# Multiplicative Utility Function and Fuzzy Operators

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**Abstract**

Our starting point is the multiplicative utility function which is extensively used in the theory of multicriteria decision making. Its associativity is shown and as its generalization a fuzzy operator class is introduced with fine and useful properties. As special cases it reduces to well-known operators of fuzzy theory: min/max, product, Einstein, Hamacher, Dombi and drastic. As a consequence, we generalize the addition of velocities in Einstein's special relativity theory to multiple moving objects. Also, a new form of the Hamacher operator is given, which makes multi-argument calculations easier. We examined the De Morgan identity which connects the conjunctive and disjunctive operators by a negation. It is shown that in some special cases (min/max, drastic and Dombi) the operator class forms a De Morgan triple with any involutive negation.

## The Multiplicative Utility Function

In their seminal treatment of multiattribute utility (MAU) theory, Keeney and Raiffa show how certain conditions of independence among attributes yield the so called multiplicative multiattribute utility function

$$u_M(\vec{z}) = \frac{1}{k}\left(\prod_{i=1}^{n}(1 + kk_iu_i(z_i)) - 1\right) \tag{1}$$

where $\vec{z} = (z_1, \dots, z_n)$, $u_i : \mathbb{R} \to [0, 1]$ are utility functions, $z_i$ are evaluations, $k_i$ are weights of the ith criteria, and $k$ is a scaling constant. The formula can also be expanded as

$$\begin{aligned} u_M(\vec{z}) = \sum_{i=1}^{n} k_iu_i(z_i) + k\sum_{i<j} k_ik_ju_i(z_i)u_j(z_j) + \\ + k^2\sum k_ik_jk_lu_i(z_i)u_j(z_j)u_l(z_l) + \dots \\ + k^{n-1}k_1k_2\dots k_nu_1(z_1)\dots u_n(z_n). \end{aligned} \tag{2}$$

allowing also for $k = 0$.

**Lemma 1.** *If $k = 0$ then*

$$u_M(\vec{z}) = \sum_{i=1}^{n} k_iu_i(z_i). \tag{3}$$

**Proof.** By substituting $k = 0$ into the expanded formula of $u_M$ (2) we get the result. □

The utility function is normal if $u_M(\vec{z}) = 0$ whether $u_i(z_i) = 0$, and $u_M(\vec{z}) = 1$ whether $u_i(z_i) = 1$ for all $i \in \{1, \dots, n\}$. A normal $u_M(\vec{z})$ implies

$$1 + k = \prod_{i=1}^{n}(1 + kk_i), \tag{4}$$

i.e. assuming the normality of $u_M$, $k$ is determined only by the weights $k_i$.

## The Associativity of the Multiplicative Utility Function

Let us substitute $x_i := k_i u_i(z_i)$ in the formula (1). Then the transformed multiplicative utility function is

$$u_M^*(\vec{x}) = \frac{1}{k}\left(\prod_{i=1}^{n}(1+kx_i)-1\right). \tag{5}$$

**Theorem 1.** *The function $u_M^*$ is associative.*

**Proof.** The proof is based on the representation theorem of Aczél. It can be easily verified, that (5) can also be written in the form $F(x,y) = f^{-1}\left(f(x)+f(y)\right)$, by putting

$$f(x) = \ln(1+kx), \tag{6}$$

and

$$f^{-1}(x) = \frac{1}{k}\left(e^x - 1\right). \tag{7}$$

□

## Logical operators and the Multiplicative Utility Function

Let $g : [0,1] \to [0,\infty]$ be a generator function of a strict operator. Let

$$f(x) = \ln(1+\gamma g(x)), \tag{8}$$

and so

$$f^{-1}(x) = g^{-1}\left(\frac{1}{\gamma}e^x - 1\right). \tag{9}$$

Note, that for all $\gamma \in (0,\infty)$, $f$ fulfills the requirements of a generator function of a strict operator. By Aczél's theorem, the associative operator $o : [0,1]^n \to [0,1]$ generated by $f$ is

$$o(x_1,\ldots,x_n) = g^{-1}\left(\frac{1}{\gamma}\left(\prod(1+\gamma g(x_i))-1\right)\right). \tag{10}$$

Similarly to (2), by first expanding the argument of $g^{-1}$ to

$$\begin{aligned}\sum_{i=1}^{n} g(x_i) + \gamma\sum_{i<j} g(x_i)g(x_j) + \\ &+ \gamma^2\sum g(x_i)g(x_j)g(x_l) + \ldots \\ &+ \gamma^{n-1} g(x_1)\ldots g(x_n),\end{aligned}$$

we can put

$$o(x_1,\ldots,x_n)|_{\gamma=0} = g^{-1}\left(\sum g(x_i)\right), \tag{11}$$

thus the case $\gamma = 0$ also results in a strict operator. Next, we will show that different types of operators fit into the framework depending on the choice of function $f$. From now on, let us assume

$$g(x) = \left(\frac{1-x}{x}\right)^{\alpha},$$

the generator function of the Dombi operator.

## The Generalized Dombi operator

**Definition 1.** *The generator functions of the Generalized Dombi operator are*

$$f_c(x) = \ln\left(1+\gamma_c\left(\frac{1-x}{x}\right)^{\alpha}\right), \qquad \alpha > 0 \tag{12}$$

$$f_d(x) = \ln\left(1+\gamma_d\left(\frac{1-x}{x}\right)^{\alpha}\right), \qquad \alpha < 0 \tag{13}$$

*where* $\gamma_c, \gamma_d \in [0, \infty]$. *From*

$$c(\vec{x}) = f_c^{-1}\left(\sum_{i=1}^{n} f_c(x_i)\right),$$
$$d(\vec{x}) = f_d^{-1}\left(\sum_{i=1}^{n} f_d(x_i)\right),$$

*and*

$$f_c^{-1}(x) = \frac{1}{1+\left(\frac{1}{\gamma_c}\left(e^x - 1\right)\right)^{1/\alpha}}, \qquad \alpha > 0 \tag{14}$$
$$f_d^{-1}(x) = \frac{1}{1+\left(\frac{1}{\gamma_d}\left(e^x - 1\right)\right)^{1/\alpha}}, \qquad \alpha < 0 \tag{15}$$

*the operators are*

$$c_{GD,\gamma_c}^{(\alpha)}(\vec{x}) = \frac{1}{1+D_{\gamma_c}(\vec{x})}, \qquad \alpha > 0 \tag{16}$$
$$d_{GD,\gamma_d}^{(\alpha)}(\vec{x}) = \frac{1}{1+D_{\gamma_d}(\vec{x})}, \qquad \alpha < 0 \tag{17}$$

*where* $\gamma_c, \gamma_d \in [0, \infty]$ *and*

$$D_\gamma(\vec{x}) = \left(\frac{1}{\gamma}\left(\prod_{i=1}^{n}\left(1+\gamma\left(\frac{1-x_i}{x_i}\right)^\alpha\right) - 1\right)\right)^{1/\alpha}. \tag{18}$$

Equations (16) and (17) differ only in the sign of $\alpha$ and so the Generalized Dombi operator class is:

$$o_{GD,\gamma}^{(\alpha)}(\vec{x}) = \frac{1}{1+\left(\frac{1}{\gamma}\left(\prod_{i=1}^{n}\left(1+\gamma\left(\frac{1-x_i}{x_i}\right)^\alpha\right) - 1\right)\right)^{1/\alpha}} \tag{19}$$

In the forthcoming sections, we will show that $o_{GD,\gamma}^{(\alpha)}$ is a strict operator for $\alpha \in (-\infty, \infty)$ and $\gamma \in (0, \infty)$.

## The Dombi operator case

The Dombi operator has the form

$$o_D^{(\alpha)}(\vec{x}) = \frac{1}{1+\left(\sum_{i=1}^{n}\left(\frac{1-x_i}{x_i}\right)^\alpha\right)^{1/\alpha}} \tag{20}$$

and if $\alpha > 0$ then the operator is conjunctive and if $\alpha < 0$ then the operator is disjunctive. The next corollary follows from lemma 1, by the substitution $k = \gamma$.

**Corollary 1.** *The Dombi operator is a special case of the Generalized Dombi operator, i.e. if* $\gamma_c = \gamma_d = 0$ *then*

$$c_{GD,0}^{(\alpha)}(\vec{x}) = c_D^{(\alpha)}(\vec{x}), \tag{21}$$
$$d_{GD,0}^{(\alpha)}(\vec{x}) = d_D^{(\alpha)}(\vec{x}). \tag{22}$$

## Conclusions

In this lecture we have

1. proved the associativity of the multiplicative utility function,

2. introduced the generalized operator:

$$\frac{1}{1+\left(\frac{1}{\gamma}\left(\prod_{i=1}^{n}\left(1+\gamma\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)-1\right)\right)^{1/\alpha}}$$

3. presented new forms of rational involutive negations:

$$n_{\nu_*}(x)=\frac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\right)^{2}\left(\frac{1-x}{x}\right)^{-1}}$$
$$n_{\nu,\nu_0}(x)=\frac{1}{1+\frac{1-\nu_0}{\nu_0}\frac{1-\nu}{\nu}\left(\frac{1-x}{x}\right)^{-1}}$$

4. proved that the new operator connectives form a De Morgan triple with a negation iff

$$\frac{\gamma_d}{\gamma_c}=\left(\frac{1-\nu_0}{\nu_0}\cdot\frac{1-\nu}{\nu}\right)^{\alpha}$$

5. proved that the Dombi operators form a De Morgan triple with any rational involutive negation
6. showed that the generalized operator has the following limits

| Type of operator | Value of $\gamma$ | Value of $\alpha$ conj. | Value of $\alpha$ disj. |
|---|---|---|---|
| Dombi | 0 | $0<\alpha$ | $\alpha<0$ |
| Product | 1 | 1 | $-1$ |
| Einstein | 2 | 1 | $-1$ |
| Hamacher | $\gamma\in(0,\infty)$ | 1 | $-1$ |
| Drastic | $\infty$ | $0<\alpha$ | $\alpha<0$ |
| Min-max | 0 | $\infty$ | $-\infty$ |

7. introduced new forms of the Hamacher operators

$$o_H^{(\alpha)}(\vec{x})=\frac{1}{1+\left(\frac{1}{\gamma_d}\left(\prod_{i=1}^{n}\left(1+\gamma_d\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)-1\right)\right)^{1/\alpha}}$$

8. presented new forms of the Einstein operators

$$o_{GD,2}^{(\alpha)}(\vec{x})=\frac{1}{1+2\left(\prod_{i=1}^{n}\left(1+2\left(\frac{1-x_i}{x_i}\right)^{\alpha}\right)-1\right)^{1/\alpha}}$$

9. showed that the addition of several velocities in the framework of special relativity is:

$$v=\frac{c}{1+2\left(\prod_{i=1}^{n}\left(1+2\frac{v_i}{c-v_i}\right)-1\right)^{-1}}.$$

This new parametric operator family has some useful applications. The two parameters offer more freedom in the sense that by adopting two, instead of just one parameter, the operator can be made to fit the problem in question better. Because we have two parameters to play with instead of one.