Stochastic Relaxation, Gibbs distributions, and the Bayesian Restoration of Images

Stuart Geman† and Donald Geman‡

† Division of Applied Mathematics, Brown University, Providence

‡ Department of Mathematics and Statistics, University of Massachusetts

Slides: Lajos Rodek, 2003

Contents

- Basic idea
- *A priori* image model
- Degraded image model
- Graphs and neighborhoods
- Markov random fields
- Gibbs distribution
- Restoration algorithm
- Stochastic relaxation
- Gibbs sampler
- Results

Basic Idea

- Consider images as a statistical mechanics system ⇒ intensity levels become states of atoms or molecules
- Assign an energy function to the system ⇒ Gibbs distribution ⇒ Markov random field (MRF) image model
- Restoration in a Bayesian framework ⇒ objective is to maximize the posterior distribution
- Isolate low energy states of the system by annealing ⇒ these correspond to maximum *a posteriori* (MAP) estimates of the original image

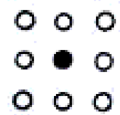
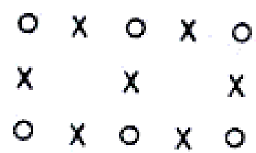
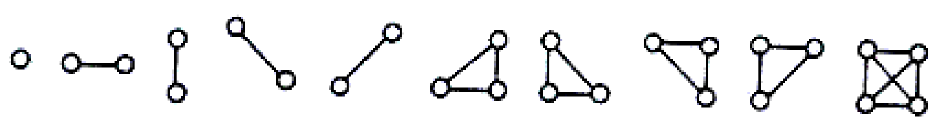
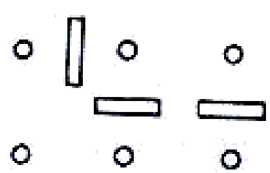
A Priori Image Model

- Hierarchical (layered stochastic processes for various image attributes)
- Image model: $\mathbf{X} = (\mathbf{F}, \mathbf{L})$, where
  - F: $m \times m$ matrix of pixel intensities (intensity process)
  - L: dual matrix of edge elements (line process)
- *A priori* knowledge is captured by distribution $P(\mathbf{X} = \boldsymbol{x})$

Degraded Image Model

- Degraded images: (G, L) with $G = \phi(H(F)) \odot N$, where
  - H: blurring
  - ϕ: nonlinear distorsion
  - $\odot$: invertible operation (e.g. addition, multiplication)
  - N: noise (Gaussian, with mean μ and std. dev. σ)
- Requirement: F and N (also L and N) are independent

Graphs and Neighborhoods

- $S = Z_m$: set of sites for F, the $m \times m$ integer lattice
- $S = D_m$: set of sites for L, the dual lattice
- $S = Z_m \cup D_m$: set of sites for $(\mathbf{F}, \mathbf{L})$
- $\mathcal{G} = \{\mathcal{G}_s, s \in S\}$: neighborhood system for S
- 8-neighborhood for F: 
- 6-neighborhood for L:  (X: line site)
- Cliques for F: 
- Cliques for L: e.g. 

Markov Random Fields

- Ω : set of all possible configurations
- Set of random variables $X = \{X_s,\ s \in S\}$ is an MRF w.r.t. neighborhood system $\mathcal{G}$ if
  1. $P(X = \omega) > 0$ for all $\omega \in \Omega$, and
  2. $P(X_s = x_s | X_r = x_r, r \neq s) = P(X_s = x_s | X_r = x_r, r \in \mathcal{G}_s)$
- Major drawback: it is too difficult to embed local characteristics
- X is an MRF $\Leftrightarrow$ $P(X = \omega)$ is a Gibbs distribution

Gibbs Distribution

- Originates in statistical mechanics
- Defined as $\pi(\omega) = \frac{1}{Z} e^{-U(\omega)/T}$, where

  - $U(\omega) = \sum_{C \in \mathcal{C}} V_C(\omega)$: energy function
  - $\mathcal{C}$: set of cliques
  - $V_C(\omega)$: clique potentials
  - $Z \doteq \sum_{\omega} e^{-U(\omega)/T}$: normalizing constant (partition function)
  - T: temperature
- Major advantage: an easy way to specify MRF's using potentials

Restoration Algorithm

- Bayesian framework
- Objective function is the posterior distribution

$$P(X=\omega \mid G=g)=\frac{P(G=g \mid X=\omega)\,P(X=\omega)}{P(G=g)}$$

- $P(X=\omega \mid G=g)$ and $P(X=\omega)$ are Gibbsian
- The distribution of G need not be known
- Maximize the objective function to get a maximum *a posteriori* (MAP) estimate of the original image
- $|\Omega|$ is huge $\Rightarrow$ optimization is based on stochastic relaxation, Gibbs sampler and simulated annealing

Stochastic Relaxation

- A nondeterministic, stochastic, iterative algorithm to find one of the lowest energy states of a mechanics system
- Based on Boltzmann distribution: $\pi(\omega) = e^{-\beta \mathcal{E}(\omega)} \Big/ \sum_{\omega} e^{-\beta \mathcal{E}(\omega)}$ where
  - $\mathcal{E}(\omega)$: potential energy
  - $\beta = 1/KT$ with K denoting Boltzmann's constant
- At iteration t a new configuration $X(t+1)$ is generated from $X(t)$ as follows:
  1. Choose a random configuration η
  2. η is accepted with probability $q = \dfrac{\pi(\eta)}{\pi(X(t))} = e^{-\beta \Delta \mathcal{E}}$, where $\Delta \mathcal{E} = \mathcal{E}(\eta) - \mathcal{E}(X(t))$
  3. $X(t+1)$ is set to $X(t)$ or η depending on q

Gibbs Sampler I.

- A stochastic relaxation algorithm, which generates new configurations from a given Gibbs distribution $\pi(\omega)$
- Used
  - to generate samples from $\pi(\omega)$, and
  - to minimize the objective function
- Supports parallel computation
- At iteration *t* the new configuration $X(t+1)$ is chosen from $\pi(\omega)$ using the local characteristics (neighborhood) of site *s*
- At most one site undergoes a change for every *t*
- Sites are visited in a fixed order
- The result does not depend on $X(0)$
- The distribution of $X(t)$ converges to $\pi(\omega)$

Gibbs Sampler II.

- Optimal solution is achieved by simulated annealing $\Rightarrow T(t)$
- $X(t)$ converges to one of the configurations of lowest energy provided that
  1. every site is visited infinitely often,
  2. $T(t)$ is monotonically decreasing,
  3. $T(t) \to 0$ as $t \to \infty$, and
  4. $T(t) \geqslant N\Delta/\log t$ for all $t \geqslant t_0 \geqslant 2$, where

$$\Delta = \max_{\omega} U(\omega) - \min_{\omega} U(\omega)$$

  and N is the number of sites
- In practice: $T(0) = 4$, $T(k) = \dfrac{C}{\log(1+k)}$ with $C = 3$ or 4

Results I.

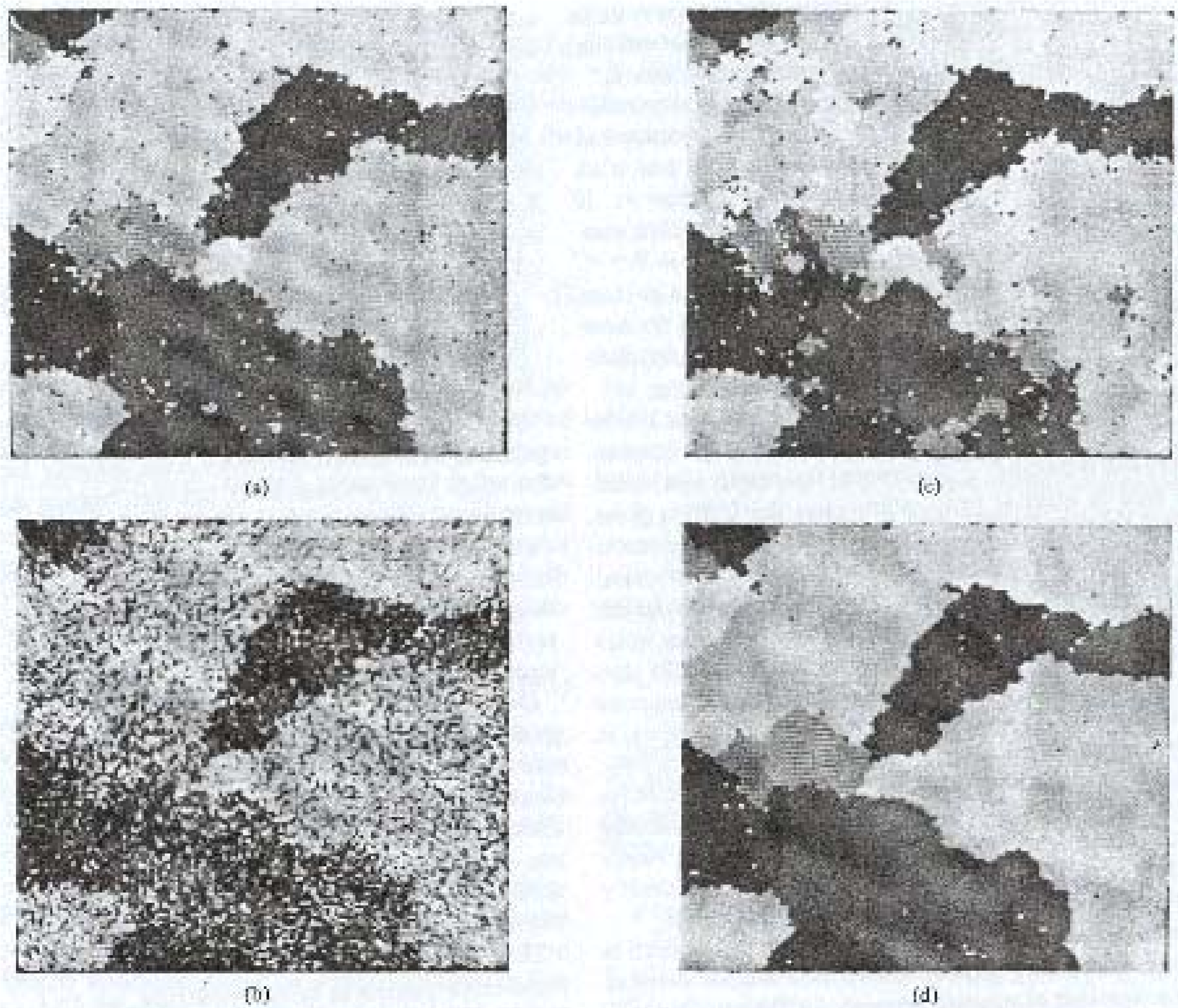


Fig. 2. (a) Original image: Sample from MRF. (b) Degraded image: Additive noise. (c) Restoration: 25 iterations. (d) Restoration: 300 iterations.

Results II.

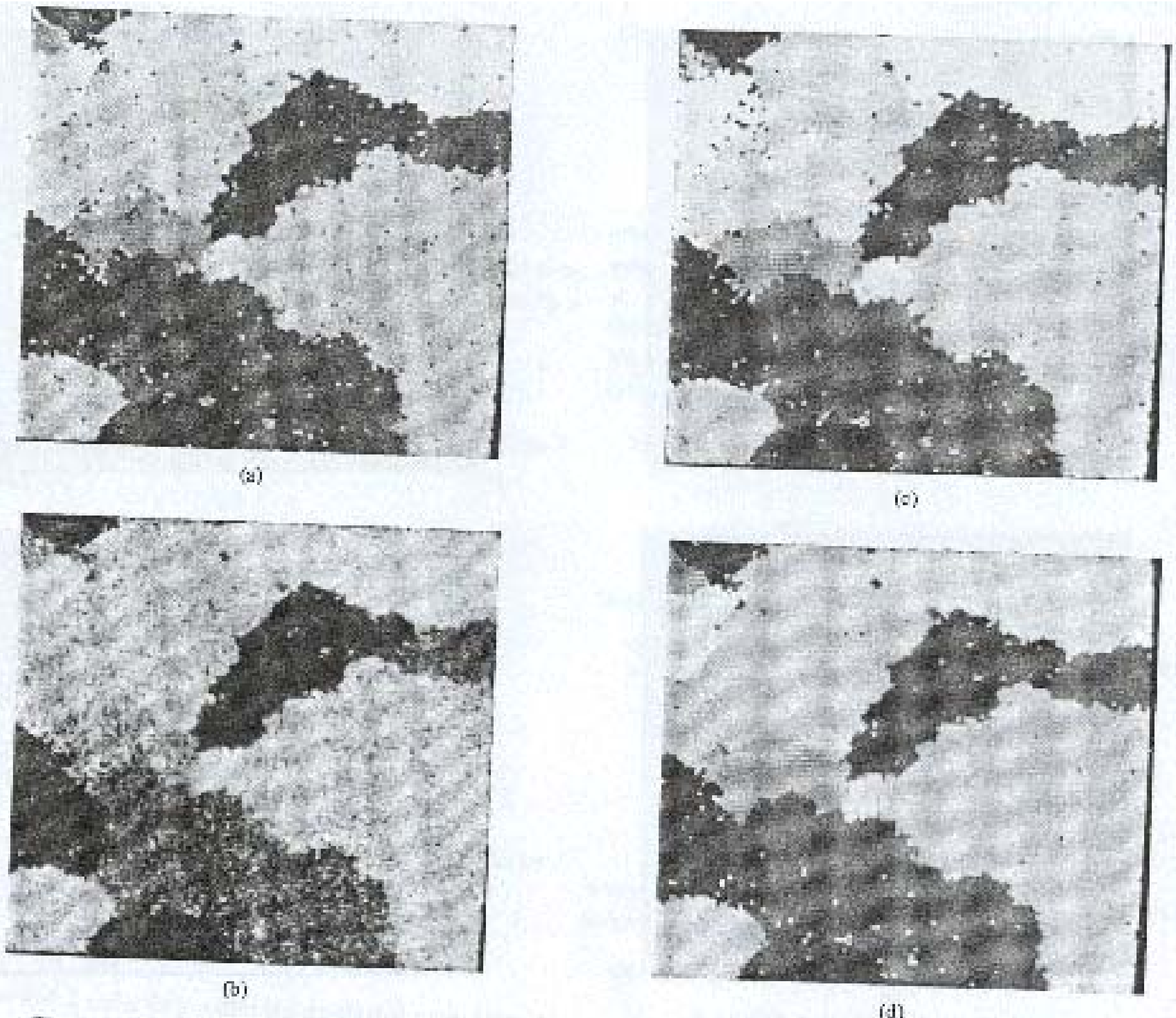


Fig. 3. (a) Original image: Sample from MRF. (b) Degraded image: Blur, nonlinear transformation, multiplicative noise. (c) Restoration: 25 iterations. (d) Restoration: 300 iterations.

Results III.

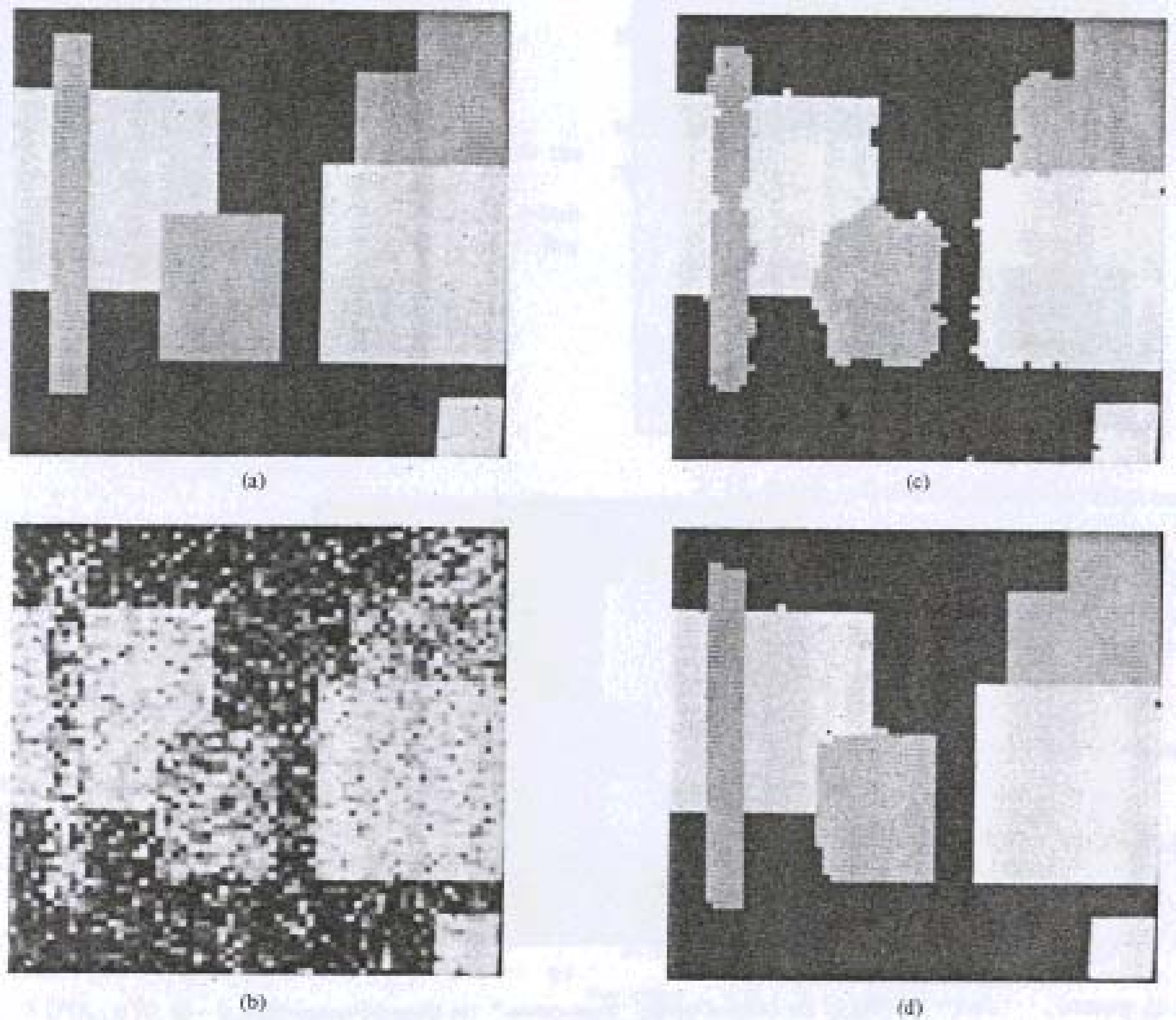


Fig. 4. (a) Original image: "Hand-drawn." (b) Degraded image: Additive noise. (c) Restoration: Without line process; 1000 iterations. (d) Restoration: Including line process; 1000 iterations.

Results IV.

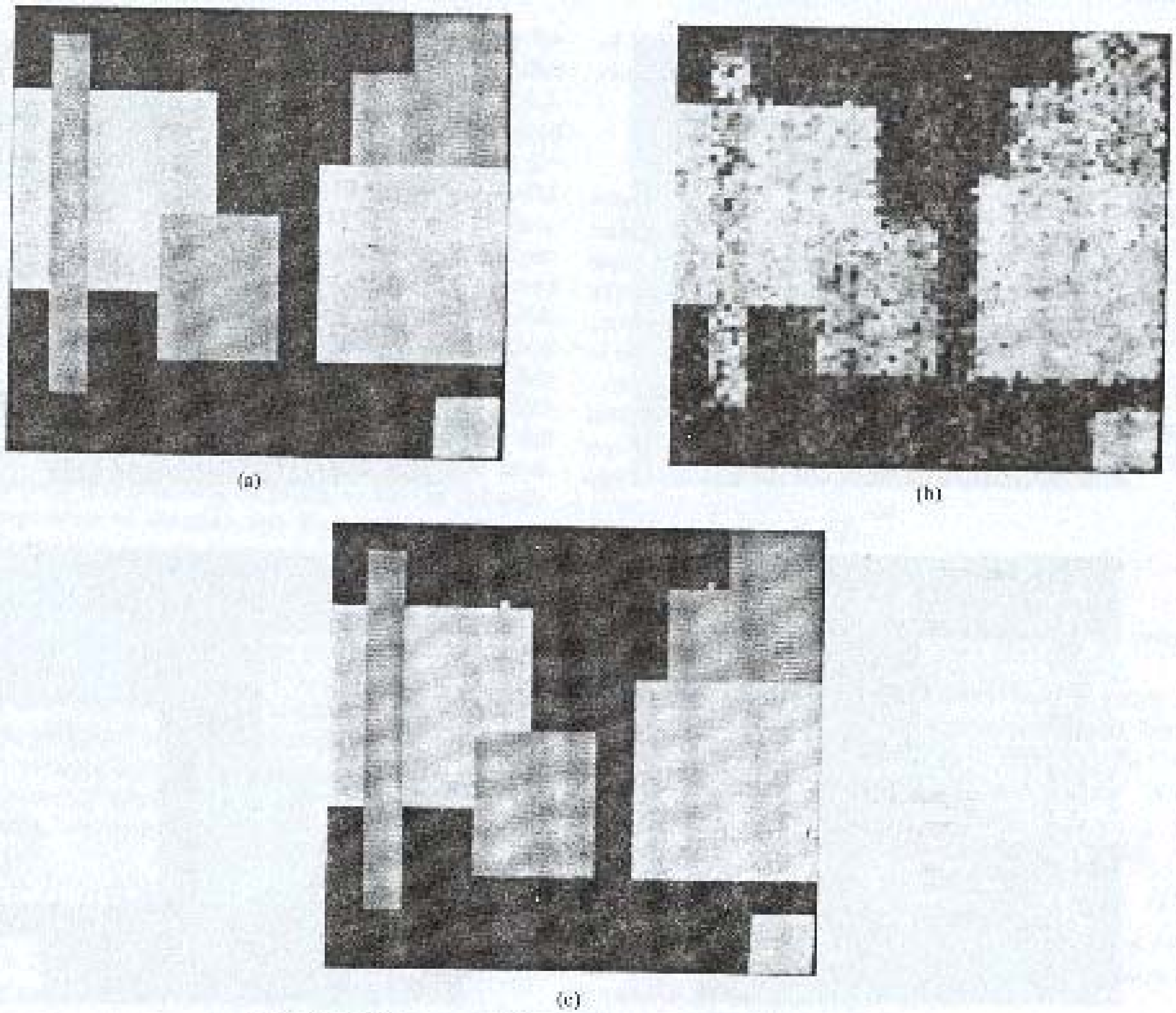


Fig. 6. (a) Original image: "Hand-drawn." (b) Degraded image: Blur, nonlinear transformation, multiplicative noise. (c) Restoration: including line process; 1000 iterations.

Results V.

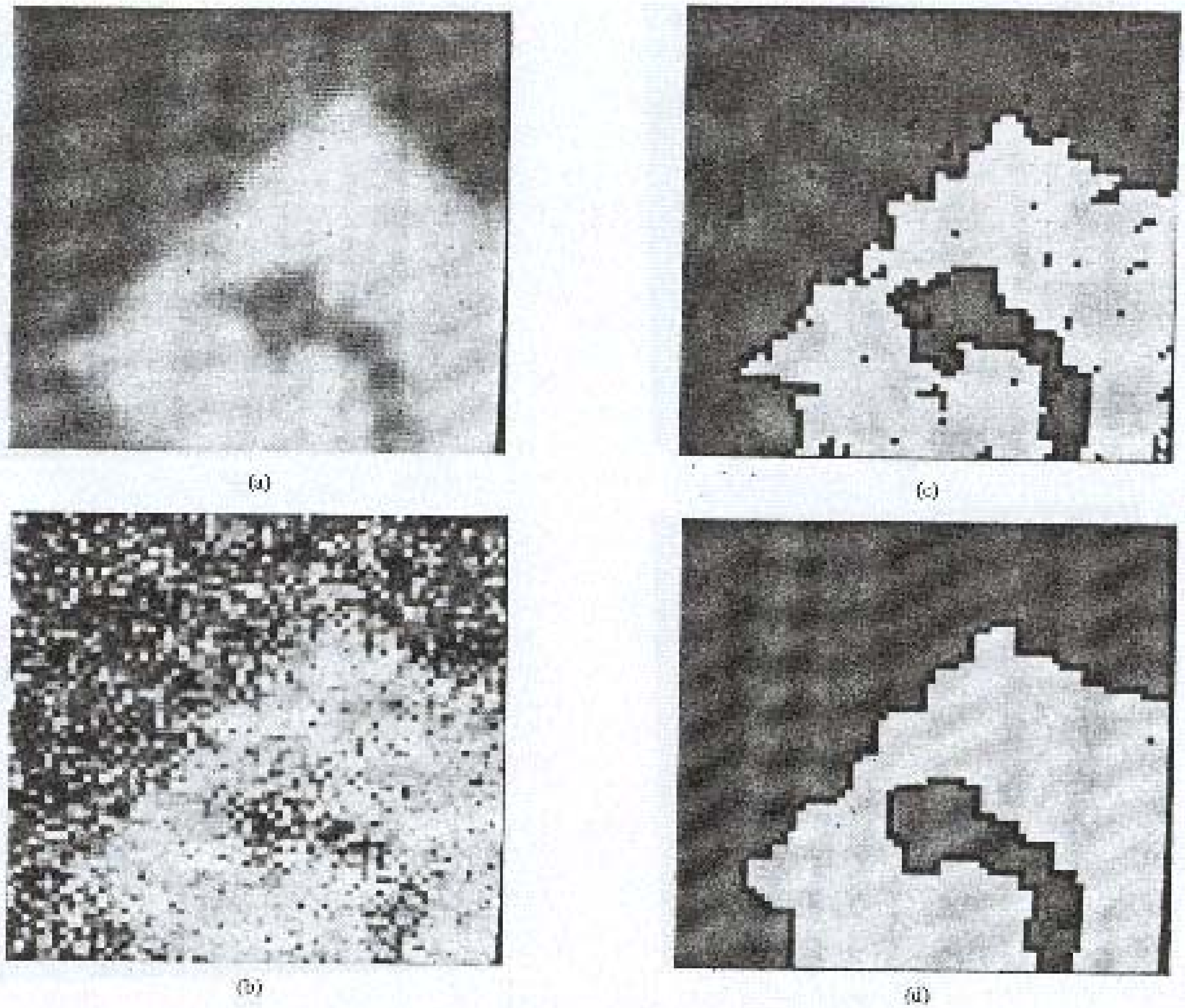


Fig. 7. (a) Blurred image (roadside scene). (b) Degraded image: Additive noise. (c) Restoration including line process; 100 iterations. (d) Restoration including line process: 1000 iterations.