Fuzziness measure in the Pliant system:
The Vagueness measure

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Abstract

The basic idea of fuzzy sets is the introduction of the membership function, which replaces the classical characteristic function. It is an interesting question to learn how close the membership function is to the characteristic function, when we use a certain class of membership functions. This measure is called the fuzziness measure.

Below we shall present an operator-dependent fuzziness measure called the vagueness measure. We will show that this measure satisfies the usual classical assumptions for the fuzziness measue. In addition, we will show that there is a connection between this measure and the entropy function.

1 Introduction

In our view, one of the most important concepts, because on this basis we can prove "convergence theorems" in the sense that "if there is less fuzziness in the input variables, then there will be less fuzziness in the result".

In the fuzzy literature we do not find such theorems, because membership functions, operators and fuzziness measures are unrelated so it seems hopeless to prove such convergence theorems.

In the Pliant concept we have the distending function instead of the membership function based on the Pliant operator, and now we will define a vagueness measure by using the generator function of Pliant operator. On the basis of this consistent concept we can derive a convergence theorem.

First, we will take a closer look at the fuzziness measure: Let $\mu(x)$ be the membership function and $d(\mu)$ be the fuzziness measure.

Fuzziness measure

We shall give a list of desirable properties for a measure of fuzziness, i.e. the kind of properties we expect it to have. Various combinations of these can be found in the literature. It is remarkable that different authors demand different properties for the d measure.

(P1) *Sharpness*: $d(\mu) = 0$ iff $\mu(x)\epsilon\{0,1\}$ for all $x\epsilon X$ (i.e. μ is "sharp").

(P2) *Maximality*: $d(\mu)$ is maximum iff $\mu(X) = \{\frac{1}{2}\}$.

(P3) *Resolution*: $d(\mu^*) \leq d(\mu)$ if μ^* is any sharpened version of μ, that is: $\mu^*(x) \leq \mu(x)$ if $\mu(x) \leq \frac{1}{2}$ and $\mu^*(x) \geq \mu(x)$ if $\mu(x) \geq \frac{1}{2}$.

(P4) *Symmetry* (about $\frac{1}{2}$): $d(\mu) = d(\eta(\mu))$,
where $\eta(\mu)$ is the negation function.

(P5) *Valuation*: the function d is a valuation on $[0,1]^X$, i.e.:

$$d(\mu_1 \cup \mu_2) + d(\mu_1 \cap \mu_2) = d(\mu_1) + d(\mu_2),$$

where $\cap$ is the min operator and $\cup$ is the max operator.

(P6) *Generalized additivity*: there exist mappings $s, t : [0,1] \rightarrow [0,\infty)$ such that:

$$d(\mu_1\mu_2) = d(\mu_1)t(P(\mu_2)) + d(\mu_2)s(P(\mu_1)).$$

where

$$P(\mu) = \sum_{i=1}^{n} \mu(x_i) \qquad \text{or} \qquad P(\mu) = \int \mu(x)dx$$

for all $\mu_1\epsilon[0,1]^X$ and $\mu_2\epsilon[0,1]^Y$, where X and Y finite sets.

The properties (P1)-(P4) are natural requirements for a measure of fuzziness. All the measures introduced so far satisfy these properties. But the meaning of (P5) and (P6) are not completely clear and therefore including them is debatable.

A survey of the existing fuzziness measures

In this section we will give a brief summary of the various fuzziness measures investigated so far and describe their properties.

1. *DeLuca* and *Termini* (1972) [2]
   Measure:

$$d(\mu) = -K \sum_{i=1}^{N} \{\mu(x_i)log(\mu(x_i)) + (1 - \mu(x_i)log(1 - \mu(x_i)))\}$$

2. *Kaufmann* (1975)

(a) Measure (using the generalized relative Hamming distance):

$$d(\mu) = \frac{2}{N} \sum_{i=1}^{N} |\mu(x_i) - \mu_{\frac{1}{2}}(x_i)|,$$

where the $\frac{1}{2}$-cut of μ is defined by:

$$\mu_{\frac{1}{2}}(x) = 0, \qquad \text{if} \qquad \mu(x) < \frac{1}{2},$$

$$\mu_{\frac{1}{2}}(x) = 1, \qquad \text{if} \qquad \mu(x) \geq \frac{1}{2}$$

(b) Measure (using the generalized relative Euclidean distance):

$$d(\mu) = \frac{2}{N^{\frac{1}{2}}} \left\{ \sum_{i=1}^{N} (\mu(x_i) - \mu_{\frac{1}{2}}(x_i))^2 \right\}^{\frac{1}{2}}$$

3. *Loo* has proposed a general mathematical form for d, thus creating a large class of measures.

Measure:

$$d(\mu) = F \left\{ \sum_{i=1}^{N} c_i F_i(\mu(x_i)) \right\}$$

where $c_i \epsilon R^+$; for all i and F_i is a real-valued function such that $F_i(0) = F_i(1) = 0$, $F_i(u) = F_i(1-u)$ for all $u \epsilon [0,1]$, F_i is strictly increasing on $[0, \frac{1}{2}]$; and F is a positive, increasing function.

Properties: (P1)-(P4) are valid and if F is linear, then (P5) also holds.

The fuzziness measure introduced by DeLuca and Termini can be obtained as special case if $F(x) = x$ and for all $i: \; c_i = K$ and

$$F_i(x) = -(x log(x) + (1-x) log(1-x))$$

4. *Trillas* and *Riera* (1978) have proposed another large class of fuzziness measures.

Measure:

$$d(\mu) = \sum_{i=1}^{N} w(x_i) F(\mu(x_i)),$$

where μ is a real-valued function on $[0,1]$, $F(x) = 0$ iff $x \epsilon \{0,1\}$, $F(x)$ is nondecreasing on $[0, \frac{1}{2}]$, $F(x)$ is nonincreasing on $[\frac{1}{2}]$, and w is a positive weight function from X to R.

5. *Emptoz* (1981) [4]
   Measure:

$$d(\mu) = \frac{1}{N}\sum_{i=1}^{N} F(\mu(x_i)),$$

   where F is a real-valued function on $[0,1]$ such that $F(0)=0$, $F(x)=F(1-x)$ and F is increasing on $[0,\frac{1}{2}]$.

6. *Ebanks* (1983) [3]
   Measure:

$$d(\mu) = \sum_{i=1}^{N} \mu(x_i)(1-\mu(x_i)),$$

   where $\mu\epsilon[0,1]^X$.

As we are interested in fuzziness measures, we will now introduce the vagueness measure.

2 Vagueness measure induced by Pliant operators

In the Pliant system the logical values basically arise from inequalities. If we are on the border of the inequality, i.e. just the equalities are fullfiled we are not sure whether we are inside or outside a region. If we move away from the border we are more certain to be inside or outside the region. Why are we so vague on the border? Because small changes can radically change the logical value. If we demand stable statements, we should avoid being just on the border. Hence it is important to measure the vagueness and also to know how it depends on the vagueness on the input values.

As we mentioned above the idea and construction of a vagueness measure can be derived from the fuzziness measure. In 1972 DeLuca and Termini [1] introduced a fuzziness measure which is used for the membership function. In the Pliant concept we will use the vagueness measure as the operand of the distending function.

From the property of $d(\mu)$, we can define $\mathcal{V}(x)$ in the following way.

Definition 1. *The Vagueness measure in the Pliant system is*

$$\mathcal{V}^*(\delta(\underline{x})) = \frac{1}{n}\sum_{i=1}^{n} \bar{c}(\delta(x_i), \eta(\delta(x_i))) \tag{1}$$

and

$$\mathcal{V}(x) = \bar{c}(x, \eta(x)) \tag{2}$$

Here $\mathcal{V}(x)$ is called the vagueness function and $c(x, y)$ is the conjunitive operator and $\eta(x)$ is a Pliant negation. I.e.

$$\bar{c}(x, y) = f^{-1}(\frac{1}{2}(f(x) + f(y))) \tag{3}$$

$$\eta(x) = f^{-1}(f(x_0)\frac{f(\nu)}{f(x)}) \tag{4}$$

f is the generator function of the conjunitive operator. [5]

The normalized vagueness measure is

$$\mathcal{V}^N(x) = \frac{1}{\bar{c}(\nu_0, \eta(\nu_0))}\bar{c}(x, \eta(x)) \tag{5}$$

Because

$$\mathcal{V}(x) = \bar{c}(x, \eta(x)) = f^{-1}\left(\frac{1}{2}\left(f(x) + f(\nu_0)\frac{f(\nu)}{f(x)}\right)\right) \tag{6}$$

is the representation of vagueness and if we demand that the maximum value should be at ν_0, we have to find the minimum of:

$$Y = X + \frac{A}{X}, \tag{7}$$

which is at $X = \sqrt{A}$, i.e. $f(x) = \sqrt{f(\nu_0)f(\nu)}$, and the maximum is at ν_0, so $f(\nu_0) = f(\nu)$, and we get

$$\mathcal{V}_{\nu_0}(x) = f^{-1}\left(\frac{1}{2}\left(f(x) + \frac{f^2(\nu_0)}{f(x)}\right)\right) \tag{8}$$

if $f(\nu_0) = 1$

$$\mathcal{V}(x) = f^{-1}\left(\frac{1}{2}\left(f(x) + \frac{1}{f(x)}\right)\right) \tag{9}$$

Now we will give a list of properties for the measure of vagueness:

1. Sharpness (No vagueness) (P1)

   $\mathcal{V}(x) = 0 \qquad iff \qquad x\epsilon\{0, 1\}$

2. Maximality (maximal vagueness) (P2)

   $\frac{1}{\nu_0}\mathcal{V}(x) = 1 \qquad iff \qquad x = \nu_0$

3. Symmetry (P4)

   $\mathcal{V}(x) = \mathcal{V}(\eta(x))$

4. Monotonicity (P3)

$$\mathcal{V}(x_1) < \mathcal{V}(x_2) \quad \text{if}$$

$$x_1 < x_2 \quad \text{and} \quad x_1 \leq \nu_0$$

or

$$x_1 > x_2 \quad \text{and} \quad x_1 \geq \nu_0$$

Vagueness measure in the Dombi operator case

As above, the vagueness function is defined in the following way:

$$\mathcal{V}(x) = f^{-1}\left(\frac{1}{2}\left(f(x) + \frac{1}{f(x)}\right)\right)$$

Let $f(x) = \left(\frac{1-x}{x}\right)^{\alpha}$, namely the Dombi operator. Then

$$F_{\alpha}(x) = \frac{1}{1 + \left(\frac{1}{2}\left(\frac{1-x}{x}\right)^{\alpha} + \frac{1}{2}\left(\frac{x}{1-x}\right)^{\alpha}\right)^{\frac{1}{\alpha}}} \tag{10}$$

If $\alpha = 1$ then

$$\mathcal{V}(x) = 2x(1-x) \tag{11}$$

Vagueness measure of the sigmoid function when $\alpha = 1$

The sigmoid function is

$$\delta^{(\lambda)}(x) = \frac{1}{1 + e^{-\lambda x}}$$

So we have

$$\mathcal{V}(\delta^{(\lambda)}(x)) = 2\delta^{(\lambda)}(x)(1 - \delta^{(\lambda)}(x)) \tag{12}$$

$$\mathcal{V}^*(\delta^{(\lambda)}(x)) = 2\int_{-\infty}^{\infty} \frac{(\delta^{(\lambda)}(x))'}{\lambda} dx = \frac{2}{\lambda}\left[\delta^{(\lambda)}(x)\right]_{-\infty}^{\infty} = \frac{2}{\lambda} \tag{13}$$

3 Vagueness measure and entropy

In the case of multiplicative Pliant systems we can define the vagueness function using the conjunctive operator and negation

$$\mathcal{V}(x) = \bar{c}(x, \eta(x)) \tag{14}$$

The normalized $\mathcal{V}(x)$ is

$$\mathcal{V}^N(x) = \frac{1}{\bar{c}(\nu_0, \eta(\nu_0))}\bar{c}(x, \eta(x)) \tag{15}$$

In the additive Pliant case (The bounded or nilpotent operator case)

$$c[x, \eta(x)] = 0$$

and (15) has no meaning.

We will use the following notation:

$$[x] = \begin{cases} 1 \text{ if } x > 1 \\ x \text{ if } 0 \leq x \leq 1 \\ 0 \text{ if } x < 0 \end{cases} \tag{16}$$

First we try to find a proper vagueness measure for the Lukasiewicz operator case.

$$c[x, y] = [x + y - 1]$$

The generator function of $c[x, y]$ is $f(x) = 1 - x$.

Let us define $c_\alpha[x, y]$ using the $(1 - x)^\alpha$ generator function, where $\alpha > 0$.

$$c_\alpha[x, y] = 1 - [((1 - x)^\alpha + (1 - y)^\alpha)^{\frac{1}{\alpha}}]$$

Therefore

$$c_1[x, y] = [x + y - 1]$$

Let $\eta(x) = 1 - x$, which is the simplest definition of $\eta(x)$. Then

$$c_\alpha[x, 1 - x] = 1 - ((1 - x)^\alpha + x^\alpha)^{\frac{1}{\alpha}} \neq 0$$

Let us define the Vagueness measure for the additive Pliant case based on (15) in the following way:

$$\mathcal{V}[x] = \lim_{\alpha \to 1} \mathcal{V}_\alpha(x) = \lim_{\alpha \to 1} \frac{1 - ((1 - x)^\alpha + x^\alpha)^{\frac{1}{\alpha}}}{1 - \left(\frac{1}{2}\right)^{\frac{1}{\alpha} - 1}} \tag{17}$$

Theorem 2.

$$\mathcal{V}[x] = \frac{1}{ln(2)}(xlnx + (1 - x)ln(1 - x)) \tag{18}$$

*i.e., we get the **Shannon entropy** as a measure of knowledge disorder and this theorem, which may be drived from the Lukosiewicz system.*

Proof. To calculate the limes of (17) we will use the L' Hospital rule, and we find the derivative at α.

First, we transform the nominator into the following form:

$$\frac{1-e^{\frac{1}{\alpha}\ln((1-x)^{\alpha}+x^{\alpha})}}{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}-1}}$$

The derivative of the denominator at $\alpha = 1$ is.

$$-\left(-\frac{1}{\alpha^2}\ln\left((1-x)^{\alpha}+x^{\alpha}\right)+\frac{1}{\alpha}\frac{1}{(1-x)^{\alpha}+x^{\alpha}}\left[x^{\alpha}\ln x+(1-x)^{\alpha}\ln(1-x)\right]\right)\Big|_{\alpha=1}=$$

$$-\left(x\ln x+(1-x)\ln(1-x)\right)$$

where

$$\frac{d(x^{\alpha})}{d\alpha}\Big|_{\alpha=1}=x^{\alpha}\ln x\Big|_{\alpha=1}=x\ln x \tag{19}$$

The derivative of the denominator is

$$-2\left(\frac{1}{2}\right)^{\alpha}\ln\left(\frac{1}{2}\right)\Big|_{\alpha=1}=\ln 2$$

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Theorem 2 can be generalized by substituting x into $f(x)$. Then we get

$$\mathcal{V}[x]=-\frac{1}{\ln(2)}\left(f(x)\ln(f(x))+(1-f(x))\ln(1-f(x))\right) \tag{20}$$

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Brief Biography: József Dombi He obtained his degrees at University of Szeged.

His Academic degrees are: doctorate degree (1977, Summa cum laude); Candidate in mathematical studies (1994, CSc, Title of the dissertation: Fuzzy set structure in the area of multicriteria decision making); Academic doctor title in 2011.

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