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Formal Series of General Algebras over a Field and Their Syntactic Algebras

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Abstract

Any mapping $S : C \rightarrow K$, where K is a field and $\mathcal{C} = (C, \Sigma)$ is a Σ-algebra, is called a $K\mathcal{C}$-series. These series are natural generalizations of both formal series on strings over a field and the tree series introduced by Berstel and Reutenauer [1]. We consider various operations on $K\mathcal{C}$-series and their effects on the linear extensions of the series. We also study some algebraic aspects of the $K\Sigma$-algebra formed by the $K\mathcal{C}$-polynomials (i.e., the $K\mathcal{C}$-series with finitely many non-zero coefficients); a $K\Sigma$-algebra is a Σ-algebra based on a K-vector space in which all the Σ-operations are multilinear. The syntactic congruence of a $K\mathcal{C}$-series S is a congruence on this $K\Sigma$-algebra of $K\mathcal{C}$-polynomials, and the syntactic $K\Sigma$-algebra $\mathrm{SA}(S)$ of S is the corresponding quotient algebra. These syntactic algebras generalize Reutenauer's syntactic K-algebras of string series [13] and the syntactic $K\Sigma$-algebras of tree series studied by Bozapalidis *et al.* [5, 4, 3]. It is shown that $\mathrm{SA}(S)$ is finite-dimensional iff the series S is recognizable. We also characterize the subdirectly irreducible $K\Sigma$-algebras and show that all of them are syntactic. Furthermore, we show how various operations on $K\mathcal{C}$-series relate to the syntactic $K\Sigma$-algebras.

Keywords: formal series; recognizable series; $K\Sigma$-algebras; syntactic algebras

1 Introduction

It is quite convenient to prove the basic common properties of syntactic semigroups of string languages (cf. [8, 12], for example) and syntactic algebras of tree languages (cf. [16], for example) for syntactic algebras of subsets of general algebras. This way one may also obtain generalizations of the variety theories of string and tree languages (cf. [15, 17], for example). In this paper, we develop a similar generalization for the

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syntactic algebras of various series over a field. More precisely, we study syntactic congruences and syntactic algebras of what we call $K\mathcal{C}$-series, i.e., mappings $S : C \to K$, where K is a field and $\mathcal{C} = (C, \Sigma)$ is a Σ-algebra. These series generalize both formal series on strings over a field and the tree series introduced by Berstel and Reutenauer [1]; the former are obtained when $\mathcal{C}$ is a finitely generated free monoid, and the latter when $\mathcal{C}$ is a finitely generated term algebra of finite type.

We consider various operations on $K\mathcal{C}$-series and their effects on the linear extensions of the series. We also study some algebraic aspects of the $K\Sigma$-algebra formed by the $K\mathcal{C}$-polynomials (i.e., the $K\mathcal{C}$-series with finitely many non-zero coefficients); a $K\Sigma$-algebra is a Σ-algebra based on a K-vector space in which all the Σ-operations are multilinear. The syntactic congruence of a $K\mathcal{C}$-series S is a congruence on this $K\Sigma$-algebra of $K\mathcal{C}$-polynomials, and the syntactic $K\Sigma$-algebra $\mathrm{SA}(S)$ of S is the corresponding quotient algebra. These syntactic algebras generalize Reutenauer's [13] syntactic K-algebras of string series and the syntactic $K\Sigma$-algebras of tree series studied by Bozapalidis *et al.* [5, 4, 3]. Accordingly, some of our results generalize results presented in those papers. However, we give our own complete proofs. It is shown that $\mathrm{SA}(S)$ is finite-dimensional iff the series S is recognizable. We also characterize the subdirectly irreducible $K\Sigma$-algebras and show that they are syntactic. Furthermore, we show how various operations on $K\mathcal{C}$-series relate to syntactic congruences and syntactic $K\Sigma$-algebras. The choice of these operations reflect the fact that this work ultimately aims at a variety theory, in first place for tree series, but perhaps also for more general families of series over a field.

The paper is organized as follows. In Section 2 we recall a few basic definitions and fix some notation. Throughout the paper, K is a field and Σ is a ranked alphabet, both arbitrarily chosen but fixed. In Section 3 we recall the notion of a $K\Sigma$-algebra introduced in [3, 4]. It is equivalent to the "representations" considered already in [1], but the explicit formulation as algebras suits us well. We describe the translations of a $K\Sigma$-algebra and use them for characterizing its congruences.

In Section 4 we consider the $K\mathcal{C}$-series $S : C \to K$ over a given general Σ-algebra $\mathcal{C} = (C, \Sigma)$. The sums $S + T$ and the scalar multiples aS of $K\mathcal{C}$-series are defined by the usual pointwise conditions, and assuming that $\mathcal{C}$ satisfies a certain simple condition, we can also define the natural Σ-operations for them. As special cases we obtain both the usual formal K-series over an alphabet X (by letting $\mathcal{C}$ be the monoid X^*) and the tree series $S : T_\Sigma(X) \to K$ (by letting $\mathcal{C}$ be the term algebra $\mathcal{T}_\Sigma(X)$), and many basic facts about string and tree series hold also for general $K\mathcal{C}$-series. In particular, the $K\mathcal{C}$-series and the $K\mathcal{C}$-polynomials form the $K\Sigma$-algebras $\mathcal{S}_K(\mathcal{C}) = (K\langle\langle C\rangle\rangle, +, \tilde{0}, \Sigma)$ and $\mathcal{P}_K(\mathcal{C}) = (K\langle C\rangle, +, \tilde{0}, \Sigma)$, respectively. We study some further operations on $K\mathcal{C}$-series such as their images and pre-images under translations or homomorphisms of the underlying Σ-algebras, and we show how such operations affect the linear extensions $\overline{S} : K\langle C\rangle \to K$ of $K\mathcal{C}$-series. Furthermore, we establish some useful algebraic facts about the polynomial algebras $\mathcal{P}_K(\mathcal{C})$. Let us note that in [10] W. Kuich considered series of general algebras over semirings as solutions of systems of equations.

In Section 5 we define the syntactic congruence $\equiv_S$ of a $K\mathcal{C}$-series S and its syntactic $K\Sigma$-algebra $\mathrm{SA}(S) = \mathcal{P}_K(\mathcal{C})/{\equiv_S}$. The recognizability of a $K\mathcal{C}$-series by a $K\Sigma$-algebra can be defined similarly as for ordinary power series in [13] or tree series in [1, 4, 3], and we show that also here $\mathrm{SA}(S)$ is in a natural sense the least $K\Sigma$-algebra that recognizes S (Proposition 5.6), and that S is recognizable if and only if $\mathrm{SA}(S)$ has finite dimension (Corollary 5.7). Moreover, we characterize the syntactic $K\Sigma$-algebras, i.e., the $K\Sigma$-algebras isomorphic to the syntactic $K\Sigma$-algebra of some $\mathcal{C}$-series for some Σ-algebra $\mathcal{C}$. As a consequence, we obtain the useful fact that every subdirectly irreducible $K\Sigma$-algebra is syntactic. Let us note that C. Mathissen [11] considers recognizable series in Σ-algebras over a commutative semiring as well as their syntactic algebras, and in some places we could have made use of his results. However, our discussion of the topic is broader, and we have retained our own presentation for the sake of uniformity and the convenience of the reader.

Finally, in Section 6 we review our results and consider some possible topics for future research.

2 Preliminaries

We frequently write $A := B$ to indicate that some object A is defined to be B. For any integer $n \geq 0$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. In particular, $[0] = \emptyset$.

For any relation $\rho \subseteq A \times B$, the fact that $(a, b) \in \rho$ for some $a \in A$ and $b \in B$, will usually be expressed by writing $a\rho b$. For any $a \in A$, let $a\rho = \{b \in B \mid a\rho b\}$ and, for any $A' \subseteq A$, let $A'\rho = \{b \in B \mid (\exists a \in A')\, a\rho b\}$. The *converse* of ρ is the relation $\rho^{-1} := \{(b, a) \in B \times A \mid a\rho b\}$ from B to A. The *composition* of relations $\rho \subseteq A \times B$ and $\rho' \subseteq B \times C$ is the relation $\rho \circ \rho' := \{(a, c) \in A \times C \mid (\exists b \in B)\, a\rho b \text{ and } b\rho' c\}$. In the particular case of an equivalence relation, we write $[a]_\rho$, or just $[a]$, for $a\rho$. As usual, A/ρ denotes the quotient set $\{[a]_\rho \mid a \in A\}$. The *diagonal relation* $\{(a, a) \mid a \in A\}$ of a set A is denoted by Δ_A. A mapping $\varphi : A \to B$ may also be viewed as a relation ($\subseteq A \times B$), and $a\varphi$ ($a \in A$) denotes either the image $\varphi(a)$ of a or the set formed by it. Especially homomorphisms will usually be written this way as right operators that are composed from left to right omitting the symbol $\circ$. Thus the composition of two mappings $\varphi : A \to B$ and $\psi : B \to C$ is the mapping $\varphi\psi : A \to C$, $a \mapsto a\varphi\psi$, where $a\varphi\psi = (a\varphi)\psi = \psi(\varphi(a))$. The *kernel* of a mapping $\varphi : A \to B$ is the equivalence relation $\ker\varphi := \{(a_1, a_2) \in A^2 \mid a_1\varphi = a_2\varphi\}$ on A.

In what follows, Σ is always a *ranked alphabet*, i.e., a finite set of symbols each of which has a given nonnegative integer arity. For any $k \geq 0$, the set of k-ary symbols in Σ is denoted by Σ_k. We use Σ as a set of operation symbols; a Σ-*algebra* $\mathcal{C}$ consists of a nonempty set C of elements and a Σ-indexed family of operations $(\sigma^{\mathcal{C}} \mid \sigma \in \Sigma)$ on C such that if $\sigma \in \Sigma_k$ is a k-ary symbol, then $\sigma^{\mathcal{C}} : C^k \to C$ is a k-ary operation on C. In particular, any nullary symbol $\omega \in \Sigma_0$ fixes a constant in C that we write as $\omega^{\mathcal{C}}$. We write simply $\mathcal{C} = (C, \Sigma)$ without any symbol for the assignment $\sigma \mapsto \sigma^{\mathcal{C}}$. Subalgebras, homomorphisms (that we may call Σ-*homomorphisms*) and congruences of such Σ-algebras are defined as usual (cf. [6] or [7], for example).

A mapping $p : C \to C$ is called an *elementary translation* of $\mathcal{C} = (C, \Sigma)$ if there exist a $k > 0$, a $\sigma \in \Sigma_k$, an $i \in [k]$ and elements $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_k \in C$ such that $p(d) = \sigma^{\mathcal{C}}(c_1, \ldots, c_{i-1}, d, c_{i+1}, \ldots, c_k)$ for every $d \in C$. Let $\mathrm{ETr}(\mathcal{C})$ denote the set of elementary translations of $\mathcal{C}$. The set $\mathrm{Tr}(\mathcal{C})$ of all *translations* of $\mathcal{C}$ is the least set of unary operations on C that contains the identity map $1_C : C \to C, c \mapsto c$, and all the elementary translations, and is closed under composition. It is well known (cf. [6] or [7], for example) that any congruence of an algebra $\mathcal{C} = (C, \Sigma)$ is invariant with respect to every translation of $\mathcal{C}$, and that an equivalence on C is a congruence on $\mathcal{C}$ if it is invariant with respect to every elementary translation of $\mathcal{C}$.

The following lemma (cf. [16]) will be needed several times.

Lemma 2.1 *Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a homomorphism between two Σ-algebras $\mathcal{C} = (C, \Sigma)$ and $\mathcal{D} = (D, \Sigma)$. For every translation $p \in \mathrm{Tr}(\mathcal{C})$ of $\mathcal{C}$ there is a translation $p_\varphi \in \mathrm{Tr}(\mathcal{D})$ of $\mathcal{D}$ such that $p(c)\varphi = p_\varphi(c\varphi)$ for every $c \in C$. If φ is surjective, then for every $q \in \mathrm{Tr}(\mathcal{D})$ there exists a $p \in \mathrm{Tr}(\mathcal{C})$ such that $q = p_\varphi$.*

Especially in the later parts of the paper, we shall use a few further concepts and results of universal algebra. These can be found, for instance, in Chapter II of [6].

3 $K\Sigma$-algebras

Throughout this paper, $(K, +, \cdot, 0, 1)$ is a field, arbitrarily chosen but fixed. We shall call it simply K. The product of any elements $a, b \in K$ is usually written as $a \cdot b$. We shall consider vector spaces over K equipped with Σ-operations that are linear in all components. Given a K-vector space $(M, +, 0)$, we will denote the *scalar product* of a field element $a \in K$ and a vector $u \in M$ by au. Following [4, 3], we define a $K\Sigma$-*algebra*[1] to be a system $\mathcal{M} = (M, +, 0, \Sigma)$ where $(M, +, 0)$ is a K-vector space

[1]In [1] these are called *linear representations* (of term algebras). In the terminology of [9] they would be called *K-Σ-vector spaces.*

and (M, Σ) is a Σ-algebra in which all operations are *multilinear*, that is to say, for any $k > 0$, $\sigma \in \Sigma_k$, $i \in [k]$, $u_1, \ldots, u_{i-1}, u, v, u_{i+1}, \ldots, u_k \in M$ and $a, b \in K$,

$$\begin{aligned}\sigma^{\mathcal{M}}&(u_1, \ldots, u_{i-1}, au + bv, u_{i+1}, \ldots, u_k) \\ &= a\sigma^{\mathcal{M}}(u_1, \ldots, u_{i-1}, u, u_{i+1}, \ldots, u_k) + b\sigma^{\mathcal{M}}(u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_k).\end{aligned}$$

The Σ-algebra part (M, Σ) of $\mathcal{M}$ is denoted by $\mathcal{M}^\bullet$, but we will write $\sigma^{\mathcal{M}}$ instead of $\sigma^{\mathcal{M}^\bullet}$ ($\sigma \in \Sigma$). The *dimension* of $\mathcal{M}$ is the dimension $\dim M$ of the underlying vector space $(M, +, 0)$, and $\mathcal{M}$ is said to be *finite-dimensional* if $\dim M$ is finite.

Subalgebras, homomorphisms and direct products of $K\Sigma$-algebras are defined as one would expect; in each case both the K-vector space and Σ-algebra aspect are taken into account. Let $\mathcal{M} = (M, +, 0, \Sigma)$ and $\mathcal{N} = (N, +, 0, \Sigma)$ be any $K\Sigma$-algebras. Then $\mathcal{M}$ is a *subalgebra* of $\mathcal{N}$ if $M \subseteq N$ and every operation of $\mathcal{M}$ is the restriction of the corresponding operation of $\mathcal{N}$, i.e., $\mathcal{M}^\bullet$ is a subalgebra of the Σ-algebra $\mathcal{N}^\bullet$, and for any $u, u' \in M$ and $a \in K$, the vector sum $u + u'$ and the scalar product au get the same values in $\mathcal{M}$ as in $\mathcal{N}$. A mapping $\varphi : M \to N$ is a *homomorphism of $K\Sigma$-algebras*, or a *$K\Sigma$-homomorphism* for short, which we express by writing $\varphi : \mathcal{M} \to \mathcal{N}$, if

(1) $(u + u')\varphi = u\varphi + u'\varphi$ for all $u, u' \in M$,

(2) $(au)\varphi = a(u\varphi)$ for all $u \in M$ and $a \in K$, and

(3) $\sigma^{\mathcal{M}}(u_1, \ldots, u_k)\varphi = \sigma^{\mathcal{N}}(u_1\varphi, \ldots, u_k\varphi)$ for all $k \geq 0$, $\sigma \in \Sigma$ and $u_1, \ldots, u_k \in M$.

A homomorphism is called an *epimorphism*, a *monomorphism* or an *isomorphism* if it is, respectively, surjective, injective or bijective. If there is an isomorphism $\varphi : \mathcal{M} \to \mathcal{N}$, then $\mathcal{M}$ and $\mathcal{N}$ are *isomorphic*, $\mathcal{M} \cong \mathcal{N}$ in symbols, and if there is an epimorphism $\varphi : \mathcal{M} \to \mathcal{N}$, then $\mathcal{N}$ is an *(epimorphic) image* of $\mathcal{M}$, $\mathcal{N} \twoheadleftarrow \mathcal{M}$ in symbols. A monomorphism $\varphi : \mathcal{M} \to \mathcal{N}$ is also called an *embedding*. Such an embedding exists exactly in case $\mathcal{M}$ is isomorphic to a subalgebra of $\mathcal{N}$, and we express this situation by writing $\mathcal{M} \subseteq \mathcal{N}$. Furthermore, $\mathcal{N}$ is said to *cover* $\mathcal{M}$ if $\mathcal{M}$ is an image of some subalgebra of $\mathcal{N}$. This we express by writing $\mathcal{M} \preceq \mathcal{N}$. Clearly, $\preceq$ generalizes both the subalgebra relation $\subseteq$ and the epimorphic image relation $\twoheadleftarrow$.

An equivalence θ on M is a *congruence* on $\mathcal{M} = (M, +, 0, \Sigma)$ if it is a congruence on the Σ-algebra $\mathcal{M}^\bullet$ such that for all $u, u', v, v' \in M$ and $a \in K$,

(1) if $u \,\theta\, v$ and $u' \,\theta\, v'$, then $u + u' \,\theta\, v + v'$, and

(2) if $u \,\theta\, v$, then $au \,\theta\, av$.

Let $\mathrm{Con}(\mathcal{M})$ denote the set of congruences on $\mathcal{M}$. The *quotient $K\Sigma$-algebra* $\mathcal{M}/\theta = (M/\theta, +, [0]_\theta, \Sigma)$ with respect to a congruence $\theta \in \mathrm{Con}(\mathcal{M})$ is defined as usual. Thus

$\sigma^{\mathcal{M}/\theta}([u_1], \ldots, [u_k]) = [\sigma^{\mathcal{M}}(u_1, \ldots, u_k)]$ for all $k \geq 0$, $\sigma \in \Sigma_k$ and $u_1, \ldots, u_k \in M$, and $[u] + [v] = [u + v]$ and $a[u] = [au]$ for all $u, v \in M$ and $a \in K$.

Let us now consider the translations of $K\Sigma$-algebras. First we note the following consequence of the multilinearity of the Σ-operations.

Lemma 3.1 *Let $\mathcal{M} = (M, +, 0, \Sigma)$ be a $K\Sigma$-algebra. For any translation p of the Σ-algebra $\mathcal{M}^\bullet = (M, \Sigma)$, and for all $a, b \in K$ and $u, v \in M$,*

$$p(au + bv) \ = \ ap(u) + bp(v).$$

The set $\mathrm{ETr}(\mathcal{M})$ of elementary translations of a $K\Sigma$-algebra $\mathcal{M} = (M, +, 0, \Sigma)$ consists of the elementary translations of $\mathcal{M}^\bullet$ and the mappings

$$M \to M,\ \xi \mapsto \xi + u, \quad \text{and} \quad M \to M,\ \xi \mapsto a\xi,$$

where $u \in M$ and $a \in K$ are any given elements. By using Lemma 3.1, it is easy to prove the following normal form result for the general translations of a $K\Sigma$-algebra.

Proposition 3.2 *Let $\mathcal{M} = (M, +, 0, \Sigma)$ be a $K\Sigma$-algebra. A mapping $p : M \to M$ is a translation of $\mathcal{M}$ if and only if it can be expressed in the form $p(\xi) = aq(\xi) + u$, where $a \in K$, $q \in \mathrm{Tr}(\mathcal{M}^\bullet)$ and $u \in M$.*

The above description of the elementary translations of a $K\Sigma$-algebra and Proposition 3.2 yield the following lemma.

Lemma 3.3 *Let $\mathcal{M} = (M, +, 0, \Sigma)$ be a $K\Sigma$-algebra. An equivalence $\equiv$ on M is a congruence of $\mathcal{M}$ if and only if, for all $u, u' \in M$,*

(1) *if $u \ \equiv \ u'$, then $p(u) \ \equiv \ p(u')$ for every $p \in \mathrm{ETr}(\mathcal{M}^\bullet)$,*

(2) *if $u \ \equiv \ u'$, then $u + v \ \equiv \ u' + v$ for every $v \in M$, and*

(3) *if $u \ \equiv \ u'$, then $au \ \equiv \ au'$ for every $a \in K$.*

On the other hand, if $\equiv$ is a congruence of $\mathcal{M}$, then $u \ \equiv \ u'$ implies $aq(u) + v \ \equiv \ aq(u') + v$ for all $a \in K$, $q \in \mathrm{Tr}(\mathcal{M}^\bullet)$ and $v \in M$.

4 Formal series and polynomials of Σ-algebras

Let K be our given field and C be a nonempty set. Any mapping $S : C \to K$ is also called a *formal C-series over K*, or just a *KC-series*. The *coefficient* $S(c)$ of an element $c \in C$ is usually denoted by (S, c), and one may view S as the formal sum $\sum_{c \in C}(S, c).c$. The set of all KC-series is denoted by $K\langle\langle C\rangle\rangle$.

The *support* of a KC-series S is the set $\mathrm{supp}(S) := \{c \in C \mid (S, c) \neq 0\}$. A *$KC$-polynomial* is a KC-series S such that $\mathrm{supp}(S)$ is finite, and the set of these is denoted by $K\langle C\rangle$. If P is a KC-polynomial such that $\mathrm{supp}(P) = \{c_1, \ldots, c_m\}$ and $(P, c_i) = a_i$ $(i = 1, \ldots, m)$, then we may write $P = a_1.c_1 + \ldots + a_m.c_m$[2]. If $\mathrm{supp}(P)$ is a singleton, then P is called a *KC-monomial.* We may regard C as a subset of $K\langle C\rangle$ by identifying any element $c \in C$ with the monomial $1.c$, and c may denote also this monomial. For any $a \in K$, we let $\widetilde{a}$ denote the *constant KC-series* for which $(\widetilde{a}, c) = a$ for every $c \in C$. Note that $\widetilde{0}$ is a KC-polynomial.

The *sum* $S + T$ of two KC-series $S, T \in K\langle\langle C\rangle\rangle$ and the *scalar multiple* aS of a KC-series S by a scalar $a \in K$, are defined in the natural way: $(S + T, c) = (S, c) + (T, c)$ and $(aS, c) = a \cdot (S, c)$ for every $c \in C$. It is well known (and easy to see that) $K\langle\langle C\rangle\rangle$ forms a K-vector space $(K\langle\langle C\rangle\rangle, +, \widetilde{0})$ under these operations. Moreover, the KC-polynomials form the subspace $(K\langle C\rangle, +, \widetilde{0})$.

If $\mathcal{C} = (C, \Sigma)$ is a Σ-algebra, then KC-series and KC-polynomials are also called *$K\mathcal{C}$-series* and *$K\mathcal{C}$-polynomials*, respectively. A $K\Sigma$-algebra $\mathcal{M} = (M, +, 0, \Sigma)$ is said to *recognize* a $K\mathcal{C}$-series S if there exist a homomorphism $\varphi : \mathcal{C} \to \mathcal{M}^\bullet$ and a linear form $\gamma : M \to K$ such that $(S, c) = \gamma(c\varphi)$ for every $c \in C$. A $K\mathcal{C}$-series is *recognizable* if it is recognized by a finite-dimensional $K\Sigma$-algebra. The triple $(\mathcal{M}, \varphi, \gamma)$ is called a *representation* of S. Let $\mathrm{Rec}_K(\mathcal{C})$ denote the set of recognizable $K\mathcal{C}$-series. This definition generalizes the usual notions of recognizable string or tree series over a field (cf. [13] and [1, 5, 4, 3], resp.). In [11] the corresponding generalization is presented for $K\mathcal{C}$-series where K is a commutative semiring; then the $K\Sigma$-algebras are based on K-semimodules rather than on vector spaces.

At this point we introduce our running example. The underlying algebra was obtained from an example considered in [14] by omitting the edge labels.

Example 4.1 The set SP of *series-parallel graphs* is defined inductively as follows:

(1) SP contains the directed graph $\downarrow$ consisting of two nodes, called the *source* and the *sink*, and a directed edge from the source to the sink;

(2) if $g_1, g_2 \in \mathrm{SP}$, then SP also contains

   (a) the *parallel composition* $g_1 \| g_2$ obtained by pasting together the sources and the sinks of g_1 and g_2, and

   (b) the *series composition* $g_1 \cdot g_2$ obtained by pasting the sink of g_1 to the source of g_2.

[2] This notation is used even when some of the c-elements may be equal; this happens especially when the polynomial is obtained by an operation from some given polynomials. In such cases the expression represents the polynomial obtained by joining together all terms with the same c-element. For example, $a_1.c + a_2.d + a_3.c$ represents $(a_1 + a_3).c + a_2.d$ (assuming that $c \neq d$).

Isomorphic graphs are regarded as identical. It should be clear that a series-parallel graph is acyclic. We shall consider the algebra $\mathcal{SP} = (\mathrm{SP}, \Sigma)$, where Σ consists of the two binary symbols $\|$ and $\cdot$ that are interpreted as the above graph operations.

Let $\mathbb{Q}$ be the field of rational numbers and let $\mathrm{Paths} \in \mathbb{Q}\langle\langle \mathrm{SP} \rangle\rangle$ be the $\mathbb{Q}\mathcal{SP}$-series such that for any $g \in \mathrm{SP}$, (Paths, g) is the number of paths from the source to the sink in g. Since every series-parallel graph is acyclic, (Paths, g) is always a well-defined positive integer.

To show that the $\mathbb{Q}\mathcal{SP}$-series Paths is recognizable, we define the $\mathbb{Q}\Sigma$-algebra $\mathcal{M} = (\mathbb{Q}^2, +, 0, \Sigma)$ as follows. Let $e_1 = (1,0)$ and $e_2 = (0,1)$ be the usual base vectors of $\mathbb{Q}^2$. If we define the Σ-operations of $\mathcal{M}$ as the multilinear extensions of the partial operations defined on the base vectors by $e_1\|e_1 = e_1$, $e_1\|e_2 = e_2\|e_1 = e_2$, $e_2\|e_2 = 0$, $e_1 \cdot e_1 = e_1$, $e_1 \cdot e_2 = e_2 \cdot e_1 = 0$, and $e_2 \cdot e_2 = e_2$, where we used the infix notation and omitted the superscript $\mathcal{M}$, we get

$$(a_1, b_1)\|(a_2, b_2) = (a_1a_2, a_1b_2 + a_2b_1) \quad \text{and} \quad (a_1, b_1) \cdot (a_2, b_2) = (a_1a_2, b_1b_2),$$

for all $(a_1, b_1), (a_2, b_2) \in \mathbb{Q}^2$. Let us note that $\mathcal{M}$ is essentially a reduct of the algebra used in Example 4.2 of [1] for evaluating arithmetic expressions.

Next, we show that the mapping

$$\varphi : \mathrm{SP} \to \mathbb{Q}^2,\ g \mapsto e_1 + (\mathrm{Paths}, g)e_2,$$

is a homomorphism $\mathcal{SP} \to \mathcal{M}^\bullet$. For this, consider any $g_1, g_2 \in \mathrm{SP}$ with $n_1 = (\mathrm{Paths}, g_1)$ and $n_2 = (\mathrm{Paths}, g_2)$. Since $(\mathrm{Paths}, g_1\|g_2) = n_1 + n_2$, we get

$$\begin{aligned}
(g_1\|g_2)\varphi &= e_1 + (n_1 + n_2)e_2 \\
&= (e_1\|e_1) + n_1(e_1\|e_2) + n_2(e_2\|e_1) + n_1n_2(e_2\|e_2) \quad (e_2\|e_2 = 0) \\
&= (e_1 + n_1e_2)\|(e_1 + n_2e_2) \quad \text{(by multilinearity)} \\
&= g_1\varphi\,\|g_2\varphi.
\end{aligned}$$

In a similar way, we can show that $(g_1 \cdot g_2)\varphi = g_1\varphi \cdot g_2\varphi$.

If define the linear form $\gamma : \mathbb{Q}^2 \to \mathbb{Q}$ by $\gamma(e_1) = 0$ and $\gamma(e_2) = 1$, then it is clear that $\gamma(g\varphi) = (\mathrm{Paths}, g)$ for every $g \in \mathrm{SP}$, and hence Paths is recognizable. $\square$

A Σ-algebra $\mathcal{C} = (C, \Sigma)$ has the *finite decomposition property (FDP)* if

$$\{(c_1, \ldots, c_k) \in C^k \mid \sigma^{\mathcal{C}}(c_1, \ldots, c_k) = c\}$$

is finite for every choice of $c \in C$, $k > 0$ and $\sigma \in \Sigma_k$. Note that for any X, both the term algebra $\mathcal{T}_\Sigma(X)$ and the free monoid X^* have this property. Also the algebra $\mathcal{SP}$ of Example 4.1 has the FDP.

Let $\mathcal{C} = (C, \Sigma)$ be a Σ-algebra with the FDP. For any $k \geq 0$, $\sigma \in \Sigma_k$ and $S_1, \ldots, S_k \in K\langle\langle C\rangle\rangle$, the $K\mathcal{C}$-series $\sigma(S_1, \ldots, S_k)$ is defined by setting

$$(\sigma(S_1, \ldots, S_k), c) = \sum\{(S_1, c_1) \cdot \ldots \cdot (S_k, c_k) \mid c_1, \ldots, c_k \in C, \sigma^{\mathcal{C}}(c_1, \ldots, c_k) = c\}$$

for every $c \in C$. Note that the sum is always finite because of the FDP. We can now turn the K-vector space $(K\langle\langle C\rangle\rangle, +, \widetilde{0})$ into a $K\Sigma$-algebra $\mathcal{S}_K(\mathcal{C}) = (K\langle\langle C\rangle\rangle, +, \tilde{0}, \Sigma)$ in which for any $k \geq 0$ and $\sigma \in \Sigma_k$ the Σ-operation is defined by

$$\sigma^{\mathcal{S}_K(\mathcal{C})} : K\langle\langle C\rangle\rangle^k \to K\langle\langle C\rangle\rangle,\ (S_1, \ldots, S_k) \mapsto \sigma(S_1, \ldots, S_k).$$

It is straightforward to verify that these operations are multilinear. We call $\mathcal{S}_K(\mathcal{C})$ the *$K\Sigma$-algebra of $K\mathcal{C}$-series.* Clearly, the set $K\langle C\rangle$ is closed under addition, multiplications by scalars and the $\sigma^{\mathcal{S}_K(\mathcal{C})}$-operations, and thus the $K\mathcal{C}$-polynomials form a subalgebra $\mathcal{P}_K(\mathcal{C}) = (K\langle C\rangle, +, \tilde{0}, \Sigma)$ of $\mathcal{S}_K(\mathcal{C})$, the *$K\Sigma$-algebra of $K\mathcal{C}$-polynomials.*

Remark 4.2 Note that $\sigma^{\mathcal{P}_K(\mathcal{C})}(P_1, \ldots, P_k)$ is a well-defined $K\mathcal{C}$-polynomial for any $k \geq 0$, $\sigma \in \Sigma_k$ and $P_1, \ldots, P_k \in K\langle C\rangle$ even if $\mathcal{C}$ does not have the FDP. Moreover, it is easy to see that if $P_i = a_{i1}.c_{i1} + \ldots + a_{im_i}.c_{im_i}$ $(i = 1, \ldots, k)$, then

$$\sigma^{\mathcal{P}_K(\mathcal{C})}(P_1, \ldots, P_k) = \sum\{a_{1j_1} \cdot \ldots \cdot a_{kj_k}.\sigma^{\mathcal{C}}(c_{1j_1}, \ldots, c_{kj_k}) \mid j_1 \in [m_1], \ldots, j_k \in [m_k]\}.$$

Let $\mathcal{C} = (C, \Sigma)$ be an algebra with the FDP. If S is a $K\mathcal{C}$-series and $p \in \mathrm{Tr}(\mathcal{C})$ a translation of $\mathcal{C}$, then the $K\mathcal{C}$-series $p(S)$ is defined by setting

$$(p(S), c) \ = \ \sum\{(S, d) \mid d \in C,\ p(d) = c\},$$

for each $c \in C$. Again the FDP guarantees that the sum is finite. In particular, for a $K\mathcal{C}$-polynomial $P = a_1.c_1 + \ldots + a_m.c_m$, we get $p(P) = a_1.p(c_1) + \ldots + a_m.p(c_m)$, and this is a well-defined polynomial even if the FDP is not assumed. On the other hand, $p^{-1}(S)$ is defined as the $K\mathcal{C}$-series such that $(p^{-1}(S), c) = (S, p(c))$ for every $c \in C$. Again, it is clear that if $\mathcal{C}$ has the FDP, then $p^{-1}(S)$ is a $K\mathcal{C}$-polynomial for every $K\mathcal{C}$-polynomial S. It is also easy to prove the following useful observations.

Lemma 4.3 *Let $\mathcal{C} = (C, \Sigma)$ be a Σ-algebra with the FDP. For any $p, q \in \mathrm{Tr}(\mathcal{C})$, $a \in K$ and $S, T \in K\langle\langle C\rangle\rangle$,*

(a) $p(aS) = ap(S)$, (d) $p^{-1}(aS) = ap^{-1}(S)$, *and*

(b) $p(S + T) = p(S) + p(T)$, (e) $p^{-1}(S + T) = p^{-1}(S) + p^{-1}(T)$.

(c) $p(q)(S) = p(q(S))$,

If S and T are $K\mathcal{C}$-polynomials, these identities hold also in case $\mathcal{C}$ that does not have the FDP.

Lemma 4.4 *Let $\mathcal{C} = (C, \Sigma)$ be a Σ-algebra. For any translation $p \in \mathrm{Tr}(\mathcal{C})$, there is a translation $\widehat{p}$ of $\mathcal{P}_K(\mathcal{C})^\bullet$ such that $\widehat{p}(P) = p(P)$ for every $P \in K\langle C\rangle$. In particular, if $p(\xi) = \sigma^{\mathcal{C}}(c_1, \ldots, \xi, \ldots, c_k)$ is an elementary translation of $\mathcal{C}$, then*

$$\widehat{p} : P \mapsto \sigma^{\mathcal{P}_K(\mathcal{C})}(1.c_1, \ldots, P, \ldots, 1.c_k), \qquad (P \in K\langle C\rangle)$$

is an elementary translation of $\mathcal{P}_K(\mathcal{C})^\bullet$ such that $\widehat{p}(P) = p(P)$ for every $P \in K\langle C\rangle$. On the other hand, any elementary translation of $\mathcal{P}_K(\mathcal{C})$ of the form

$$\Pi(\xi) \;=\; \sigma^{\mathcal{P}_K(\mathcal{C})}(Q_1, \ldots, \xi, \ldots, Q_k) \qquad (k > 0, \sigma \in \Sigma_k, Q_1, \ldots, Q_k \in K\langle C\rangle)$$

is a linear combination of such elementary translations $\widehat{p}$ with $p(\xi) \in \mathrm{ETr}(\mathcal{C})$.

Proof. The first statement follows from the second one, and since this is quite obvious, we prove just the last statement. Assume that ξ occurs in the ith argument in the elementary translation $\Pi(\xi) = \sigma^{\mathcal{P}_K(\mathcal{C})}(Q_1, \ldots, \xi, \ldots, Q_k)$ and let $Q_l = \sum_{j=1}^{m_l} a_{lj}.c_{lj}$ for $1 \le l \le k, l \ne i$. Then for any $P \in K\langle C\rangle$ and $c \in C$, we get

$$\begin{aligned}
&(\sigma^{\mathcal{P}_K(\mathcal{C})}(Q_1, \ldots, P, \ldots, Q_k), c)\\
&= \sum\{(Q_1, c_1) \cdot \ldots \cdot (P, d) \cdot \ldots \cdot (Q_k, c_k) \mid c_l, d \in C\, (l \ne i),\\
&\qquad\qquad\qquad\qquad \sigma^{\mathcal{C}}(c_1, \ldots, d, \ldots, c_k) = c\}\\
&= \sum\{a_{1j_1} \cdot \ldots \cdot (P, d) \cdot \ldots \cdot a_{kj_k} \mid j_l \in [m_l]\,(l \ne i), d \in C,\\
&\qquad\qquad\qquad\qquad \sigma^{\mathcal{C}}(c_{1j_1}, \ldots, d, \ldots, c_{kj_k}) = c\}\\
&= \sum\{a_{1j_1} \cdot \ldots \cdot (P, d) \cdot \ldots \cdot a_{kj_k} \cdot (\sigma(1.c_{1j_1}, \ldots, 1.d, \ldots, 1.c_{kj_k}), c) \mid\\
&\qquad\qquad\qquad\qquad j_l \in [m_l]\,(l \ne i), d \in C\}\\
&= \Big(\sum\{a_{1j_1} \cdot \ldots \cdot (P, d) \cdot \ldots \cdot a_{kj_k}\sigma(1.c_{1j_1}, \ldots, 1.d, \ldots, 1.c_{kj_k}) \mid\\
&\qquad\qquad\qquad\qquad j_l \in [m_l]\,(l \ne i), d \in C\}, c\Big)\\
&= \Big(\sum\{a_{1j_1} \cdot \ldots \cdot (P, d) \cdot \ldots \cdot a_{kj_k}\sigma^{\mathcal{P}_K(\mathcal{C})}(1.c_{1j_1}, \ldots, 1.d, \ldots, 1.c_{kj_k}) \mid\\
&\qquad\qquad\qquad\qquad j_l \in [m_l]\,(l \ne i), d \in C\}, c\Big)\\
&= \Big(\sum\{a_{1j_1} \cdot \ldots \cdot a_{kj_k}\sigma^{\mathcal{P}_K(\mathcal{C})}(1.c_{1j_1}, \ldots, (P, d).d, \ldots, 1.c_{kj_k}) \mid\\
&\qquad\qquad\qquad\qquad j_l \in [m_l]\,(l \ne i), d \in C\}, c\Big)\\
&= \Big(\sum\{a_{1j_1} \cdot \ldots \cdot a_{kj_k}\sigma^{\mathcal{P}_K(\mathcal{C})}(1.c_{1j_1}, \ldots, P, \ldots, 1.c_{kj_k}) \mid j_l \in [m_l]\,(l \ne i)\}, c\Big),
\end{aligned}$$

and hence the original translation can be represented as a linear combination of the required kind. □

Lemma 4.4 means that when dealing with congruences of the polynomial algebra $\mathcal{P}_K(\mathcal{C})$, we may in general operate with translations of the Σ-algebra $\mathcal{C}$ itself. This is expressed also in the following corollary of Lemma 3.3.

Corollary 4.5 *Let $\mathcal{C} = (C, \Sigma)$ be a Σ-algebra. An equivalence $\equiv$ on $K\langle C\rangle$ is a congruence of the $K\Sigma$-algebra $\mathcal{P}_K(\mathcal{C})$ of $K\mathcal{C}$-polynomials, if and only if, for all $P, Q \in K\langle C\rangle$,*

(1) *if $P \equiv Q$, then $p(P) \equiv p(Q)$ for every $p \in \mathrm{ETr}(\mathcal{C})$,*

(2) *if $P \equiv Q$, then $P + R \equiv Q + R$ for every $R \in K\langle C\rangle$, and*

(3) *if $P \equiv Q$, then $aP \equiv aQ$ for every $a \in K$.*

On the other hand, if $\equiv$ is any congruence on $\mathcal{P}_K(\mathcal{C})$, then $P \equiv Q$ implies $p(P) \equiv p(Q)$ for every $p \in \mathrm{Tr}(\mathcal{C})$.

Before considering images and pre-images of series under homomorphisms, let us note the close connection between the homomorphisms defined on a Σ-algebra $\mathcal{C}$ and the homomorphisms of $K\Sigma$-algebras defined on the polynomial algebra $\mathcal{P}_K(\mathcal{C})$.

Lemma 4.6 *Let $\mathcal{C} = (C, \Sigma)$ be a Σ-algebra and $\mathcal{M} = (M, +, 0, \Sigma)$ be a $K\Sigma$-algebra.*

(a) *For any Σ-homomorphism $\varphi : \mathcal{C} \to \mathcal{M}^\bullet$, there exists a unique $K\Sigma$-homomorphism $\overline{\varphi} : \mathcal{P}_K(\mathcal{C}) \to \mathcal{M}$ such that $(1.c)\overline{\varphi} = c\varphi$ for every $c \in C$.*

(b) *If $\eta : \mathcal{P}_K(\mathcal{C}) \to \mathcal{M}$ is a $K\Sigma$-homomorphism, then $\widehat{\eta} : \mathcal{C} \to \mathcal{M}^\bullet$, $c \mapsto (1.c)\eta$, is a Σ-homomorphism.*

(c) *$\widehat{\overline{\varphi}} = \varphi$ for any Σ-homomorphism $\varphi : \mathcal{C} \to \mathcal{M}^\bullet$.*

(d) *$\overline{\widehat{\eta}} = \eta$ for any $K\Sigma$-homomorphism $\eta : \mathcal{P}_K(\mathcal{C}) \to \mathcal{M}$.*

Proof. It clear that in (a) we must set $P\overline{\varphi} = a_1(c_1\varphi) + \ldots + a_m(c_m\varphi)$ for any $P = a_1.c_1 + \ldots + a_m.c_m$ in $K\langle C\rangle$, and that this gives the required homomorphism. Also the remaining statements can be proved by straightforward computations. □

Note also that, when we identify each monomial $1.c$ with the element $c \in C$, then the $\overline{\varphi}$ of Lemma 4.6 (a) becomes the linear extension of φ from C to $K\langle C\rangle$, while the $\widehat{\eta}$ of Lemma 4.6 (b) is the restriction of η to C.

Let us now consider any Σ-algebras $\mathcal{C} = (C, \Sigma)$ and $\mathcal{D} = (D, \Sigma)$, and assume that there is a homomorphism $\varphi : \mathcal{C} \to \mathcal{D}$. Since $\mathcal{D}$ may be regarded as a subalgebra of $\mathcal{P}_K(\mathcal{D})^\bullet$, we may view φ as a homomorphism $\varphi \colon \mathcal{C} \to \mathcal{P}_K(\mathcal{D})^\bullet$. Then its linear extension $\overline{\varphi} \colon \mathcal{P}_K(\mathcal{C}) \to \mathcal{P}_K(\mathcal{D})$ introduced in Lemma 4.6 maps a $K\mathcal{C}$-polynomial $P = a_1.c_1 + \ldots + a_m.c_m$ to the $K\mathcal{D}$-polynomial $a_1.c_1\varphi + \ldots + a_m.c_m\varphi$ because now the operations in the image $P\overline{\varphi}$ are those of the polynomial algebra $\mathcal{P}_K(\mathcal{D})$. Usually, we write just $P\varphi$ for $P\overline{\varphi}$.

Let us say that the homomorphism $\varphi : \mathcal{C} \to \mathcal{D}$ is *locally finite* if $d\varphi^{-1}$ is finite for every $d \in D$. If this is the case, we may define $S\varphi$ for any $S \in K\langle\langle C\rangle\rangle$ by

$$(S\varphi, d) = \sum\{(S,c) \mid c \in d\varphi^{-1}\} \qquad (d \in D).$$

Furthermore, for any $\varphi : \mathcal{C} \to \mathcal{D}$ and any $K\mathcal{D}$-series $S \in K\langle\langle D\rangle\rangle$ the $K\mathcal{C}$-series $S\varphi^{-1}$ is defined by

$$(S\varphi^{-1}, c) = (S, c\varphi) \qquad (c \in C).$$

It is easy to see that if φ is locally finite, then $P\varphi^{-1}$ is a $K\mathcal{C}$-polynomial for every $K\mathcal{D}$-polynomial $P \in K\langle D\rangle$.

Let us note that Lemma 2.1 can be extended to polynomials as follows. We omit the straightforward proof.

Lemma 4.7 *Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a homomorphism of Σ-algebras. Then $p(P)\varphi = p_\varphi(P\varphi)$ for any $K\mathcal{C}$-polynomial P and any translation $p \in \mathrm{Tr}(\mathcal{C})$.*

Moreover, we note the following facts.

Lemma 4.8 *Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a locally finite homomorphism of Σ-algebras. For any $S, S' \in K\langle\langle C\rangle\rangle$, $T, T' \in K\langle\langle D\rangle\rangle$ and $a \in K$,*

(a) $(aS)\varphi = a(S\varphi)$, (c) $(aT)\varphi^{-1} = a(T\varphi^{-1})$, *and*

(b) $(S + S')\varphi = S\varphi + S'\varphi$, (d) $(T + T')\varphi^{-1} = T\varphi^{-1} + T'\varphi^{-1}$.

The linear extension of a $K\mathcal{C}$-series $S : C \to K$ to the K-vector space $(K\langle C\rangle, +, \tilde{0})$ is the linear form $\overline{S} : K\langle C\rangle \to K$ defined by

$$\overline{S}(P) \;=\; \sum_{c \in C}(P,c)\cdot(S,c) \qquad (P \in K\langle C\rangle).$$

For $P = a_1.c_1 + \ldots + a_m.c_m$, this can be written as

$$\overline{S}(P) \;=\; a_1 \cdot (S, c_1) + \ldots + a_m \cdot (S, c_m).$$

In particular, $\overline{S}(1.c) = (S,c)$ for every $c \in C$. Note also that if $(\mathcal{M}, \varphi, \gamma)$ is a representation of S, then $\overline{S}(P) = \gamma(P\overline{\varphi})$ for every $P \in K\langle C\rangle$.

In the following lemma we list some basic properties of these linear forms.

Lemma 4.9 *Let $\mathcal{C}$ and $\mathcal{D}$ be Σ-algebras, and let S be a $K\mathcal{C}$-series.*

(a) $\overline{S}(aP) = a \cdot \overline{S}(P)$ *for any $a \in K$ and $P \in K\langle C\rangle$.*

(b) $\overline{S}(P + Q) = \overline{S}(P) + \overline{S}(Q)$ *for any $P, Q \in K\langle C\rangle$.*

(c) $\overline{S}(p(P)) = \overline{p^{-1}(S)}(P)$ *for any $P \in K\langle C\rangle$ and any translation $p \in \mathrm{Tr}(\mathcal{C})$.*

(d) *If* $\varphi : \mathcal{C} \to \mathcal{D}$ *is a homomorphism of* Σ*-algebras, then* $\overline{T}(P\varphi) = \overline{T\varphi^{-1}}(P)$ *for any* $K\mathcal{D}$*-series* $T \in K\langle\langle D\rangle\rangle$ *and any* $K\mathcal{C}$*-polynomial* $P \in K\langle C\rangle$*.*

Proof. Statements (a) and (b), that just express the fact that $\overline{S}$ is a linear form, are quite obvious. To show (c) and (d), let $P = a_1.c_1 + \ldots + a_m.c_m$. Then

$$\begin{aligned}\overline{S}(p(P)) &= \overline{S}(a_1.p(c_1) + \ldots + a_m.p(c_m))\\ &= a_1 \cdot (S, p(c_1)) + \ldots + a_m \cdot (S, p(c_m))\\ &= a_1 \cdot (p^{-1}(S), c_1) + \ldots + a_m \cdot (p^{-1}(S), c_m)\\ &= \overline{p^{-1}(S)}(P)\end{aligned}$$

proves (c), and similarly

$$\begin{aligned}\overline{T\varphi^{-1}}(P) &= a_1 \cdot (T\varphi^{-1}, c_1) + \ldots + a_m \cdot (T\varphi^{-1}, c_m)\\ &= a_1 \cdot (T, c_1\varphi) + \ldots + a_m \cdot (T, c_m\varphi)\\ &= \overline{T}(a_1.c_1\varphi + \ldots + a_m.c_m\varphi)\\ &= \overline{T}(P\varphi)\end{aligned}$$

proves (d) □

The following facts also have very simple proofs.

Lemma 4.10 *Let* $\mathcal{C} = (C, \Sigma)$ *be a* Σ*-algebra and let* $S, T \in K\langle\langle C\rangle\rangle$ *be any* $K\mathcal{C}$*-series. Then for any* $a \in K$ *and any* $P \in K\langle C\rangle$*,*

(a) $\overline{S+T}(P) = \overline{S}(P) + \overline{T}(P)$ *and* (b) $\overline{aS}(P) = a \cdot \overline{S}(P)$.

5 Syntactic congruences and $K\Sigma$-algebras of $K\mathcal{C}$-series

In this section $\mathcal{C} = (C, \Sigma)$ is again any given Σ-algebra. When applied to string or tree series, the following notion is equivalent to the notion based on syntactic ideals presented in [13], [4] or [3], for example. It allows us to develop for general power series a basic common theory of syntactic congruences and syntactic algebras. As mentioned in the Introduction, some of the notions and facts of this section and the following one also appear in [11] in a somewhat more general setting.

The *syntactic congruence* $\equiv_S$ of a $K\mathcal{C}$-series S is the relation on $K\langle C\rangle$ defined by

$$P \equiv_S Q \;\Leftrightarrow\; (\forall p \in \mathrm{Tr}(\mathcal{C}))\, \overline{S}(p(P)) = \overline{S}(p(Q)) \qquad (P, Q \in K\langle C\rangle).$$

For the sake of convenience, we write simply $[P]_S$ for $[P]_{\equiv_S}$. It is easy to see that

$$\gamma_S : K\langle C\rangle/\!\equiv_S \to K,\ [P]_S \mapsto \overline{S}(P),$$

is a well-defined linear form. We call it the *syntactic linear form* of S.

Example 5.1 Let us compute the syntactic congruence of the series Paths of Example 4.1. First we prove by induction on p that for every $p \in \mathrm{Tr}(\mathcal{SP})$, there are integers $k \geq 1$ and $l \geq 0$ such that $(\mathrm{Paths}, p(g)) = k(\mathrm{Paths}, g) + l$ for every $g \in \mathrm{SP}$.

1. For $p = 1_{\mathrm{SP}}$ we choose $k = 1$ and $l = 0$.
2. If $p = \xi \| g'$ or $p = g' \| \xi$ for some $g' \in \mathrm{SP}$, then $k = 1$ and $l = (\mathrm{Paths}, g')$ satisfy the condition. If $p = \xi \cdot g'$ or $p = g' \cdot \xi$ for some $g' \in \mathrm{SP}$, then we choose $k = (\mathrm{Paths}, g')$ and $l = 0$.
3. Let $p(\xi) = p_1(p_2(\xi))$ be the composition of some $p_1, p_2 \in \mathrm{Tr}(\mathcal{SP})$ and let k_i and l_i be the integers belonging to p_i $(i = 1, 2)$. It is easy to verify that p is defined by $k = k_1 k_2$ and $l = k_1 l_2 + l_1$.

Also the converse holds: for all integers $k \geq 1$ and $l \geq 0$, there is a $p \in \mathrm{Tr}(\mathcal{SP})$ such that $(\mathrm{Paths}, p(g)) = k(\mathrm{Paths}, g) + l$ for every $g \in \mathrm{SP}$. In fact, if $l > 0$, we may choose $p = (\xi \cdot g_1) \| g_2$, where $g_1, g_2 \in \mathrm{SP}$ are such that $(\mathrm{Paths}, g_1) = k$ and $(\mathrm{Paths}, g_2) = l$. If $l = 0$, then we set $p = \xi \cdot g_1$, where $(\mathrm{Paths}, g_1) = k$.

For any polynomials $P = a_1.f_1 + \ldots + a_m.f_m$ and $Q = b_1.g_1 + \ldots + b_n.g_n$ in $\mathbb{Q}\langle \mathrm{SP} \rangle$, $P \equiv_{\mathrm{Paths}} Q$ if and only if, for every $p \in \mathrm{Tr}(\mathcal{SP})$,

$$a_1(\mathrm{Paths}, p(f_1)) + \ldots + a_m(\mathrm{Paths}, p(f_m)) = b_1(\mathrm{Paths}, p(g_1)) + \ldots + b_n(\mathrm{Paths}, p(g_n)).$$

By the above observations, this holds if and only if, for all $k \geq 1$ and $l \geq 0$,

$$a_1(k(\mathrm{Paths}, f_1) + l) + \ldots + a_m(k(\mathrm{Paths}, f_m) + l) = \\ b_1(k(\mathrm{Paths}, g_1) + l) + \ldots + b_n(k(\mathrm{Paths}, g_n) + l),$$

i.e., if and only if, for all $k \geq 1$ and $l \geq 0$,

$$k(a_1(\mathrm{Paths}, f_1) + \ldots + a_m(\mathrm{Paths}, f_m)) + l(a_1 + \ldots + a_m) = \\ k(b_1(\mathrm{Paths}, g_1) + \ldots + b_n(\mathrm{Paths}, g_n)) + l(b_1 + \ldots + b_n).$$

However, this latter condition is equivalent to the pair of equations

$$a_1(\mathrm{Paths}, f_1) + \ldots + a_m(\mathrm{Paths}, f_m) = b_1(\mathrm{Paths}, g_1) + \ldots + b_n(\mathrm{Paths}, g_n)$$

$$a_1 + \ldots + a_m = b_1 + \ldots + b_n.$$

This means that the $\equiv_{\mathrm{Paths}}$-class of P is determined by the two numbers $a_1 + \ldots + a_m$ and $a_1(\mathrm{Paths}, f_1) + \ldots + a_m(\mathrm{Paths}, f_m)$.

On the other hand, for any pair of rational numbers $a, A \in \mathbb{Q}$, there is a $\mathbb{Q}\mathcal{SP}$-polynomial $a_1.f_1 + \ldots + a_m.f_m$ such that $a_1 + \ldots + a_m = a$ and $a_1(\mathrm{Paths}, f_1) + \ldots + a_m(\mathrm{Paths}, f_m) = A$. In fact, we may choose

$$P_{a,A} := (2a - A).\downarrow + (A - a).(\downarrow \| \downarrow)$$

as the canonical representative of the class corresponding to the pair a, A. Then we have $\mathbb{Q}\langle \mathrm{SP}\rangle/{\equiv_{\mathrm{Paths}}} = \{[P_{a,A}]_{\mathrm{Paths}} \mid a, A \in \mathbb{Q}\}$. □

A congruence $\theta \in \mathrm{Con}(\mathcal{P}_K(\mathcal{C}))$ *saturates* a $K\mathcal{C}$-series $S \in K\langle\langle C\rangle\rangle$ if there exists a linear form $\gamma : K\langle C\rangle/\theta \to K$ such that $(S, c) = \gamma([c]_\theta)$ for every $c \in C$ (again we represent $1.c$ by c). Note that then $\overline{S}(P) = \gamma([P]_\theta)$ for every polynomial $P \in K\langle C\rangle$. Note also that if $(\mathcal{M}, \varphi, \gamma)$ is any representation of S, then $\ker \overline{S}$ saturates S.

The following proposition generalizes a result that transpires from [4] or [3]. For the reader's convenience we give here a complete proof.

Proposition 5.2 *Let $\mathcal{C} = (C, \Sigma)$ be a Σ-algebra. For any $K\mathcal{C}$-series $S \in K\langle\langle C\rangle\rangle$, the relation $\equiv_S$ is the greatest congruence on the $K\Sigma$-algebra $\mathcal{P}_K(\mathcal{C})$ of $K\mathcal{C}$-polynomials that saturates S.*

Proof. Obviously $\equiv_S$ is an equivalence relation. That it is a congruence on $\mathcal{P}_K(\mathcal{C})$ follows from Corollary 4.5 and Lemmas 4.3 and 4.9, and it saturates S because, for the syntactic linear form γ_S, we get $\gamma_S([c]_S) = \overline{S}(1.c) = (S, c)$ for every $c \in C$.

Let θ be any congruence on $\mathcal{P}_K(\mathcal{C})$ that saturates S, and let $\gamma : K\langle C\rangle/\theta \to K$ be a linear form such that $(S, c) = \gamma([c]_\theta)$ for every $c \in C$. For any $P, Q \in K\langle C\rangle$,

$$\begin{aligned} P\,\theta\,Q &\Rightarrow (\forall p \in \mathrm{Tr}(\mathcal{C}))\, p(P)\,\theta\,p(Q) \quad \text{(by Corollary 4.5)} \\ &\Rightarrow (\forall p \in \mathrm{Tr}(\mathcal{C}))\, \gamma([p(P)]_\theta) = \gamma([p(Q)]_\theta) \\ &\Rightarrow (\forall p \in \mathrm{Tr}(\mathcal{C}))\, \overline{S}(p(P)) = \overline{S}(p(Q)) \\ &\Rightarrow P \equiv_S Q, \end{aligned}$$

and hence $\theta \subseteq {\equiv_S}$. □

Let us note how some basic operations on series affect the syntactic congruences.

Lemma 5.3 *Let $\mathcal{C} = (C, \Sigma)$ and $\mathcal{D} = (D, \Sigma)$ be Σ-algebras.*

(a) ${\equiv_S} \cap {\equiv_T} \subseteq {\equiv_{S+T}}$ *for all $S, T \in K\langle\langle C\rangle\rangle$.*

(b) ${\equiv_{aS}} = {\equiv_S}$ *for every $S \in K\langle\langle C\rangle\rangle$ and every $a \in K, a \neq 0$.*

(c) ${\equiv_S} \subseteq {\equiv_{p^{-1}(S)}}$ *for every $S \in K\langle\langle C\rangle\rangle$ and every $p \in \mathrm{Tr}(\mathcal{C})$.*

(d) *If $\varphi : \mathcal{C} \to \mathcal{D}$ is a homomorphism of Σ-algebras, then $\varphi \circ {\equiv_S} \circ \varphi^{-1} \subseteq {\equiv_{S\varphi^{-1}}}$ for every $S \in K\langle\langle D\rangle\rangle$. If φ is an epimorphism, then equality holds.*

Proof. Let P and Q be any $K\mathcal{C}$-polynomials.

(a) If $P \equiv_S \cap \equiv_T Q$, then for every translation $p \in \mathrm{Tr}(\mathcal{C})$, both $\overline{S}(p(P)) = \overline{S}(p(Q))$ and $\overline{T}(p(P)) = \overline{T}(p(Q))$ hold, and hence also $\overline{S+T}(p(P)) = \overline{S+T}(p(Q))$. This means that $P \equiv_{S+T} Q$.

(b) For any $a \neq 0$,

$$\begin{aligned} P \equiv_{aS} Q &\Leftrightarrow (\forall p \in \mathrm{Tr}(\mathcal{C}))\, \overline{aS}(p(P)) = \overline{aS}(p(Q)) \\ &\Leftrightarrow (\forall p \in \mathrm{Tr}(\mathcal{C}))\, a \cdot \overline{S}(p(P)) = a \cdot \overline{S}(p(Q)) \quad \text{(Lemma 4.10}(b)) \\ &\Leftrightarrow (\forall p \in \mathrm{Tr}(\mathcal{C}))\, \overline{S}(p(P)) = \overline{S}(p(Q)) \\ &\Leftrightarrow P \equiv_S Q, \end{aligned}$$

(c) For any $p \in \mathrm{Tr}(\mathcal{C})$,

$$\begin{aligned} P \equiv_S Q &\Leftrightarrow (\forall q \in \mathrm{Tr}(\mathcal{C}))\, \overline{S}(q(P)) = \overline{S}(q(Q)) \\ &\Rightarrow (\forall q \in \mathrm{Tr}(\mathcal{C}))\, \overline{S}(p(q(P))) = \overline{S}(p(q(Q))) \\ &\Leftrightarrow (\forall q \in \mathrm{Tr}(\mathcal{C}))\, \overline{p^{-1}(S)}(q(P)) = \overline{p^{-1}(S)}(q(Q)) \quad \text{(Lemma 4.9}(c)) \\ &\Leftrightarrow P \equiv_{p^{-1}(S)} Q. \end{aligned}$$

(d) Now

$$\begin{aligned} P\, \varphi \circ \equiv_S \circ \varphi^{-1}\, Q &\Leftrightarrow P\varphi \equiv_S Q\varphi \\ &\Leftrightarrow (\forall q \in \mathrm{Tr}(\mathcal{D}))\, \overline{S}(q(P\varphi)) = \overline{S}(q(Q\varphi)) \\ &\Rightarrow (\forall p \in \mathrm{Tr}(\mathcal{C}))\, \overline{S}(p_\varphi(P\varphi)) = \overline{S}(p_\varphi(Q\varphi)) \quad \text{(Lemma 2.1)} \\ &\Leftrightarrow (\forall p \in \mathrm{Tr}(\mathcal{C}))\, \overline{S}(p(P)\varphi) = \overline{S}(p(Q)\varphi) \quad \text{(Lemma 4.7)} \\ &\Leftrightarrow (\forall p \in \mathrm{Tr}(\mathcal{C}))\, \overline{S\varphi^{-1}}(p(P)) = \overline{S\varphi^{-1}}(p(Q)) \quad \text{(Lemma 4.9}(d)) \\ &\Leftrightarrow P \equiv_{S\varphi^{-1}} Q, \end{aligned}$$

and if φ is surjective, then the only "$\Rightarrow$" also becomes an equivalence sign (again by Lemma 2.1). □

Again, let $\mathcal{C} = (C, \Sigma)$ be any Σ-algebra. The *syntactic $K\Sigma$-algebra* of a $K\mathcal{C}$-series $S \in K\langle\langle C \rangle\rangle$ is defined as the quotient $K\Sigma$-algebra $\mathrm{SA}(S) := \mathcal{P}_K(\mathcal{C})/\equiv_S$. Depending on the context, $\mathrm{SA}(S)$ may also denote either the K-vector space $(K\langle C \rangle/\equiv_S, +, [\widetilde{0}])$ or just the set $K\langle C \rangle/\equiv_S$. The natural homomorphism

$$\nu_S : \mathcal{P}_K(\mathcal{C}) \to \mathrm{SA}(S),\ P \mapsto [P]_S,$$

is called the *syntactic homomorphism* of S. Obviously, we get $\overline{S}$ as the composition of ν_S and γ_S as illustrated by Fig. 1.

$$
\begin{tikzcd}
\mathcal{P}_K(\mathcal{C}) \arrow[dr, "\nu_S"] \arrow[dd, "\overline{S}"'] & \\
& \mathrm{SA}(S) \arrow[dl, "\gamma_S"] \\
K &
\end{tikzcd}
$$

Figure 1: $\overline{S}$ as the composition of ν_S and γ_S.

Example 5.4 Let us determine the syntactic $\mathbb{Q}\Sigma$-algebra of the $\mathbb{Q}\mathcal{SP}$-series defined in Example 4.1. In what follows, we write simply $[P]$ for $[P]_{\mathrm{Paths}}$. In Example 5.1 we already showed that the set of elements of SA(Paths) is $\{[P_{a,A}] \mid a, A \in \mathbb{Q}\}$, where $P_{a,A} = (2a - A).\downarrow + (A - a).(\downarrow \parallel \downarrow)$ for each pair $a, A \in \mathbb{Q}$. By using the characterization of the syntactic congruence $\equiv_{\mathrm{Paths}}$ given in Example 5.1, it is easy to verify that for all $a, b, A, B \in \mathbb{Q}$,

(1) $[P_{a,A}] + [P_{b,B}] = [P_{a+b,A+B}]$,

(2) $b[P_{a,A}] = [P_{ba,bA}]$,

(3) $[P_{a,A}] \| [P_{b,B}] = [P_{ab,aB+Ab}]$, and

(4) $[P_{a,A}] \cdot [P_{b,B}] = [P_{ab,AB}]$,

and this also shows that $\eta : \mathrm{SA(Paths)} \to \mathbb{Q}^2$, $[P_{a,A}] \mapsto (a, A)$, is an isomorphism between SA(Paths) and the 2-dimensional $\mathbb{Q}\Sigma$-algebra $\mathcal{M}$ of Example 4.1. $\square$

The following facts correspond to those presented in Lemma 5.3.

Proposition 5.5 *Let $\mathcal{C} = (C, \Sigma)$ and $\mathcal{D} = (D, \Sigma)$ be Σ-algebras.*

(a) $\mathrm{SA}(S + T) \preceq \mathrm{SA}(S) \times \mathrm{SA}(T)$ *for all* $S, T \in K\langle\langle C\rangle\rangle$.

(b) $\mathrm{SA}(aS) = \mathrm{SA}(S)$ *for every* $S \in K\langle\langle C\rangle\rangle$ *and every* $a \in K, a \neq 0$.

(c) $\mathrm{SA}(p^{-1}(S)) \twoheadleftarrow \mathrm{SA}(S)$ *for all* $S \in K\langle\langle C\rangle\rangle$ *and* $p \in \mathrm{Tr}(\mathcal{C})$.

(d) *If* $\varphi : \mathcal{C} \to \mathcal{D}$ *is any homomorphism, then* $\mathrm{SA}(S\varphi^{-1}) \preceq \mathrm{SA}(S)$ *for every* $S \in K\langle\langle D\rangle\rangle$. *If* φ *is an epimorphism, then* $\mathrm{SA}(S\varphi^{-1}) \cong \mathrm{SA}(S)$.

Proof. The first three assertions follow directly from the corresponding statements in Lemma 5.3. To prove (d), let us first assume that $\varphi : \mathcal{C} \to \mathcal{D}$ is an epimorphism. We shall now show that

$$\psi : \mathrm{SA}(S\varphi^{-1}) \to \mathrm{SA}(S),\ [P]_{S\varphi^{-1}} \mapsto [P\varphi]_S,$$

$$\begin{tikzcd} \mathcal{P}_K(\mathcal{C}) \arrow[r, "\overline{\varphi}"] \arrow[d, "\nu_{S\varphi^{-1}}"'] & \mathcal{P}_K(\mathcal{D}) \arrow[d, "\nu_S"] \\ \mathrm{SA}(S\varphi^{-1}) \arrow[r, "\psi"'] & \mathrm{SA}(S) \end{tikzcd}$$

Figure 2: The isomorphism ψ in Proposition 5.5 (d).

is an isomorphism of $K\Sigma$-algebras. The definition of ψ is illustrated by Fig. 2. First of all, ψ is well-defined and injective. Indeed, for any $P, Q \in K\langle C\rangle$,

$$\begin{aligned} [P]_{S\varphi^{-1}}\psi = [Q]_{S\varphi^{-1}}\psi \; &\Leftrightarrow \; P\varphi \equiv_S Q\varphi \\ &\Leftrightarrow \; P \equiv_{S\varphi^{-1}} Q \qquad \text{(Lemma 5.3 (d))} \\ &\Leftrightarrow \; [P]_{S\varphi^{-1}} = [Q]_{S\varphi^{-1}}. \end{aligned}$$

Since φ is surjective, so is ψ, and straightforward computations show that it preserves the operations of $K\Sigma$–algebras. For example, if $\sigma \in \Sigma_k$ and $P_1, \ldots, P_k \in K\langle C\rangle$, then

$$\begin{aligned} \sigma^{\mathrm{SA}(S\varphi^{-1})}([P_1]_{S\varphi^{-1}}, \ldots, [P_k]_{S\varphi^{-1}})\psi &= [\sigma^{\mathcal{P}_K(\mathcal{C})}(P_1, \ldots, P_k)]_{S\varphi^{-1}}\psi \\ &= [\sigma^{\mathcal{P}_K(\mathcal{C})}(P_1, \ldots, P_k)\varphi]_S \\ &= [\sigma^{\mathcal{P}_K(\mathcal{D})}(P_1\varphi, \ldots, P_k\varphi)]_S \\ &= \sigma^{\mathrm{SA}(S)}([P_1\varphi]_S, \ldots, [P_k\varphi]_S) \\ &= \sigma^{\mathrm{SA}(S)}([P_1]_{S\varphi^{-1}}\psi, \ldots, [P_k]_{S\varphi^{-1}}\psi). \end{aligned}$$

Consider now the general case in which $\varphi : \mathcal{C} \to \mathcal{D}$ is not necessarily an epimorphism. Let $\mathcal{E} = (E, \Sigma)$ be the subalgebra of $\mathcal{D}$ obtained as the image of $\mathcal{C}$ under φ. Furthermore, let $T \in K\langle\langle E\rangle\rangle$ be the restriction of S to E, and let $\psi : \mathcal{C} \to \mathcal{E}$, $c \mapsto c\varphi$, be the epimorphism from $\mathcal{C}$ onto $\mathcal{E}$ obtained from φ by restricting its range to $E\,(= C\varphi)$. Then it is clear that $S\varphi^{-1} = T\psi^{-1}$. Furthermore, every translation $p \in \mathrm{Tr}(\mathcal{E})$ of $\mathcal{E}$ is the restriction of some translation $p' \in \mathrm{Tr}(\mathcal{D})$ of $\mathcal{D}$, and every $K\mathcal{E}$-polynomial $P \in K\langle E\rangle$ can also be regarded as a $K\mathcal{D}$-polynomial, and it is easy to see that then $\overline{T}(p(P)) = \overline{S}(p'(P))$. From this it follows that if $P \equiv_S Q$ for some $P, Q \in K\langle E\rangle$, then also $P \equiv_T Q$. This means that the restriction θ of $\equiv_S$ to $K\langle E\rangle$ is contained in $\equiv_T$, and therefore $\mathrm{SA}(T) = \mathcal{P}(\mathcal{E})/{\equiv_T}$ is an image of $\mathcal{P}(\mathcal{E})/\theta$. Since $\mathcal{P}(\mathcal{E})/\theta$ is covered by $\mathrm{SA}(S)$, it follows from the first part of the proof that

$$\mathrm{SA}(S\varphi^{-1}) = \mathrm{SA}(T\psi^{-1}) \cong \mathrm{SA}(T) \preceq \mathrm{SA}(S),$$

and hence also the general part of (d) holds. $\square$

Proposition 5.6 *Let $S \in K\langle\langle C\rangle\rangle$ be a $K\mathcal{C}$-series for some Σ-algebra $\mathcal{C} = (C, \Sigma)$.*

(a) *The syntactic algebra* $\mathrm{SA}(S)$ *recognizes* S.

(b) *If a $K\Sigma$-algebra $\mathcal{M}$ recognizes S, then* $\mathrm{SA}(S) \preceq \mathcal{M}$. *In particular, if there is a representation $(\mathcal{M}, \varphi, \gamma)$ of S such that the linear extension $\overline{\varphi}\colon \mathcal{P}_K(\mathcal{C}) \to \mathcal{M}$ of φ is an epimorphism, then* $\mathrm{SA}(S) \twoheadleftarrow \mathcal{M}$ *and, more precisely, there exists an epimorphism $\eta : \mathcal{M} \to \mathrm{SA}(S)$ of $K\Sigma$-algebras such that $\overline{\varphi}\eta = \nu_S$ and $\eta\gamma_S = \gamma$.*

Proof. If we restrict the syntactic homomorphism ν_S to C, we obtain the homomorphism $\varphi : \mathcal{C} \to \mathrm{SA}(S)^\bullet$, $c \mapsto (1.c)\nu_S$, of Σ-algebras. For the syntactic linear form γ_S, we get $\gamma_S(c\varphi) = \gamma_S([c]_S) = (S, c)$ for every $c \in C$, and hence $\mathrm{SA}(S)$ recognizes S.

To prove (b), let $\mathcal{M} = (M, +, 0, \Sigma)$ be a $K\Sigma$-algebra that recognizes S. First consider the special case, where there is a representation $(\mathcal{M}, \varphi, \gamma)$ of S such that the linear extension $\overline{\varphi}\colon \mathcal{P}_K(\mathcal{C}) \to \mathcal{M}$ of φ is an epimorphism

For any $u \in M$, let $u\eta = [P]_S$, where P is any $K\mathcal{C}$-polynomial such that $P\overline{\varphi} = u$. Let us verify that this yields the required epimorphism $\eta : \mathcal{M} \to \mathrm{SA}(S)$.

If $P\overline{\varphi} = Q\overline{\varphi}$ for some $P, Q \in K\langle C\rangle$, then for every $p \in \mathrm{Tr}(\mathcal{C})$, $p(P)\overline{\varphi} = p_\varphi(P\overline{\varphi}) = p_\varphi(Q\overline{\varphi}) = p(Q)\overline{\varphi}$, and hence $\overline{S}(p(P)) = \gamma(p(P)\overline{\varphi}) = \gamma(p(Q)\overline{\varphi}) = \overline{S}(p(Q))$. This means that $[P]_S = [Q]_S$, and hence η is well-defined. Clearly, η is surjective.

Let us show that η is a homomorphism. If $u = P\overline{\varphi}$ and $v = Q\overline{\varphi}$ with $P, Q \in K\langle C\rangle$ and $a \in K$, then $(u+v)\eta = [P+Q]_S = [P]_S + [Q]_S = u\eta + v\eta$, and $(au)\eta = [aP]_S = a[P]_S = a(u\eta)$. Let $k \geq 0$, $\sigma \in \Sigma_k$ and $u_1, \ldots, u_k \in M$. If $P_1, \ldots, P_k \in K\langle C\rangle$ are such that $P_1\overline{\varphi} = u_1, \ldots, P_k\overline{\varphi} = u_k$, then $\sigma(P_1, \ldots, P_k)\overline{\varphi} = \sigma^{\mathcal{M}}(P_1\overline{\varphi}, \ldots, P_k\overline{\varphi})$, and hence (omitting S as a subscript)

$$\sigma^{\mathcal{M}}(u_1, \ldots, u_k)\eta = [\sigma(P_1, \ldots, P_k)] = \sigma^{\mathrm{SA}(S)}([P_1], \ldots, [P_k]) = \sigma^{\mathrm{SA}(S)}(u_1\eta, \ldots, u_k\eta).$$

For all $P \in K\langle C\rangle$, $P\overline{\varphi}\eta = [P]_S = P\nu_S$, i.e., $\overline{\varphi}\eta = \nu_S$. Similarly, if $u = P\overline{\varphi} \in M$, then $u\eta\gamma_S = P\overline{\varphi}\eta\gamma_S = \gamma_S([P]_S) = \overline{S}(P) = \gamma(P\overline{\varphi}) = \gamma(u)$, and hence $\eta\gamma_S = \gamma$.

The first statement of part (b) now follows easily when we first replace $\mathcal{M}$ with its subalgebra $\mathcal{N} = (K\langle C\rangle\overline{\varphi}, +, 0, \Sigma)$ obtained as the image of $\mathcal{P}_K(\mathcal{C})$ under $\overline{\varphi}$. □

Now we get the following generalization of the corresponding facts about string series (cf. [13]) and tree series (cf. [4, 3, 11]).

Corollary 5.7 *Let $\mathcal{C}$ be any Σ-algebra. A $K\mathcal{C}$-series S is recognizable iff* $\mathrm{SA}(S)$ *is finite-dimensional.*

Proof. If $\mathrm{SA}(S)$ is finite-dimensional, then $S \in \mathrm{Rec}_K(\mathcal{C})$ by Proposition 5.6 (a). On the other hand, if S is recognized by a finite-dimensional $K\Sigma$-algebra $\mathcal{M}$, then by Proposition 5.6 (b), the dimension of $\mathrm{SA}(S)$ is at most equal to that of $\mathcal{M}$. □

Proposition 5.8 *Let $\mathcal{M} = (M, +, 0, \Sigma)$ and $\mathcal{N} = (N, +, 0, \Sigma)$ be finite-dimensional $K\Sigma$-algebras, and let $\mathcal{C} = (C, \Sigma)$ be a Σ-algebra freely generated over some variety of Σ-algebras into which $\mathcal{N}^\bullet$ belongs. If $\mathcal{M} \preceq \mathcal{N}$ and $\mathcal{M}$ recognizes some $K\mathcal{C}$-series S, then also $\mathcal{N}$ recognizes S.*

Proof. Let $\varphi\colon \mathcal{C} \to \mathcal{M}^\bullet$ be a homomorphism and $\gamma\colon M \to K$ be a linear form such that $(S, c) = \gamma(c\varphi)$ for every $c \in C$. First we prove the assertion in two special cases.

Case 1. Assume that $\mathcal{M}$ is a subalgebra of $\mathcal{N}$. Then we may view φ also as a homomorphism $\varphi\colon \mathcal{C} \to \mathcal{N}^\bullet$ and γ can be extended to a linear form $\gamma'\colon N \to K$. It is clear that $(\mathcal{N}, \varphi, \gamma')$ is a representation of S.

Case 2. Assume now that there exists an epimorphism $\eta\colon \mathcal{N} \to \mathcal{M}$ of $K\Sigma$-algebras. Then $\widetilde{\eta}\colon \mathcal{N}^\bullet \to \mathcal{M}^\bullet$, $u \mapsto u\eta$, is an epimorphism of Σ-algebras, and by our assumption concerning $\mathcal{C}$ and $\mathcal{N}^\bullet$, there exists a homomorphism $\psi\colon \mathcal{C} \to \mathcal{N}^\bullet$ such that $\psi\widetilde{\eta} = \varphi$. On the other hand, $\beta := \eta\gamma$ is a linear form of N, and it is easy to see that $(\mathcal{N}, \psi, \beta)$ is a representation of S.

The general assertion of the proposition follows from these two special cases. Indeed, if $\mathcal{M} \preceq \mathcal{N}$, then $\mathcal{M}$ is an image of some subalgebra $\mathcal{N}_1$ of $\mathcal{N}$. Also $\mathcal{N}_1^\bullet$ is in the variety mentioned in the proposition, and hence $\mathcal{N}_1$ recognizes S by Case 2, and therefore $\mathcal{N}$ recognizes S by Case 1. $\square$

Note that the following corollary applies, in particular, to the cases where $\mathcal{C}$ is the free monoid X^* or the term algebra $\mathcal{T}_\Sigma(X)$ for some alphabet X.

Corollary 5.9 *Let $\mathcal{C}$ be a Σ-algebra freely generated over some variety $\mathbf{V}$ of Σ-algebras, and let $S \in \mathrm{Rec}_K(\mathcal{C})$. If $\mathcal{M}$ is a finite-dimensional $K\Sigma$-algebra such that $\mathrm{SA}(S) \preceq \mathcal{M}$ and $\mathcal{M}^\bullet \in \mathbf{V}$, then $\mathcal{M}$ recognizes S.*

Let us call a $K\Sigma$-algebra *syntactic* if it is isomorphic to the syntactic $K\Sigma$-algebra of some $\mathcal{C}$-series for some Σ-algebra $\mathcal{C}$.

Note that since γ_S is a homomorphism of K-vector spaces, $\ker \gamma_S$ is a congruence on the K-vector space $(\mathrm{SA}(S), +, [\widetilde{0}]_S)$, but in general it is not a congruence on the $K\Sigma$-algebra $\mathrm{SA}(S)$.

Lemma 5.10 *Let $\mathcal{C} = (C, \Sigma)$ be a Σ-algebra and let $S \in K\langle\langle C\rangle\rangle$. Then there is no non-trivial congruence θ on the $K\Sigma$-algebra $\mathrm{SA}(S)$ such that $\theta \subseteq \ker \gamma_S$.*

Proof. If $\theta \in \mathrm{Con}(\mathrm{SA}(S))$ and $\theta \subseteq \ker \gamma_S$, then $\nu_S \circ \theta \circ \nu_S^{-1} \subseteq \nu_S \circ \ker \gamma_S \circ \nu_S^{-1} = \ker \overline{S}$.

Hence, for any $P, Q \in K\langle C\rangle$,

$$\begin{aligned}
[P]_S\,\theta\,[Q]_S \quad &\Rightarrow \quad (\forall q \in \mathrm{Tr}(\mathrm{SA}(S)^\bullet))\, q([P]_S)\,\theta\, q([Q]_S)\\
&\Rightarrow \quad (\forall p \in \mathrm{Tr}(\mathcal{P}_K(\mathcal{C})^\bullet))\, p_{\nu_S}(P\nu_S)\,\theta\, p_{\nu_S}(Q\nu_S)\\
&\Rightarrow \quad (\forall p \in \mathrm{Tr}(\mathcal{P}_K(\mathcal{C})^\bullet))\, p(P)\nu_S\,\theta\, p(Q)\nu_S\\
&\Rightarrow \quad (\forall p \in \mathrm{Tr}(\mathcal{C}))\, \widehat{p}(P)\nu_S\,\theta\, \widehat{p}(Q)\nu_S\\
&\Rightarrow \quad (\forall p \in \mathrm{Tr}(\mathcal{C}))\, \overline{S}(p(P)) = \overline{S}(p(Q))\\
&\Rightarrow \quad [P]_S = [Q]_S,
\end{aligned}$$

where we also made use of Lemmas 2.1 and 4.4. Hence θ is the diagonal relation. □

We will also need the following technical lemma.

Lemma 5.11 *Let $\mathcal{M} = (M, +, 0, \Sigma)$ be a $K\Sigma$-algebra. If $\gamma : M \to K$ is any linear form, then the relation θ_γ on M defined by the condition*

$$u\,\theta_\gamma\, v \;\Leftrightarrow\; (\forall p \in \mathrm{Tr}(\mathcal{M}^\bullet))\, \gamma(p(u)) = \gamma(p(v)) \quad (u, v \in M),$$

is the greatest congruence θ on $\mathcal{M}$ such that $\theta \subseteq \ker\gamma$.

Proof. It is clear that θ_γ is an equivalence on M such that $\theta_\gamma \subseteq \ker\gamma$. It is a congruence of the Σ-algebra $\mathcal{M}^\bullet$ since $u\,\theta_\gamma\, v$ implies that $\gamma(p(q(u))) = \gamma(p(q(v)))$ for all $p, q \in \mathrm{Tr}(\mathcal{M}^\bullet)$. It is also a congruence on the K-vector space $(M, +, 0)$ because γ and all the translations $p \in \mathrm{Tr}(\mathcal{M}^\bullet)$ are linear.

If θ is a congruence on $\mathcal{M}$ such that $\theta \subseteq \ker\gamma$, then for any $u, v \in M$,

$$u\,\theta\, v \;\Rightarrow\; (\forall p \in \mathrm{Tr}(\mathcal{M}^\bullet))\, p(u)\,\theta\, p(v) \;\Rightarrow\; (\forall p \in \mathrm{Tr}(\mathcal{M}^\bullet))\, \gamma(p(u)) = \gamma(p(v)) \;\Rightarrow\; u\,\theta_\gamma\, v,$$

i.e., $\theta \subseteq \theta_\gamma$. □

Proposition 5.12 *A $K\Sigma$-algebra $\mathcal{M} = (M, +, 0, \Sigma)$ is syntactic if and only if there exists a linear form $\gamma : M \to K$ such that $\ker\gamma$ does not contain any non-trivial congruences on the $K\Sigma$-algebra $\mathcal{M}$.*

Proof. Let $\mathcal{C} = (C, \Sigma)$ be any Σ-algebra and $S \in K\langle\langle C\rangle\rangle$ be any $\mathcal{C}$-series. By Lemma 5.10, $\gamma_S : \mathrm{SA}(S) \to K$ is a linear form for which there is no non-trivial congruence θ on $\mathrm{SA}(S)$ such that $\theta \subseteq \ker\gamma_S$.

Assume now that $\mathcal{M} = (M, +, 0, \Sigma)$ is a $K\Sigma$-algebra for which there exists a linear form $\gamma : M \to K$ such that $\ker\gamma$ does not contain any non-trivial congruence on $\mathcal{M}$. Then, in particular, $\theta_\gamma = \Delta_M$. Let S be the $K\mathcal{M}^\bullet$-series such that $(S, u) = \gamma(u)$ for every $u \in M$ (i.e., as mappings S and γ are equal). It is easy to verify that

$$\eta\colon \mathcal{P}_K(\mathcal{M}^\bullet) \to \mathcal{M},\; a_1.u_1 + \ldots + a_m.u_m \mapsto a_1u_1 + \ldots + a_mu_m,$$

is a $K\Sigma$-epimorphism such that $\overline{S}(P) = \gamma(P\eta)$ for every $P \in K\langle C\rangle$. Now, for any $P, Q \in K\langle M\rangle$,

$$\begin{aligned} P \equiv_S Q \quad &\Leftrightarrow \quad (\forall p \in \mathrm{Tr}(\mathcal{M}^\bullet))\, \overline{S}(p(P)) = \overline{S}(p(Q)) \\ &\Leftrightarrow \quad (\forall p \in \mathrm{Tr}(\mathcal{M}^\bullet))\, \gamma(\widehat{p}(P)\eta) = \gamma(\widehat{p}(Q)\eta) \\ &\Leftrightarrow \quad (\forall p \in \mathrm{Tr}(\mathcal{M}^\bullet))\, \gamma(\widehat{p}_\eta(P\eta)) = \gamma(\widehat{p}_\eta(Q\eta)) \\ &\Leftrightarrow \quad (\forall q \in \mathrm{Tr}(\mathcal{M}^\bullet))\, \gamma(q(P\eta)) = \gamma(q(Q\eta)) \\ &\Leftrightarrow \quad P\eta\, \theta_\gamma\, Q\eta \\ &\Leftrightarrow \quad P \ \ker \eta\, Q, \end{aligned}$$

where we used Lemmas 2.1, 4.4 and 5.11 as well as the fact that $\theta_\gamma = \Delta_M$. Hence,

$$\mathcal{M} \cong \mathcal{P}_K(\mathcal{M}^\bullet)/\ker\eta = \mathcal{P}_K(\mathcal{M}^\bullet)/{\equiv_S} = \mathrm{SA}(S),$$

which shows that $\mathcal{M}$ is syntactic. □

As a consequence of the previous proposition, we get the following useful result.

Proposition 5.13 *Every finite-dimensional subdirectly irreducible $K\Sigma$-algebra is syntactic.*

Proof. Let $\mathcal{M} = (M, +, 0, \Sigma)$ be a finite-dimensional subdirectly irreducible $K\Sigma$-algebra, and let $\{u_1, \ldots, u_n\}$ be a basis of $(M, +, 0)$. If $n = 0$, then $\mathcal{M}$ is the trivial $K\Sigma$-algebra and thus isomorphic to $\mathrm{SA}(\tilde{0})$ (for any choice of $\mathcal{C}$).

Let us now assume that $n \geq 1$. For each $i \in [n]$, we define $\gamma_i : M \to K$ by the condition that if $u = a_1u_1 + \ldots + a_nu_n$ is the representation of an element $u \in M$ as the sum the base elements, then $\gamma_i(u) = a_i$. Obviously, each γ_i is a linear form and $\ker\gamma_1 \cap \ldots \cap \ker\gamma_n = \Delta_M$. If for every $i \in [n]$, there is a non-trivial congruence θ_i such that $\theta_i \subseteq \ker\gamma_i$, then we would have $\theta_1 \cap \ldots \cap \theta_n = \Delta_M$ contradicting the assumption that $\mathcal{M}$ is subdirectly irreducible. Therefore at least one of the linear forms γ_i satisfies the condition of Proposition 5.12 and $\mathcal{M}$ is syntactic. □

6 Concluding remarks

We have defined and studied syntactic congruences and syntactic $K\Sigma$-algebras of $K\mathcal{C}$-series, where $\mathcal{C}$ is a general Σ-algebra. As a preparation, we established several facts about such series and various operations on them, and we studied the $K\Sigma$-algebra of $K\mathcal{C}$-polynomials in some detail. Much of the work done aims directly at a variety theory, and in a forthcoming paper we will use many of these results to prove a variety theorem for tree series. Whether this can be further generalized to something corresponding to the theory of varieties of recognizable subsets of free algebras in a variety presented in [15] (cf. also [17]), remains to be seen. It would also be interesting to see whether this work can be extended to series over something more general than a field, for example, along the line suggested by Mathissen [11].

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