

# Modalities

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**Abstract** Hedges play an important role in fuzzy theory, although there are relatively few articles on them. Our aim is to provide a theoretical basis not only for hedges, but also for every type of unary operator. One of them is the negation operator, which was presented in an article [14] concerning the DeMorgan class. In our study we will develop unary operators related to other binary operators by demanding that they satisfy certain properties.

## 1 Introduction

Zadeh introduced modifier functions of fuzzy sets called linguistic hedges. A number of studies [12, 11, 18] have been made which discuss fuzzy logic and fuzzy reasoning with linguistic truth values. However, a systematic view of it has not been presented in the construction of linguistic hedges, which have corresponding reverse effects, such as in the case of “very” and “more or less”.

In the early 1970s, Zadeh [32] introduced a class of powering modifiers, which defined the concept of linguistic variables and hedges. He proposed computing with words as an extension of fuzzy sets and logic theory (Zadeh [33, 37, 30]). The linguistic hedges (LHs) change the meaning of primary term values. Many theoretical studies have contributed to the computation with words and to the LH concepts (De Cock and Kerre [11]; Huynh, Ho, and Nakamori [19]; Rubin [27]; Turksen [28]).

As pointed out by Zadeh [34, 35, 36], linguistic variables and terms are closer to human thinking, (which emphasise importance more than certainty) and are used in everyday life. For this reason, words and linguistic terms can be used to model human thinking systems (Liu et al. [25]; Zadeh [31]).

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Zadeh [32] said that a proposition such as "The sea is very rough" can be interpreted as "It is very true that the sea is rough." Consequently, the sentences "The sea is very rough," "It is very true that the sea is rough," "(The sea is rough) is very true" can be considered equivalent. In fact, truth function modification permits an algorithmic approach to the calculus of deduction in approximate reasoning [2], by strengthening the liaison connection with classical logic. Since in traditional propositional logic the validity of a reasoning depends on the simple truth proof of logic propositions [5], in a fuzzy logic we have the truth values that determine the fuzzy set associated with the conclusion of a deduction [29].

Basic notions of linguistic variables were formalized in different works by Zadeh in the mid 1970s [34, 35, 36]. These papers sought to provide a mathematical model for linguistic variables.

### 1.1 Historical background

#### Linguistic hedges (LH)

LHs are special linguistic terms by which other linguistic terms are modified. "Very", "more or less", "fairly", and "extremely" are given as examples of LHs (Jang et al. [20]). For example,  $A^s$  = "very young" is secondary linguistic that may be produced from the primary linguistic term  $A$  = "young" by using LHs (Banks [3]; Jang et al. [20]; Turksen [28]).

#### Representation of LHs

In the standard (canonical) fuzzy concept an LH or modifier is any operation that changes the meaning of any linguistic term (Banks [3]; Jang et al. [20]). Let  $A$  be a continuous linguistic term for an input variable  $x$  with MF  $\mu_A(x)$ . Then  $A^s$  is interpreted as a modified version of the original linguistic term, thus

$$A^s := (x, (\mu_A(x))^p) | x \in X, \quad (1)$$

where  $p$  denotes the linguistic hedge value of the linguistic term  $A$ . Two major modifier operations are commonly used in scientific literature. One of them is the concentration (Jang et al. [20]):

$$CON(A) := A^2. \quad (2)$$

The other is the dilution operations (Jang et al. [20]):

$$DIL(A) := A^{0.5}. \quad (3)$$

Conventionally,  $\text{CON}(A)$  and  $\text{DIL}(A)$  are the results of applying hedges "very" and "more or less" to the linguistic term  $A$ , respectively. However, there are different and constant LH definitions in the literature, such as "very very" ( $p = 4$ ), "quite" ( $p = 1.25$ ), "a little less" ( $p = 0.75$ ) (Banks [3]; Chatterjee and Siarry [10]; Jang et al. [20]; Turksen [28]).

In the article of Lascio [et al.] [24] they construct a mathematical model for the truth values of the Truth linguistic variable and, on a more general basis, for the values of a Boolean linguistic generic variable, which maintains the natural order relation existing between them.

Cat Ho and Wechler in [8, 9] pointed out the discrepancy between the intuitive use made in the natural language of linguistic truth values and the numerical values obtained using CON and DIL operators.

### Shifting modifiers

Another type of fuzzy modifiers, called shifting modifiers, was casually suggested by Lakoff [23] in the 1970's. Hellendoorn [17] and Bouchon [6] then used it in a more formal manner.

## 2 Introduction to the Pliant concept

In this section, besides the min/max and the drastic operators, we will be concerned with strict operators and we will look for the general form of  $c(x, y)$  and  $d(x, y)$ . We assume that the following conditions are satisfied:

1. Continuity:

$$c: [0, 1] \times [0, 1] \rightarrow [0, 1] \quad d: [0, 1] \times [0, 1] \rightarrow [0, 1]$$

2. Strict monotonous increasing:

$$c(x, y) < c(x, y') \text{ if } y < y' \quad x \neq 0 \quad d(x, y) < d(x, y') \text{ if } y < y' \quad x \neq 0$$

3. Compatibility with two-valued logic:

$$\begin{array}{ll} c(0, 0) = 0 & c(1, 1) = 1 \\ c(0, 1) = 0 & c(1, 0) = 0 \end{array} \quad \begin{array}{ll} d(0, 0) = 0 & d(1, 1) = 1 \\ d(0, 1) = 1 & d(1, 0) = 1 \end{array}$$

4. Associativity:

$$c(x, c(y, z)) = c(c(x, y), z) \quad d(x, d(y, z)) = d(d(x, y), z)$$

5. Archimedean:

$$c(x, x) < x, \quad x \in (0, 1) \quad d(x, x) > x, \quad x \in (0, 1)$$

So

$$c(x, y) = f_c^{-1}(f_c(x) + f_c(y)). \quad (4)$$

Similarly, the strict t-conorm on  $(0, 1] \times (0, \infty]$  has the form:

$$d(x, y) = f_d^{-1}(f_d(x) + f_d(y)). \quad (5)$$

Here  $f_c(x) : [0, 1] \rightarrow [0, \infty]$  and  $(f_d(x) : [0, 1] \rightarrow [0, \infty])$  are continuous and strictly increasing (decreasing) monotone functions and they are the generator functions of the strict t-norms and strict t-conorms.

Those familiar with fuzzy logic theory will find that the terminology used here is slightly different from that used in standard texts [22, 7, 1, 4, 26, 16]. This is because I would like to distinguish between fuzzy logic and Pliant logic.

**Definition 1.** We say that  $\eta(x)$  is a negation if  $\eta : [0, 1] \rightarrow [0, 1]$  satisfies the following conditions:

- C1:  $\eta : [0, 1] \rightarrow [0, 1]$  is continuous (Continuity)
- C2:  $\eta(0) = 1, \eta(1) = 0$  (Boundary conditions)
- C3:  $\eta(x) < \eta(y)$  for  $x > y$  (Monotonicity)
- C4:  $\eta(\eta(x)) = x$  (Involution)

From C1, C2 and C3, it follows that there exists a fix point  $v_* \in [0, 1]$  of the negation where

$$\eta(v_*) = v_* \quad (6)$$

So another possible characterization of negation is when we assign a so-called decision value  $v$  for a given  $v_0$ , i.e. a point  $(v, v_0)$  can be specified that the curve must intersect. This tells us something about how strong the negation operator is.

$$\eta(v) = v_0 \quad (7)$$

If  $\eta(x)$  has a fix point  $v_*$ , we use the notation  $\eta_{v_*}(x)$  and if the decision value is  $v$ , then we use the notation  $\eta_v(x)$ . If  $\eta(x)$  is used without a suffix then the parameter has no importance in the proofs. Later on we will characterize the negation by the  $v_*$ ,  $v_0$  and  $v$  parameters.

**Definition 2.** Generalized operators based on strict t-norms and t-conorms are

$$c(\mathbf{w}, \mathbf{x}) = c(w_1, x_1; w_2, x_2; \dots; w_n, x_n) = f_c^{-1} \left( \sum_{i=1}^n w_i f_c(x_i) \right), \quad (8)$$

$$d(\mathbf{w}, \mathbf{x}) = d(w_1, x_1; w_2, x_2; \dots; w_n, x_n) = f_d^{-1} \left( \sum_{i=1}^n w_i f_d(x_i) \right), \quad (9)$$

where  $w_i \geq 0$ .

If  $w_i = 1$  we get the t-norm and t-conorm. If  $w_i = \frac{1}{n}$ , then we get mean operators. If  $\sum_{i=1}^n w_i = 1$ , then we get weighted operators.

**Definition 3.** The DeMorgan law holds for the generalized operator based on strict t-norms and strict t-conorms and for negation if and only if the following condition holds.

$$c(w_1, \eta(x_1); w_2, \eta(x_2); \dots; w_n, \eta(x_n)) = \eta(d(w_1, x_1; w_2, x_2; \dots; w_n, x_n)), \quad (10)$$

We call this later on the generalized DeMorgan law.

**Theorem 1 (General form of the negation).** We have that  $c(\mathbf{w}, \mathbf{x})$ ,  $d(\mathbf{w}, \mathbf{x})$  and  $\eta(x)$  is a DeMorgan triple if and only if

$$\eta(x) = f^{-1}(k(f(x))), \quad (11)$$

where  $f(x) = f_c(x)$  or  $f(x) = f_d(x)$  and  $k(x)$  is a strictly decreasing continuous function with the property

$$k(x) = k^{-1}(x). \quad (12)$$

where  $k: [0, \infty] \rightarrow [0, \infty]$ .

*Proof.* See [15].

**Theorem 2.**  $c(x, y)$  and  $d(x, y)$  build a DeMorgan system for  $\eta_{v_*}(x)$  where  $\eta_{v_*}(v_*) = v_*$  for all  $v_* \in (0, 1)$  if and only if

$$f_c(x)f_d(x) = 1. \quad (13)$$

*Proof.* See [15].

**Definition 4.** If  $k(x) = 1/x$ , that is

$$f_c(x)f_d(x) = 1, \quad (14)$$

then we call the generated connectives a multiplicative Pliant system.

**Theorem 3.** The general form of the multiplicative Pliant system is

$$o_\alpha(x, y) = f^{-1} \left( (f^\alpha(x) + f^\alpha(y))^{1/\alpha} \right) \quad (15)$$

$$\eta_v(x) = f^{-1} \left( f(v_0) \frac{f(v)}{f(x)} \right) \quad \text{or} \quad (16)$$

$$\eta_{v_*}(x) = f^{-1} \left( \frac{f^2(v_*)}{f(x)} \right), \quad (17)$$

where  $f(x)$  is the generator function of the strict  $t$ -norm operator and  $f : [0, 1] \rightarrow [0, \infty]$  is a continuous and strictly decreasing function.

*Proof.* See [15].

Because the generator function is determined up to a multiplicative constant, we can arrange it such that

$$f(v_0) = 1$$

and so

$$\eta_v(x) = f^{-1}\left(\frac{f(v)}{f(x)}\right) \quad (18)$$

If  $f(v_0) = 1$  and if  $v_0 = v$ , then we get:

$$\eta(x) = f^{-1}\left(\frac{1}{f(x)}\right) \quad (19)$$

**Definition 5 (Drastic negation).**

$$\eta_1(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases} \quad \eta_0(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases}$$

$\eta_0(x)$  is the strictest negation, while  $\eta_1(x)$  is the least strict negation  $v$  is the neutral value of the negation and can be interpreted as the strictness of the negation, i.e. if  $v_1 < v_2$  then  $\eta_{v_1}(x)$  is a stricter negation than  $\eta_{v_2}(x)$

### 3 Modalities induced by two negations

The linguistic hedges “very” or “very very” express the modal hedge necessity; and, similarly, the hedge “more or less” expresses the possibility hedge.

From this starting point, the hedges used in fuzzy logic are based on an extension of modal logic to the continuous case. We begin with the negation operator and we use two types of this operator; one that is strict, and one that is less strict. We will show that with these two negation operators we can define the modal hedges.

Modal logic, which is an area of mathematical logic, can be viewed as a logical system obtained by adding logical symbols and inference rules. From a semantics viewpoint, modal logic can also be viewed as a part of a logical system.

This issue is related in part to linguistic hedges to corresponding reverse effects and also to the modal operators with mutually reverse modal concepts. We will construct linguistic modal hedges called necessity and possibility hedges. The construction is based on the fact that modal operators can be realized by combining two kinds of

negation operators, i.e. negation in the reverse sense of classical logic and negation in the reverse sense of intuitionistic logic, which is a strict negation operator.

In intuitionistic logic, another kind of negation operator also has to be taken into account because the law of Excluded Middle does not hold.  $\sim_1 x$  does not imply "x is not", although "there exists a path such that x is a contradiction." In other words,  $\sim_1 x$  is a stronger negation than  $\sim_2 x$ . Because  $\sim_1 x$  in modal logic, it means "x is impossible".

$$\sim_1 x = \Box \sim_2 x \quad (20)$$

One can define the necessity hedge by  $\Box x$  and the possibility hedge by  $\Diamond x$ , which have mutually reverse effects.

We will show that both operators belong to the same class of unary functions, and that because they have a common form in the Pliant system, we will denote both of them by  $\tau_v(x)$ . Depending on the  $v$  value, we get the necessity hedge or the possibility hedge.

As we mentioned above, in modal logic we have two more operators than the classical logic case: namely necessity and possibility; and in modal logic there are two basic identities. These are:

$$\sim_1 x = \text{impossible}(x) = \text{necessity}(\text{not}(x)) = \Box \sim_2 x \quad (21)$$

$$\Diamond x = \text{possible}(x) = \text{not}(\text{impossible}(x)) = \sim_2 (\sim_1 x) \quad (22)$$

In our context, we model *impossible*(x) with a stricter negation operator than *not*(x).

In modal logic  $\Box x$  means that  $x$  is necessarily valid. If we negate  $x$ , then "necessarily not<sub>1</sub> x",  $\Box \sim_1$  has the meaning "impossible" and we suppose "not<sub>2</sub> impossible" is "possible", so

$$\Diamond x = \sim_2 \Box \sim_1 x. \quad (23)$$

This serves as a definition of the possibility operator.

If in Eq.(21) we replace  $x$  by  $\sim_2 x$  and using the fact that  $\sim_2 x$  is involutive, we get

$$\Box x = \sim_1 (\sim_2 x), \quad (24)$$

and with Eq.(23), we have

$$\diamond x = \sim_2 (\sim_1 x). \quad (25)$$

It is also obvious in modal logic that:

$$\Box x \leq x \leq \diamond x \quad (26)$$

In Pliant logic there are several types of negation operators [14, 13] that can be distinguished by the neutral value. If  $v$  is small we can say that negation operator is strict; otherwise it is not strict. Using this, we can apply Eq.(24) and Eq.(25) to the Pliant concept.

Based on the above considerations, we can formally define the necessity and possibility modifiers.

**Definition 6.** On the basis of Eq.(24), we get:

$$\tau_{\Box}(x) = \eta_{v_1}(\eta_{v_0}^{-1}(x)) = \eta_{v_1}(\eta_{v_0}(x)), \quad (27)$$

where  $v_1 < v_0$ , and  $\tau_{\Box}(x)$  the necessity operator.

We can use Eq.(16) and express  $\tau_{\Box}(x)$  like so

$$\tau_{\Box}(x) = f^{-1} \left( f(v_0) \frac{f(v_1)}{f(v_0)f(v_0)} f(x) \right)$$

then we can rewrite it in the following form:

$$\tau_{\Box}(x) = f^{-1} \left( f(v_0) \frac{f(x)}{f(v_{\Box})} \right), \quad (28)$$

where

$$v_{\Box} = f^{-1} \left( \frac{f(v_0)}{f(v_1)} \right)$$

Applying similar reasoning, we can get the possibility operator using Eq.(25). Now our results can be summarized by the following theorem:

**Theorem 4.** Let  $\tau_{\Box}(x)$  and  $\tau_{\diamond}(x)$  be the necessity modifier and possibility modifier, respectively. Then on the basis of eqs.(24) and (25), we have

$$\tau_I(x) = f^{-1} \left( \frac{f(v_0) f(x)}{f(v_I)} \right)$$

(where  $I = \{\Box, \Diamond\}$ .)

They both have a common form where  $v_\Diamond < v_0 < v_\Box$

A more general concept can be stated as a set of definitions.

**Definition 7.** A modal hedge means that

$$\tau_{v_1, v_2}(x) = \eta_{v_1}(\eta_{v_2}(x)), \quad (29)$$

where  $v_1$  and  $v_2$  are neutral values. If  $v_1 < v_2$ , then  $\tau_{v_1, v_2}(x)$  is a necessity operator and if  $v_2 < v_1$ , then  $\tau_{v_1, v_2}(x)$  is a possibility operator.

From the above definition, we get

$$\tau_{v_1, v_2}(x) = f^{-1} \left( \frac{f(v_1) f(x)}{f(v_2)} \right), \quad (30)$$

This can be rewritten as

$$\tau_{v, v_0}(x) = f^{-1} \left( \frac{f(v_0) f(x)}{f(v)} \right). \quad (31)$$

We call this the general form of the hedges and in this case it is not hard to show that if  $v_0 < v$ ,  $\tau_v(x)$  is the necessity hedge and if  $v < v_0$  it is the possibility hedge. If  $f(v_0) = 1$ , then

$$\tau_v(x) = f^{-1} \left( \frac{f(x)}{f(v)} \right). \quad (32)$$

**Definition 8.** We call a necessity (possibility) hedge a dual hedge if  $v_1 = \eta(v_2)$ , i.e.

$$v_1 = f^{-1} \left( \frac{1}{f(v_2)} \right)$$

From this, the necessity operator is

$$\tau_\Box(x) = f^{-1} \left( \frac{f(x)}{f(v)} \right),$$

and the corresponding possibility operator is (if  $v < v_0$ )

$$\tau_\Diamond = f^{-1}(f(v)f(x)).$$

If we use the definition of  $\tau(x) = \Box x$  or  $\Diamond x$  (i.e. necessity, or possibility  $x$ ), then we can introduce different necessity and possibility operators.

$$\begin{aligned}\Box^2 x &= \Box(\Box(x)) = \tau_{\Box}(\tau_{\Box}(x)) \\ \Diamond^2 x &= \Diamond(\Diamond(x)) = \tau_{\Diamond}(\tau_{\Diamond}(x))\end{aligned}\quad (33)$$

We will use the following notation:

- If  $v > v_0$  and  $f(v_0) = 1$

$$\tau_{\Box}(x) = \Box(x) = f^{-1}\left(\frac{f(x)}{f(v)}\right) = \tau_v(x) \quad \text{a necessity} \quad (34)$$

$$\tau_{\Diamond}(x) = \Diamond(x) = f^{-1}(f(v)f(x)) = \tau_{\eta(v)}(x) \quad \text{a possibility} \quad (35)$$

- If  $v < v_0$

$$\tau_{\Diamond}(x) = \Diamond(x) = f^{-1}\left(\frac{f(x)}{f(v)}\right) = \tau_v(x) \quad \text{a necessity}$$

$$\tau_{\Box}(x) = \Box(x) = f^{-1}(f(v)f(x)) = \tau_{\eta(v)}(x) \quad \text{a possibility}$$

**Definition 9.** We call graded modalities a  $k$  composition of the modalities.

$$\tau_{\Box}(\tau_{\Box}(\dots \tau_{\Box}(x))) = \Box(\underbrace{\Box(\dots \Box(x))}_{K} \dots) = \Box^K(x) \quad (36)$$

$$\tau_{\Diamond}(\tau_{\Diamond}(\dots \tau_{\Diamond}(x))) = \Diamond(\underbrace{\Diamond(\dots \Diamond(x))}_{K} \dots) = \Diamond^K(x) \quad (37)$$

**Definition 10.**

$$\text{drastic necessity} \quad \tau_1(x) = \Box_1(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases} \quad (38)$$

$$\text{drastic possibility} \quad \tau_0(x) = \Diamond_0(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases} \quad (39)$$

On the basis of eqs.(34) and (35), it is easy to verify the following properties.

#### 4 Basic properties of modalities

1.  $\tau_{\Box}(\tau_{\Diamond}(x)) = \Box(\Diamond(x)) = x$
2.  $\tau_{\Diamond}(\tau_{\Box}(x)) = \Diamond(\Box(x)) = x$

3.  $\tau_1(\tau_0(x)) = \square_1(\diamond_0(x)) = \diamond_1(x)$
4.  $\tau_0(\tau_1(x)) = \diamond_0(\square_1(x)) = \square_0(x)$
5.  $\tau_{\square}^n(\tau_{\square}^m(x)) = \square^n(\square^m(x)) = \square^{n+m}(x)$
6.  $\tau_{\diamond}^n(\tau_{\diamond}^m(x)) = \diamond^n(\diamond^m(x)) = \diamond^{n+m}(x)$
7.  $\tau_{\diamond}^n(\tau_{\square}^m(x)) = \diamond^n(\square^m(x)) = \begin{cases} \diamond^{n-m}(x) & \text{if } n-m > 0 \\ x & \text{if } n=m=0 \\ \square^{m-n}(x) & \text{if } n-m < 0 \end{cases}$
8.  $\tau_{\square}^n(\tau_{\diamond}^m(x)) = \square^n(\diamond^m(x)) = \begin{cases} \square^{n-m}(x) & \text{if } n-m > 0 \\ x & \text{if } n=m=0 \\ \diamond^{m-n}(x) & \text{if } n-m < 0 \end{cases}$
9.  $\lim_{K \rightarrow \infty} \tau_{\square}^K(x) = \tau_1(x)$
10.  $\lim_{K \rightarrow \infty} \tau_{\diamond}^K(x) = \tau_0(x)$

### 4.1 Limit of modalities

Here we introduce the drastic modality operator:

$$\tau_1(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x < 1 \end{cases}$$

and

$$\tau_0(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

**Theorem 5.** *Let*

$$\tau_{v, v_0}(\tau_{v, v_0}(\tau_{v, v_0} \dots \tau_{v, v_0}(x))) = \underbrace{\tau \circ \tau \circ \dots \tau \circ \tau}_M(x) = \tau^M(x)$$

*If*  $v_0 < v$  *then:*

$$\lim_{M \rightarrow \infty} \tau^M(x) = \tau_1(x)$$

*If*  $v_0 > v$  *then:*

$$\lim_{M \rightarrow \infty} \tau^M(x) = \tau_0(x)$$

*Proof.* Because

$$\tau^M(x) = f^{-1} \left( f(v_0) \frac{f(x)}{f(v)} \right), \quad (40)$$

where

$$v_M = f^{-1} \left( f(v_0) \left( \frac{f(v)}{f(v_0)} \right)^M \right) \quad (41)$$

If  $v < v_0$  then  $f(v_0) > f(v)$

$$\lim_{M \rightarrow \infty} v_M = 0 \quad (42)$$

In a similar way, we can get the result when  $v_0 < v$ .

If  $v = v_0$ , then  $v_M = \frac{1}{2}$

## 5 Conclusion

In this article we give a general and theoretical basis for modalities. Here we define the necessity and possibility operators and we define them by a generator function of the Pliant operators. The operator system is so called Pliant system because the construction is based on different negation operations and in the Pliant system infinitely many negations are consistent with the DeMorgan identity.

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