

Corollary 3: A logical operator is distributive with its corresponding aggregative operator, i.e.
$$a(x, c(y,z)) = c(a(x,y), a(x,z)). \tag{42}$$

Corollary 4: Let $f(x)$ be the additive generator of a logical operator. The aggregative operators gene-rated by $f(x)$ and $f_*(x) = cf(x)$ $(c > 0)$ belong to the same family.

Corollary 5: Let $f_\alpha(x) = (f(x))^\alpha$ $(\alpha > 0)$ be a generator function of a logical operator. Its corresponding aggregative operator is independent of α.

By Theorem 16 and Corollaries 4 and 5, every logical operator has infinitely many corresponding aggregative operators because its generator function is determined up to a multiplicative constant. Conversely, every aggregative operator has infinitely many corresponding logical operators because a generator function on different powers generates different logical operators and identical aggregative operators.

A straight consequence of Theroem 13 and Theorem 16 is the following representation theorem of negations.

Corollary 6 (Multiplicative form of negations): The function $\eta : [0,1] \to [0,1]$ is a negation with neutral value ν_* if and only if

$$\eta(x) = f^{-1}\left(\frac{f^2(\nu_*)}{f(x)}\right), \tag{43}$$

where f is a generator function of a logical operator.

Summarizing the above statements, there is a well defined correspondence between logical operators, aggregative operators and negations. Every logical operator has corresponding aggregative operators, and additionally corresponding negations, too. Note that the Dombi operators have the same corresponding aggregative operator and negation.

The next Theorem gives a necessary and sufficient condition on a pair of conjunctive and disjunctive logical operators to have identical corresponding aggregative operators.

Theorem 17: Let f_c be an additive generator function of a conjunctive, and f_d be an additive generator function of a disjunctive operator. Their corresponding aggregative operators a_c and a_d are equivalent on $[0,1]^2 \backslash \{(0,1),(1,0)\}$ if and only if $f_d(x) = f_c^k(x)$, where $k \in \mathbb{R}^-$.

Corollary 7: Let c and d be a conjunctive and a disjunctive operator with additive generator functions f_c and f_d. Let a_c and a_d be their corresponding aggregative operators, and let n_c and n_d be their corresponding negations. The negations η_c and η_d are equivalent if and only if $f_d(x) = f_c^k(x)$, $k \in \mathbb{R}^-$.

V. Pliant operators

If the condition $f_d(x) = f_c^k(x)$ with $k < 0$ is fulfilled then the logical operators have a common aggregative operator and negation. This set of logical operators is still broad. De Morgan's law is a condition which must be fulfilled by a "good" triplet of connectives. Requiring the validity of De Morgan's law further restricts the examined set of logical operators.

Theorem 18: Let c and d be a conjunctive and a disjunctive operator with additive generator functions f_c and f_d. Suppose their corresponding negations are equivalent (i.e. $f_d(x) = f_c^k(x)$, $k < 0$), denoted by η ($\eta(\nu_*) = \nu_*$). The three connectives c, d and n form a De Morgan triplet if and only if $k = -1$.

Definition 10: A system of logical operators which have the property $f_c(x) f_d(x) = 1$ is called multiplicative pliant system.

In multiplicative pliant systems the corresponding aggregative operators of the conjunctive and disjunctive operators are equivalent, and De Morgan's law is fulfilled with the (common) corresponding negation of the logical operators.

We can summarize the elements of multiplicative pliant system:

$$c(\mathbf{x}) = f^{-1}\left(\sum_{i=1}^{n} f(x_i)\right) \tag{44}$$

$$d(\mathbf{x}) = f^{-1}\left(\frac{1}{\sum_{i=1}^{n} \frac{1}{f(x_i)}}\right) \tag{45}$$

$$a(\mathbf{x}) = f^{-1}\left(\prod_{i=1}^{n} f(x_i)\right) \tag{46}$$

$$\eta(x) = f^{-1}\left(\frac{f^2(\nu_*)}{f(x)}\right) \tag{47}$$

where $f(x)$ is the generator function of the conjunctive operator.

A. The Dombi operator system

For another example, the Dombi operators form a pliant system. The operators are

$$c(\mathbf{x}) = \frac{1}{1 + \left(\sum_{i=1}^{n} \left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)^{1/\alpha}} \tag{48}$$

$$d(\mathbf{x}) = \frac{1}{1 + \left(\sum_{i=1}^{n} \left(\frac{1-x_i}{x_i}\right)^{-\alpha}\right)^{-1/\alpha}} \tag{49}$$

$$a(\mathbf{x}) = \frac{1}{1 + \prod_{i=1}^{n} \frac{1-x_i}{x_i}} \tag{50}$$

$$\eta(x) = \frac{1}{1 + \left(\frac{1-\nu_*}{\nu_*}\right)^2 \frac{x}{1-x}} \tag{51}$$

where $\nu_* \in]0,1[$, with generator functions

$$f_c(x) = \left(\frac{1-x}{x}\right)^{\alpha} \qquad f_d(x) = \left(\frac{1-x}{x}\right)^{-\alpha} \tag{52}$$

where $\alpha > 0$. The operators c, d and n fulfill the De Morgan identity for all ν, a and n fulfill the self De Morgan identity for all ν and the aggregative operator is distributive with the logical operators.

(48), (49), (50), (51) can be found in different articles of Dombi. (48) and (49) can be found in [4], (50) in [3] and (51) can be found in [6].

(50) called 3π operator because it can be written in the following form:

$$a(\mathbf{x}) = \frac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n} x_i + \prod_{i=1}^{n} (1 - x_i)} \tag{53}$$

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